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Research Article

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A metric graph satisfying $w_4^1 = 1$ that cannot be lifted to a curve satisfying $\dim(W_4^1) = 1$

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Abstract: For all integers $g \geq 6$ we prove the existence of a metric graph G with $w_4^1 = 1$ such that G has Clifford index 2 and there is no tropical modification G' of G such that there exists a finite harmonic morphism of degree 2 from G' to a metric graph of genus 1. Those examples show that not all dimension theorems on the space classifying special linear systems for curves have immediate translation to the theory of divisors on metric graphs.

Keywords: Metric graphs, Curves, Lifting problems, Special divisors, Clifford index, Dimension theorems

MSC: 14H51, 14T05, 05C99

1 Introduction

Let K be an algebraically closed field complete with respect to some non-trivial non-Archimedean valuation. Let R be the valuation ring of K, let m_R be its maximal ideal and $k = R/m_R$ the residue field. Let X be a smooth complete curve of genus g defined over K. Associated to a semistable formal model $\mathfrak{X}$ over R of X there exists a so-called skeleton $\Gamma = \Gamma_{\mathfrak{X}}$ which is a finite metric subgraph of the Berkovich analytification X^{an} of X together with an augmentation function $a : \Gamma \to \mathbb{Z}^+$ such that $a(v) = 0$ except for at most finitely many points (see e.g. [1]). In the case all components of the special fiber $\mathfrak{X}_k$ are rational then this augmentation function is identically zero and we can consider Γ as a metric graph. This is the situation we consider in this paper. In this situation, from the point of view of the metric graph Γ, we say the curve X is a lift of Γ.

There exists a theory of divisors and linear equivalence on Γ very similar to the theory on curves and, in case X is a lift of Γ, those theories on X and Γ are related by means of a specialisation map

$$\tau_* : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma) .$$

For a divisor E on Γ one defines a rank $\mathrm{rk}(E)$ and for a divisor D on X the specialisation theorem says (see e.g. [2], since we restrict to the case of zero augmentation map this is in principle considered in [3])

$$\dim(|D|) \leq \mathrm{rk}(\tau_*(D)) .$$

In the hyperelliptic case many classical results on linear systems on curves also do hold for linear systems on metric graphs. As an example, if the graph Γ has a very special linear system g^r_{2r} then Γ has a g^1_2 and $g^r_{2r} = rg^1_2$ (see [4, 5]). Hence the theory of linear systems g^r_d of Clifford index 0 (meaning $d - 2r = 0$) is the same for graphs as for curves (see Section 2.2 for this terminology). This is not true for the theory of linear systems of Clifford index more than 0. As an example: the theory of linear systems of Clifford index 1 concerns the theory of linear systems g^r_d satisfying $d - 2r = 1$, $r \geq 1$ and $d \leq r + g - 2$. In the case of curves, H. Martens Theorem (see [6]) states that if C is a non-hyperelliptic curve and C has a linear system g^r_d with Clifford index 1 and $d \leq g - 1$ then either

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$d = 3$ or $d = 5$. In [7] we obtain for all $r \geq 1$ and $d = 2r + 1 \leq g - 1$ the existence of a non-hyperelliptic graph Γ of genus g having a linear system g^r_d.

For a curve X the complete linear systems g^n_d with $n \geq r$ are parametrized by a closed subscheme W^r_d of the Jacobian $J(X)$ and $d - 2r - \dim(W^r_d)$ gives a kind of generalisation of the Clifford index for moving linear systems on X. In particular in the case $r \leq g-1$ then $\dim(W^r_d) \leq d - 2r$ and $\dim(W^r_d) = d - 2r$ for some $r < g-1$ if and only if X is hyperelliptic (see [8]). In [9] it is shown that using the dimension of a similar subspace of the Jacobian $J(\Gamma)$ of a metric graph Γ this statement is not true. Moreover in that paper the authors do introduce a much better invariant w^r_d as a replacement for $\dim(W^r_d)$ in the case of graphs which is more close to the definition of the rank of a divisor on a graph (see Section 2.2). In [10] it is proved that $w^r_d \geq \dim(W^r_d)$ in the case Γ is a metric graph and the curve X is a lift of Γ. At the moment is seems not known whether $w^r_d \geq d - 2r$ for some $0 < r < g-1$ implies Γ is hyperelliptic.

Concerning the next case, in [11, Appendix] one finds Mumford's classification of all curves X such that $\dim(W^r_d) = d - 2r - 1$. In the case C is a non-hyperelliptic curve such that $\dim(W^r_d) = d - 2r - 1$ for some $r \geq 1$ and $d \leq g - 1$ then either C is a trigonal curve or C is a smooth plane curve of degree 5 or C is a bi-elliptic curve (meaning that there exists a double covering $\pi : C \to E$ with E an elliptic curve). In [11] the author assumes $\text{char}(k) \neq 2$ although it is not so clear whether this is also necessary for the arguments in the appendix of that paper. In the appendix of this paper I give a very short proof that in the case C is neither hyperelliptic, not trigonal of genus $g \geq 10$ and if C has two different linear systems g^1_4 then C is bi-elliptic not using any assumption on $\text{char}(k)$. Many more generalisations are proved by different authors (see e.g. [12]). In this paper we show that the theory of curves satisfying $\dim(W^1_4) = 1$ is different from the theory of graphs satisfying $w^1_4 = 1$. In particular for all genus $g \geq 10$ we prove the existence of a metric graph G_n of genus g ($n = g - 3$) satisfying $w^1_4 = 1$ that cannot be lifted to a curve satisfying $\dim(W^1_4) = 1$.

Related to this result it should be mentioned that non-liftable linear systems on graphs are also known. We say that a linear system g^r_d on a graph G is liftable if there exists a lift X of G and a divisor D on X with $\dim(|D|) = r$ such that $\tau_*(D) \in g^r_d$. As an example, from [13, Theorem 4.8] it is known that the linear system g^1_2 on a hyperelliptic graph having a vertex v adjacent to at least 3 different bridges (a bridge of a graph G is an edge e such that $G \setminus e$ is disconnected) is not liftable to a hyperelliptic curve because of the violation of some Hurwitz condition. In [14, Example 5.13] one finds an example due to Luo of a graph with a g^1_3 that cannot be lifted to a curve. Also in [7] one finds lots of types of linear systems g^r_d that cannot be lifted to curves, e.g. so-called free linear systems g^2_5 of graphs that cannot be lifted to a curve because a curve with a plane model of degree 5 has genus at most 6.

In Section 2 we recall some generalities on graphs and the theory of divisors on graphs. For generalities on the specialisation map and the relation between the metric graphs and skeleta inside Berkovich curves we refer to the references. It is not needed to understand the arguments used in this paper, it is important for the motivation. In Section 3 we give the description of the graph denoted by G_n (n an integer at least equal to 2) and we prove it satisfies $w^1_4 = 1$ and it has Clifford index equal to 2. In Section 4 first we explain that in the case G_n could be lifted to a curve satisfying $\dim(W^1_4) = 1$ then for some tropical modification $\tilde{\Gamma}$ of G_n there would exist a finite harmonic morphism (see Section 2.3) $\pi : \tilde{\Gamma} \to \Gamma$ of degree 2 with Γ a metric graph of genus 1. Finally in Section 4 we prove that such harmonic morphism does not exist.

2 Generalities

2.1 Graphs

A *topological graph* Γ is a compact topological space such that for each $P \in \Gamma$ there exists $n_P \in \mathbb{Z}^+$ and $\epsilon \in \mathbb{R}^+_0$ such that some neighborhood U_P of P in Γ is homeomorphic to $\{z = re^{2\pi ik/n_P} : 0 \leq r \leq \epsilon$ and k is an integer satisfying $0 \leq k \leq n_P - 1\} \subset \mathbb{C}$ with P corresponding to 0. Such a topological graph Γ is called *finite* in the case there are only finitely many points $P \in \Gamma$ satisfying $n_P \neq 2$. We only consider finite topological graphs. We call n_P the *valence* of P on Γ. Those finitely many points P of Γ with $n_P \neq 2$ are called the *essential vertices* of Γ. The *tangent space* $T_P(\Gamma)$ of Γ at P is the set of n_P connected components of $U_P \setminus \{P\}$

for U_P as above. In this definition, using another such neighborhood U'_P then we identify connected components of $U_P \setminus \{P\}$ and $U'_P \setminus \{P\}$ in the case their intersection is not empty.

A metric graph Γ is a finite topological graph Γ together with a finite subset $V_\infty(\Gamma)$ of the set of 1-valent points of Γ and a complete metric on $\Gamma \setminus V_\infty(\Gamma)$. A *vertex set* V of a metric graph Γ is a finite subset of Γ containing all essential vertices. The pair (Γ, V) is called a *metric graph with vertex set* V. The elements of V are called the vertices of (Γ, V). The connected components of $\Gamma \setminus V$ are called the *edges* of (Γ, V). The elements of $\overline{e} \setminus e$ are called the *end vertices* of e ($\overline{e}$ is the closure of e). We always choose V such that each edge has two different end vertices. Using the metric on $\Gamma \setminus V_\infty(\Gamma)$ each edge e of Γ has a length $l(e) \in \mathbb{R}^+_0 \cup \{\infty\}$. Moreover $l(e) = \infty$ if and only if some end vertex of e belongs to $V_\infty(\Gamma)$. We write $E(\Gamma, V)$ to denote the set of edges of (Γ, V). The *genus* of (Γ, V) is defined by $|E(\Gamma)| - |V(\Gamma)| + 1$ and it is independent of the choice of V. Therefore it is denoted by $g(\Gamma)$ and called the genus of Γ.

A *subgraph* of a metric graph (Γ, V) with vertex set is a closed subset $\Gamma' \subset \Gamma$ such that $(\Gamma', \Gamma' \cap V)$ is a metric graph with vertex set. In the case Γ' is homeomorphic to the unit circle S^1 in $\mathbb{C}$ then it is called a *loop* in (Γ, V). A metric graph Γ is called a *tree* if $g(\Gamma) = 0$.

2.2 Linear systems on graphs

We refer to Section 2 of [7] for the definitions of a divisor, an effective divisor, a rational function, linear equivalence of divisors, the canonical divisor and the rank of a divisor on a metric graph. The rank of a divisor D on a metric graph Γ is denoted by $\text{rk}(D)$ and if it is necessary to add the graph then we write $\text{rk}_\Gamma(D)$. For a divisor D on a graph Γ we write $D \geq 0$ to indicate it is an effective divisor on Γ. A very important tool in the study of divisors on a metric graph Γ is the concept of a reduced divisor at some point P of Γ (see [7]*Section 2.1) and the burning algorithm to decide whether a given divisor on Γ is reduced at P (see [7, Section 2.2]).

For a divisor D on a metric graph Γ we write $|D|$ to denote the set of effective divisors linearly equivalent to D. As in the case of curves we call it the *complete linear system* defined by D. The rank $\text{rk}(D)$ replaces the concept of the dimension of a complete linear system on a curve. As in the case of curves we say the complete linear system $|D|$ is a linear system g^r_d on Γ if $\deg(D) = d$ and $\text{rk}(D) = r$. From of the Riemann-Roch Theorem for divisors on graphs it follows as in the case of curves that divisors D on Γ such that $\text{rk}(D)$ cannot be computed in a trivial way are exactly those divisors satisfying $\text{rk}(D) > \max\{0, \deg(D) - g(\Gamma) + 1\}$. Those divisors are called *very special*. The *Clifford index* of a very special divisor D on Γ is defined by $c(D) = \deg(D) - 2\,\text{rk}(D)$. Clifford's Theorem for metric graphs implies $c(D) \geq 0$ for all very special divisors D on Γ. The Clifford index $c(\Gamma)$ of Γ is the minimal value $c(D)$ for a very special divisor D on Γ.

Motivated by the definition of the rank of a divisor on a metric graph one introduces the following replacement for the dimension of the space W^r_d parametrizing linear systems g^r_d on a curve. In the case Γ has no linear system g^r_d then $w^r_d = -1$. Otherwise w^r_d is the maximal integer $w \geq 0$ such that for each effective divisor F of degree $r + w$ there exists an effective divisor E of degree d with $\text{rk}(E) \geq r$ such that $E - F \geq 0$.

2.3 Harmonic morphism

Let Γ and Γ' be two metric graphs and let $\phi : \Gamma' \to \Gamma$ be a continuous map. In the case V (resp. V') is a vertex set of Γ (resp. Γ') then ϕ is called a *morphism* from (Γ', V') to (Γ, V) if $\phi(V') \subset V$ and for each $e \in E(\Gamma, V)$ the set $\phi^{-1}(\overline{e})$ is a union of closures of edges of (Γ', V'). Moreover if $e' \in E(\Gamma', V')$ with $e' \subset \phi^{-1}(\overline{e})$ then either $\phi(e')$ is a vertex in V' or the restriction $\phi_{e'} : e' \to e$ is a dilation with some factor $d_{e'}(\phi) \in \mathbb{Z}^+_0$. In the case $\phi(e')$ is a vertex then we write $d_{e'}(\phi) = 0$. We call $d_{e'}(\phi)$ the *degree* of ϕ along e'.

We say ϕ is a morphism of metric graphs if there exist vertex sets V (resp. V') of Γ (resp. Γ') such that ϕ is a morphism from (Γ', V') to (Γ, V). In that case, for $P \in \Gamma'$, $v' \in T_{P'}(\Gamma')$ and e' an edge of (Γ', V') such that v' is defined by some connected component of $e' \setminus \{P'\}$ we set $d_{v'}(\phi) = d_{e'}(\phi)$. Such morphism is called *finite* in the case $d_{e'}(\phi) > 0$ for all $e' \in E(\Gamma', V')$. There is a natural map $d_\phi(P') : T_{P'}(\Gamma') \setminus \{v' : d_{v'}(\phi) = 0\} \to T_{\phi(P')}(\Gamma)$

defined as follows. The connected component of $e' \setminus \{P'\}$ defining $v' \in T_{P'}(\Gamma')$ with $d_{v'}(\phi) \neq 0$ is mapped to a connected component of $\phi(e') \setminus \{\phi(P')\}$ and this defines $v \in T_{\phi(P')}(\Gamma)$, then $d_\phi(P')(v') = v$.

The morphism $\phi : \Gamma' \to \Gamma$ of metric graphs is called *harmonic* at $P' \in \Gamma'$ if for each $v \in T_{\phi(P')}(\Gamma)$ the number

$$\Sigma\{d_{v'}(\phi) : v' \in T_{P'}(\Gamma') \text{ and } d_\phi(P')(v') = v\}$$

is independent of v. In that case this sum is denoted by $d_{P'}(\phi)$ and it is called the *degree* of ϕ at P'. We say the morphism ϕ is harmonic if ϕ is surjective and ϕ is harmonic at each point $P' \in \Gamma'$. In this case for $P \in \Gamma$ one has $\Sigma(d_{P'}(\phi) : \phi(P') = P)$ is independent of P and it is called the *degree* of ϕ denoted by $\deg(\phi)$.

An *elementary tropical modification* of a metric graph Γ is a metric graph Γ' obtained by attaching an infinite closed edge to Γ at some point $P \in \Gamma \setminus V_\infty(\Gamma)$. A metric graph obtained from Γ as a composition of finitely many elementary tropical modifications is called a *tropical modification* of Γ. Two metric graphs Γ_1 and Γ_2 are called *tropically equivalent* if there is a common tropical modification Γ of Γ_1 and Γ_2. This terminology can be found in e.g. [14] together with some examples.

3 The example

In the proof of this section we are going to use some lemmas concerning linear systems on graphs.

Lemma 3.1 (Lemma 1 in [7]). *Let Γ_0 be a metric graph and let Γ be a graph obtained from Γ_0 by attaching loops at some different points of valence 2 on Γ_0. Let γ be such a loop attached to Γ_0. Let E and E' be linearly equivalent divisors on Γ_0 or on γ then E and E' are linearly equivalent divisors on Γ.*

Lemma 3.2 (Lemma 2 in [7]). *Assume Γ_0 and Γ are as in Lemma 3.1. Let E and E' be effective divisors on Γ_0 such that E and E' are linearly equivalent as divisors on Γ. Then E and E' are linearly equivalent as divisors on Γ_0.*

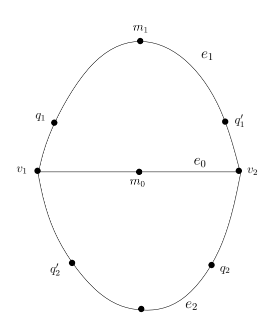
The proofs of Lemmas 3.1 and 3.2 do not depend on the particular graph Γ_0 used in [7].

Lemma 3.3 (Corollary 1 in [7]). *Let Γ_0 be a metric graph and let Γ be the graph obtained from Γ_0 by attaching a loop γ at some point $v \in \Gamma_0$. Let P be a point of $\gamma \setminus \{v\}$ and let D be an effective divisor on Γ_0. If* $\mathrm{rk}_\Gamma(D+P) \geq r$ *then* $\mathrm{rk}_{\Gamma_0}(D) \geq r$.

Lemma 3.4 (Main Theorem in [4]). *Let Γ be a metric graph of genus $g \geq 4$ and let r be an integer satisfying $2 \leq r \leq g-2$ such that Γ has a linear system g^r_{2r} then Γ has a linear system g^1_2.*

The metric graph G_0 we start with has genus 2 and can be seen in figure 1.

Fig. 1. The graph G_0



Here v_1 and v_2 are two points of valence 3 (all other points have valence 2) and they are connected by three edges e_0, e_1 and e_2 of mutually different lengths. For $0 \le i \le 2$ the point m_i is the midpoint of e_i.

Lemma 3.5. *The graph G_0 has a unique g_2^1 given by $|v_1 + v_2| = |2m_0| = |2m_1| = |2m_2|$ and in the case $v \in G_0$ such that $2v \in |v_1 + v_2|$ then $v = m_i$ for some $0 \le i \le 2$.*

Proof. Clearly $2m_i \in |v_1 + v_2|$ for $0 \le i \le 2$ and in the case $v \in e_i \setminus \{v_1, v_2, m_i\}$ then taking v' on e_i such that the distance on e_i from v to v_1 is equal to the distance of v' to v_2, then $v + v' \in |v_1 + v_2|$ (clearly $v \neq v'$). This proves $\mathrm{rk}(v_1 + v_2) = 1$ (it cannot have rank 2 because $g(G_0) \neq 0$). It is well-known that a graph of genus at least 2 has at most one g_2^1 (see [15, Proposition 5.5], the proof given for a finite graph also holds for a metric graph). Indeed, for this graph G_0, if $v \neq v_2$ then $v_1 + v$ is clearly v_2-reduced, hence $|v_1 + v - v_2| = \emptyset$ and therefore $\mathrm{rk}(v_1 + v) = 0$. This proves the uniqueness of g_2^1 on G_0.

Finally for $v + v' \in |v_1 + v_2|$ as before (including the possibility $v + v' = v_1 + v_2$), since v' is a v-reduced divisor one has $|v' - v| = \emptyset$ hence $|v_1 + v_2 - 2v| = \emptyset$. This proves $2v \in |v_1 + v_2|$ implies $v = m_i$ for some $0 \le i \le 2$. □

As indicated in figure 1 we fix $q_i \in]v_i, m_i[\subset e_i$ for $i = 1, 2$.

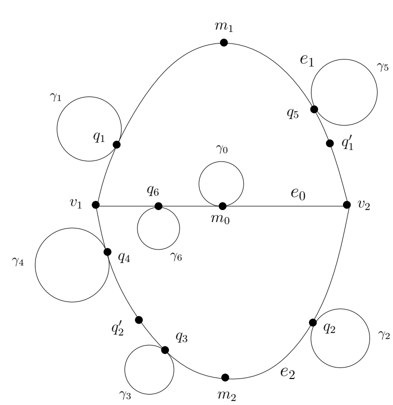
Lemma 3.6. *There is no g_3^1 on G_0 such that $|g_3^1 - 2m_0| \neq \emptyset$, $|g_3^1 - 2q_1| \neq \emptyset$ and $|g_3^1 - 2q_2| \neq \emptyset$.*

Proof. For $i = 1, 2$ we take $q_i' \in e_i$ such that the distance on e_i from v_1 to q_i is equal to the distance on e_i from v_2 to q_i'. Assume g_3^1 on G_0 with $|g_3^1 - 2m_0| \neq \emptyset$, hence there exists $v \in G_0$ such that $g_3^1 = |g_2^1 + v|$.

First assume $v \in (e_0 \cup e_2) \setminus \{v_1\}$. Then $q_1 + q_1' + v \in g_3^1$ and clearly $q_1' + v$ is a q_1-reduced divisor. This implies $|q_1' + v - q_1| = |g_3^1 - 2q_1| = \emptyset$. In the case $v \notin (e_0 \cup e_2) \setminus \{v_1\}$ then certainly $v \in (e_0 \cup e_1) \setminus \{v_2\}$ and using similar arguments we find $|g_3^1 - 2q_2| = \emptyset$. This finishes the proof of the lemma. □

Now for an integer $n \ge 1$ we make a graph G_n as follows. In case $n \ge 3$ we fix some more different points $q_3, \cdots, q_n$ on $G_0 \setminus \{v_1, v_2, m_0, m_1, m_2, q_1, q_3\}$. Then, for all $n \ge 1$, the graph G_n is obtained from G_0 by attaching a loop γ_0 at m_0 and loops γ_i at q_i for each $1 \le i \le n$ (we also are going to denote m_0 by q_0). As an example see a possible picture of G_6 in figure 2. Clearly $g(G_n) = n + 3$. We prove that the Clifford index of G_n is at least 2 in case $n \ge 2$.

Fig. 2. The graph G_6



Proposition 3.7. *Let r be an integer with $1 \le r \le n$. Then G_n has no g_{2r+1}^r in the case $n \ge 2$.*

Proof. First we show G_n has no linear system g_2^1 in the case $n \ge 1$. Assume there is a g_2^1 on G_n. Take $1 \le i \le n$ and let $v \in \gamma_i \setminus \{q_i\}$ and $v' \in G_n$ such that $v + v' \in g_2^1$. Let G_n^0 be the closure of $G_n \setminus \gamma_i$ and assume $v' \in G_n^0$. It follows from Lemma 3.3 that $\mathrm{rk}_{G_n^0}(v') \ge 1$, but because $g(G_n^0) > 0$ this is impossible. Hence $v' \in \gamma_i$. On γ_i there exists v'' such that $v + v'$ is linearly equivalent to $q_i + v''$ as divisors on γ_i, hence $q_i + v'' \in g^1_2$ because of Lemma 3.1. But we proved this implies $v'' = q_i$, hence $2q_i \in g^1_2$. This implies $\mathrm{rk}_{G_0}(2q_i) = 1$. Indeed, take $p \in G_0 \setminus \{q_i\}$ and let D_p be the p-reduced divisor on G_0 linearly equivalent to $2q_i$. We need to show $D_p - p \geq 0$. The burning algorithm applied to G_n implies D_p is a p-reduced divisor on G_n too. Since $\mathrm{rk}_{G_n}(2q_i) = 1$ it follows $D_p - p \geq 0$. So we obtain $\mathrm{rk}_{G_0}(2q_i) = 1$, but from Lemma 3.5 we know this cannot be true, hence G_n has no linear system g^1_2.

From now on assume $n \geq 2$. Fix some integer r satisfying $1 \leq r \leq n-1$ and assume G_n has a linear system g^r_{2r+1}. For $0 \leq i \leq r+1$ fix $v_i \in \gamma_i$ with $v_i \neq q_i$. For $0 \leq i \leq 2$ there exists $E_i \in g^r_{2r+1}$ satisfying

$$E_i - (v_i + v_3 + \cdots v_{r+1}) \geq 0\,.$$

For $j \in \{3, \cdots, r+1\} \cup \{i\}$ let $D_{i,j} = E_i \cap (\gamma_j \setminus \{q_j\})$, hence $D_{i,j} - v_j \geq 0$. In the case that for some j the q_j-reduced divisor on γ_j linearly equivalent to $D_{i,j}$ contains a point v'_j different from q_j (then the point v'_j is unique) then there is an effective divisor E' on $\overline{G_n \setminus \gamma_j}$ of degree $2r$ such that $E' + v'_j \in g^r_{2r+1}$. From Lemma 3.3 it follows $\mathrm{rk}_{\overline{G_n \setminus \gamma_j}}(E') = r$, hence $\overline{G_n \setminus \gamma_j}$ has a linear system g^r_{2r}. Since $g(\overline{G_n \setminus \gamma_j}) = n+2$ and $2 \leq 2r \leq 2n-2$ this would imply $\overline{G_n \setminus \gamma_j}$ is a hyperelliptic graph (Lemma 3.4). But we proved $\overline{G_n \setminus \gamma_j}$ is not hyperelliptic (it is a graph G_{n-1}; the proof of that argument also works on $\overline{G_n \setminus \gamma_0}$ in the case $j = 0$). Since v_j is not linearly equivalent to q_j as a divisor on γ_j it follows $D_{i,j}$ is linearly equivalent to $m_{i,j}q_j$ for some $m_{i,j} \geq 2$ on γ_j.

So we obtain $E'_i \in g^r_{2r+1}$ on G_n such that

$$E'_i - (2q_i + 2q_3 + \cdots + 2q_{r+1}) \geq 0$$

for $0 \leq i \leq 2$ and E'_i is contained in G_0. Because of Lemma 3.2 those divisors E'_i are linearly equivalent as divisors on G_0. It follows $E"_i = E'_i - (2q_3 + \cdots + 2q_{r+1})$ with $0 \leq i \leq 2$ are effective linearly equivalent divisors on G_0 of degree 3 with $E"_i - 2q_i \geq 0$. Since $g(G_0) = 2$ each divisor of degree 3 on G_0 defines a g^1_3 and we obtain a contradiction to Lemma 3.6.

Finally, if G_n has an g^n_{2n+1} then because of the Theorem of Riemann-Roch $|K_{G_n} - g^n_{2n+1}| = g^1_3$, we already excluded this case. □

Proposition 3.8. *On G_n we have $w^1_4 = 1$.*

In order to prove this proposition we need the existence of many linear systems g^1_4 on G_n. Those can be obtained from divisors of degree 4 on G_0 using the following lemma.

Lemma 3.9. *Let D be an effective divisor of degree 4 on G_0. Then $\mathrm{rk}_{G_n}(D) \geq 1$.*

Proof. Since D is an effective divisor of degree 4 on G_0 from the Riemann-Roch Theorem it follows $\mathrm{rk}_{G_0}(D) = 2$. This implies for each $0 \leq i \leq n$ there is an effective divisor $D' \geq 2q_i$ linearly equivalent to D on G_0. Because of Lemma 3.1 the divisor D' is linearly equivalent to D on G_n. Moreover, for $v \in \gamma_i$ there is an effective divisor on γ_i linearly equivalent to $2q_i$ containing v and using the same lemma we obtain the existence of an effective divisor on G_n linearly equivalent to D containing v. Similarly, for $v \in G_0$ we obtain an effective divisor on G_0 linearly equivalent to D and containing v and again this divisor is also linearly equivalent to D as a divisor on G_n. This proves $\mathrm{rk}_{G_n}(D) \geq 1$. □

Proof of Proposition 3.8. Fix $v_1, v_2 \in G_n$. We need to prove that there exists a g^1_4 on G_n such that $|g^1_4 - v_1 - v_2| \neq \emptyset$.

In the case $v_1, v_2 \in G_0$ we can use any effective divisor D on G_0 containing $v_1 + v_2$. Then we have $\mathrm{rk}_{G_n}(D) \geq 1$ because of Lemma 3.9 and $|D - v_1 - v_2| \neq \emptyset$. Next assume $v_1 \in G_0$ and $v_2 \in \gamma_i \setminus \{q_i\}$ for some $0 \leq i \leq n$. On G_0 take any effective divisor D of degree 4 containing $v_1 + 2q_i$. Since $2q_i$ is linearly equivalent to $v_2 + v'_2$ for some $v'_2 \in \gamma_i$ as a divisor on γ_i, again using Lemma 3.1 we find that $D - 2q_i + v_2 + v'_2$ is an effective divisor linearly equivalent to D on G_n. Hence $\mathrm{rk}_{G_n}(D) \geq 1$ because of Lemma 3.9 and $|D - v_1 - v_2| \neq \emptyset$. Assume $v_1 \in \gamma_{i_1} \setminus \{q_{i_1}\}$ and $v_2 \in \gamma_{i_2} \setminus \{q_{i_2}\}$ for some $i_1 \neq i_2$. Then one makes a similar argument using the divisor $2q_{i_1} + 2q_{i_2}$ on G_0. Finally, if $v_1, v_2 \in \gamma_i \setminus \{q_i\}$ for some i then one uses any effective divisor D of degree 4 on G_0 containing $3q_i$. Since $3q_i$ is linearly equivalent on γ_i to an effective divisor containing $v_1 + v_2$ again we find $\mathrm{rk}_{G_n}(D) \geq 1$ and $|D - v_1 - v_2| \neq \emptyset$. □

4 The lifting problem

We now consider the following lifting problem associated to G_n. Let K be an algebraically closed complete non-archimedean valued field and let X be a smooth algebraic curve of genus g. Let X^{an} be the analytification of X (as a Berkovich curve). Let R be the valuation ring of K, assume $\mathfrak{X}$ is a strongly semistable model of X over R (meaning the special fiber is nodal with smooth irreducible components) such that the special fiber has only rational components and let Γ be the associated skeleton. Is it possible to obtain this situation such that $\Gamma = G_n$ and $\dim(W_4^1(X)) = 1$? In that case, taking into account the result from [10] mentioned in the introduction, this would give a geometric explanation for $w_4^1(G_n) = 1$. This lifting problem will be the motivation for considering the existence of a certain harmonic morphism associated to G_n. Making this motivation we are going to refer to some suited papers for terminology and some definitions. The definitions necessary to understand the question on the existence of the harmonic morphism are given in Section 2.3. Finally we are going to prove that the harmonic morphism does not exist, proving that the lifting problem has no solution. In particular we obtain that the classification of metric graphs satisfying $w_4^1 = 1$ is different from the classification of smooth curves satisfying $\dim(W_4^1) = 1$.

Assume the lifting problem has a solution. The curve X of that solution cannot be hyperelliptic since G_n is not hyperelliptic. This follows from the specialisation Theorem from [2] or [3] already mentioned in the introduction. From [11] one obtains the following classification in case $\mathrm{char}(k) \neq 2$ of non-hyperelliptic curves X of genus at least 6 satisfying $\dim(W_4^1(X)) = 1$ (for arbitrary characteristic and $g \geq 10$ see the Appendix): X is trigonal (has a g_3^1), X is a smooth plane curve of degree 5 (hence has genus 6 and has a g_5^2) or X is bi-elliptic (there exists a double covering $\pi : X \to E$ with $g(E) = 1$). From Proposition 3.7 we know G_n has no g_3^1 and no g_5^2, hence the curve X has to be bi-elliptic.

So assume there exists a morphism $\pi : X \to E$ with $g(E) = 1$ of degree 2. This induces a map $\pi^{an} : X^{an} \to E^{an}$ between the Berkovich analytifications. In the case E is not a Tate curve then each strong semistable reduction of E contains a component of genus 1 in its special fiber, in particular the augmentation map of the associated skeleton has a unique point with value 1. Otherwise such skeleton can be considered as a metric graph of genus 1. Each skeleton associated to a semistable reduction of X is tropically equivalent to the graph G_n, in particular it can be considered as a metric graph. From the results in [16, Section 4, especially Corollaries 4.26 and 4.28] it follows that there exist skeletons $\tilde{\Gamma}$ (resp. Γ) of X (resp. E) such that π induces a finite harmonic morphism $\tilde{\Gamma} \to \Gamma$ of degree 2 (Section 4 of [16] uses no assumption on the characteristic of k). Since $\tilde{\Gamma}$ is a metric graph (augmentation map identically zero) this is also the case for Γ hence Γ is a metric graph. So in the case G_n is liftable to smooth curve X satisfying $\dim(W_4^1(X)) = 1$ then there exist a tropical modification $\tilde{\Gamma}$ of G_n and a metric graph Γ of genus 1 such that there exists a finite harmonic morphism $\tilde{\pi} : \tilde{\Gamma} \to \Gamma$ of degree 2. We are going to prove that such finite harmonic morphism does not exist. In the proof the following lemma will be useful.

Lemma 4.1. *Let $\phi : (\Gamma_1, V_1) \to (\Gamma_2, V_2)$ be a finite harmonic morphism between metric graphs with vertex sets. Let $(T', V') \subset (\Gamma_1, V_1)$ be a subgraph such that T' is a tree, $\overline{\Gamma_1 \setminus T'} \subset \Gamma_1$ is connected and $\overline{\Gamma_1 \setminus T'} \cap T'$ consists of a unique point t (in particular $t \in V_1$). There is no subtree (T, V) of (T', V') different from a point such that $\phi(T)$ is contained in a loop $\Gamma \subset \Gamma_2$.*

Proof. Assume T is a subtree of T' not being one point and assume $\phi(T)$ is contained in a loop Γ of Γ_2. Let $l(T)$ (resp. $l(\Gamma)$) be the sum of the lengths of all the edges of T (resp. Γ). By definition one has $l(T) \leq \deg(\phi) l(\Gamma)$, in particular $l(T)$ is finite. We are going to prove that we have to be able to enlarge T such that $l(T)$ grows with a fixed lower bound. Repeating this a few times gives a contradiction to the upper bound $\deg(\phi) l(\Gamma)$.

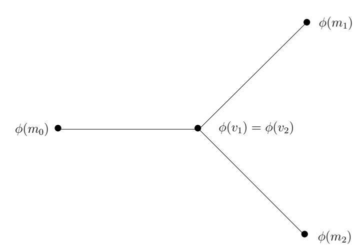
Let $q \in V$ be a point of valence 1 on T such that $q \neq t$ and let f be the edge of T having q as a vertex point. This edge f defines $v \in T_q(\Gamma_1)$, let $w = d_\phi(q)(v)$, hence $\phi(q) \in \Gamma$ and $w \in T_{\phi(q)}(\Gamma)$. Since Γ is a loop there is a unique $w' \in T_{\phi(q)}(\Gamma)$ with $w' \neq w$ and since ϕ is harmonic there exists $v' \in T_q(\Gamma_1)$ with $d_\phi(q)(v') = w'$. Let f' be the edge in Γ_1 having q as a vertex point and defining v'. From $d_\phi(q)(v') \in T_{\phi(q)}(\Gamma)$ it follows $\phi(f') \subset \Gamma$, hence $l(f')$ is finite. Since $f' \neq f$ and $q \neq t$ one has f' is an edge of $\overline{T' \setminus T}$. Since T' is a tree, also $T \cup f'$ is a tree and one has $\phi(T \cup f') \subset \Gamma$. Moreover $l(T \cup f') = l(T) + l(f')$ and $l(f')$ has as a fixed lower bound the minimal length of an edge contained in Γ_1. ☐

Theorem 4.2. *There does not exist a tropical modification $\tilde{\Gamma}$ of G_n such that there exists a graph Γ with $g(\Gamma) = 1$ and a finite harmonic morphism $\phi : \tilde{\Gamma} \to \Gamma$ of degree 2.*

Proof. Assume $\tilde{\Gamma}$ is a tropical modification of G_n and $\phi : \tilde{\Gamma} \to \Gamma$ is a finite harmonic morphism of degree 2 of metric graphs with $g(\Gamma) = 1$. Since $g(\Gamma) = 1$ we know $g(\phi(G_0)) \leq 1$. In case $g(\phi(G_0)) = 0$ then in step 1 we are going to prove that the restriction of ϕ to G_0 has a very particular description and next in step 2 we are going to prove that this does not occur.

Step 1: Assume $g(\phi(G_0)) = 0$. Then $\phi(G_0)$ looks as in Figure 3 with $\phi\big|_{e_i} : e_i \to [\phi(v_1), \phi(m_i)]$ having degree 2, $\sharp((\phi\big|_{e_i})^{-1}(q)) = 2$ for all $q \in [\phi(v_1), \phi(m_i)[$ and $\left(\phi|_{e_i}\right)^{-1}(\phi(m_i)) = \{m_i\}$ for $0 \leq i \leq 2$.

Fig. 3. In the case $g(\phi(G_0) = 0$



Assume $g(\phi(G_0)) = 0$. Consider the loop $c_1 = e_1 \cup e_0$ (remember Figure 1). Since $\phi(G_0)$ has genus 0 it follows $\phi(c_1)$ is a subtree T_1 of $\phi(G_0)$. Since T_1 is the image of a loop and $\deg(\phi) = 2$, it follows that $\sharp((\phi\big|_{c_1})^{-1}(q')) = 1$ for some $q' \in \phi(c_1)$ if and only if q' is a point of valence 1 of $\phi(c_1)$. Since $\deg(\phi) = 2$ it follows $d_q(\phi) = 2$ for $q \in c_1$ such that $\phi(q)$ has valence 1 on $\phi(c_1)$ and $d_q(\phi) = 1$ for $q \in c_1$ if $\phi(q)$ does not have valence 1 on $\phi(c_1)$. In the case $\phi(c_1)$ would have a point q' of valence 3 then there exist at least 3 different points q on e_1 with $\phi(q) = q'$, contradicting $\deg(\phi) = 2$. Hence $\phi(c_1)$ can be considered as a finite edge with two vertices. Also for each $q \in c_1$ and $v \in T_q(c_1)$ one has $d_v(\phi) = 1$.

Assume $\phi(v_1) \neq \phi(v_2)$. Then $\phi(e_2)$ is a path from $\phi(v_1)$ to $\phi(v_2)$ outside of $\phi(c_1)$. This would imply $g(\phi(G_0)) \geq 1$, contradicting $g(\phi(G_0)) = 0$, hence $\phi(v_1) = \phi(v_2)$. This also implies that $\phi(m_0)$ and $\phi(m_1)$ are the two points of valence 1 on $\phi(c_1)$. Repeating the previous arguments for the loop $c_2 = e_2 \cup e_0$ one obtains the given description for $\phi\big|G_0 : G_0 \to \phi(G_0)$.

Step 2: $g(\phi(G_0)) = 1$.

In case $g(\phi(G_0)) = 0$ then we have the description for $\phi\big|_{G_0} : G_0 \to \phi(G_0)$ obtained in Step 1. We are going to prove that this description cannot hold. Since $g(\phi(G_0)) \leq 1$, this implies $g(\phi(G_0)) = 1$.

Consider $\phi(q_1) \in \phi(e_1)$ and $q_1' \neq q_1$ on e_1 with $\phi(q_1) = \phi(q_1')$ (see Figure 2). Assume $\phi(\gamma_1)$ is a tree. If $\phi(\gamma_1)$ would have valence 1 at $\phi(q_1)$ then from the arguments used in Step 1 it follows $d_{q_1}(\phi) = 2$. But from Step 1 we know $d_{q_1}(\phi) = 1$, therefore $\phi(q_1)$ cannot be a point of valence 1 on $\phi(\gamma_1)$. Hence there exists $q_1'' \in \gamma_1 \setminus \{q_1\}$ with $\phi(q_1) = \phi(q_1'')$ and we obtain $\sharp(\phi^{-1}(\phi(q_1))) \geq 3$, contradicting $\deg(\phi) = 2$. It follows $g(\phi(\gamma_1)) = 1$, hence $\phi(\gamma_1)$ contains a loop e_1' in Γ. In case $\phi(q_1) \notin e_1'$ then again we obtain $\sharp(\phi^{-1}(\phi(q_1))) \geq 3$, contradicting $\deg(\phi) = 2$. Therefore $\phi(q_1) \in e_1'$ and $e_1' \cap \phi(G_0) = \{\phi(q_1)\}$.

Repeating the arguments using q_2 and γ_2 we obtain a loop e_2' in Γ such that $e_2' \cap \phi(G_0) = \{\phi(q_2)\}$, hence $e_1' \cap e_2' = \emptyset$. Since $g(\Gamma) = 1$ this is impossible. As a conclusion we obtain $g(\phi(G_0)) = 1$ finishing the proof of step 2.

In the case $\phi(c_1)$ would have genus 0 (c_1 as in the proof of Step 1), then from the arguments used in Step 1 it follows that for $q \in e_2 \setminus \{v_1, v_2\}$ one has $\phi(q) \notin \phi(c_1)$. In the case $\phi(v_1) \neq \phi(v_2)$ it implies $\phi(c_2)$ has genus 1 (again c_2 as in the proof of Step 1). In the case $\phi(v_1) = \phi(v_2)$ and $g(\phi(c_2)) = 0$ too, it would imply $g(\phi(G_0)) = 0$

so this cannot occur. Therefore without loss of generality, we can assume $\phi(c_1)$ has genus 1 (but then $\phi(c_2)$ could have genus 0).

Step 3: $\phi\big|_{c_1} : c_1 \to \phi(c_1)$ is an isomorphism (meaning it is finite harmonic of degree 1)

Since $g(\phi(c_1)) = 1$ it follows there is a loop e in $\phi(c_1)$, finitely many points $r_1, \cdots, r_t$ on e and finitely many trees T_i inside $\phi(c_1)$ such that

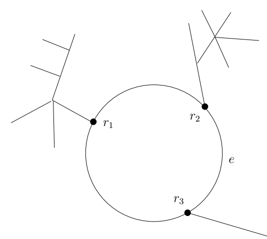
$T_i \cap e = \{r_i\}$

$T_i \cap T_j = \emptyset$ in the case $i \neq j$

$\phi(c_1) = e \cup T_1 \cup \cdots \cup T_t$

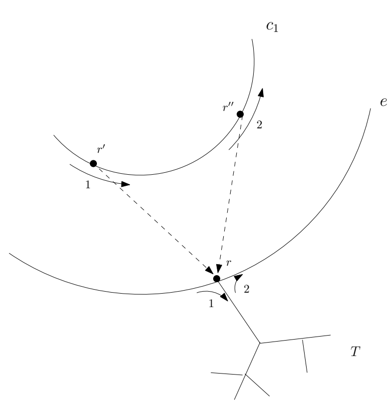
(of course $t = 0$, hence $\phi(c_1)$ is a loop, is also possible; we are going to prove that $t = 0$).

Fig. 4. $g(\phi(c_1)) = 1$



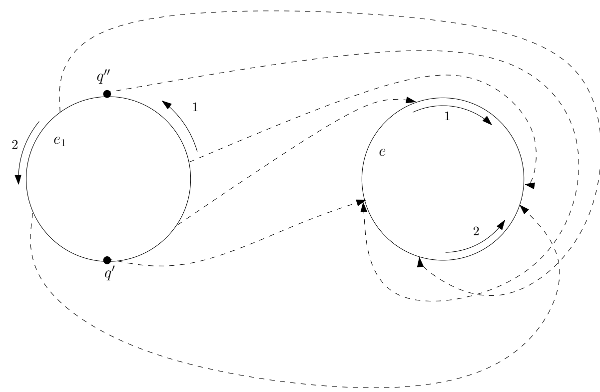
In the case $\mathrm{val}_{\phi(c_1)}(r_i) > 3$ for some $1 \leq i \leq t$ then $\phi^{-1}(r_i)$ contain at least 3 different points on c_1, contradicting $\deg(\phi) = 2$. So we obtain a situation like in Figure 4. Let $r \in \{r_1, \cdots, r_t\}$ and let T be the associated subtree of $\phi(c_1)$. Then $\phi^{-1}(r) = \{r', r''\} \subset c_1$ with $r' \neq r''$. The tangent space $T_r(\phi(c_1))$ consists of 3 elements (see Figure 5). Hence there exists $w \in T_{r'}(G_0) \setminus T_{r'}(c_1)$ such that $d_\phi(r')(w) \in T_r(e)$. (In Figure 5 it is the tangent vector corresponding to the direction on e indicated by the number 2.) Let f be the edge of G'_n defining w hence $\phi(f) \subset e$. Because of Lemma 4.1 this implies r' is one of the points q_i on G_0 and $f \subset \gamma_i$. Since $\deg(\phi) = 2$ and $g(\Gamma) = 1$ it follows $e \subset \phi(\gamma_i)$. Repeating the same argument using r'' instead of r' one obtains a contradiction to $\deg(\phi) = 2$. This proves $t = 0$, hence $\phi(c_1) = e$ is a loop.

Fig. 5. In the case $t \neq 0$



Because $\deg(\phi) = 2$ we cannot go back and forth on e moving along c_1 and taking the image under ϕ. In principle it could be the case that there exist different points q', q'' on c_1 such that the image of the closure of both components of $c_1 \setminus \{q', q''\}$ is equal to e with $\phi(q') = \phi(q'')$ and $d_{q'}(\phi) = d_{q''}(\phi) = 2$. This would correspond to something like shown in Figure 6. This figure has to be understood as follows. Moving along c_1 from q' to q'' in the direction indicated by 1 (left hand side of the figure) the image under ϕ is equal to e while moving in the direction indicated by 1 (right hand side of the figure). Moving on c_1 from q'' to q' in the direction indicated by 2 (left hand side of the figure) the image under ϕ is equal to e while moving in the direction indicated by 2 (right hand side of the figure).

Fig. 6. A case that cannot occur



In that case there should exist $v \in T_{q'}(G'_n)$ with $d_\phi(q')(v) \in T_{\phi(q')}(e) \setminus d_\phi(q')(T_{q'}(c_1))$. Hence the edge f of G'_n defining v satisfies $\phi(f) \subset e$, contradicting $\deg(\phi) = 2$. Hence the situation from Figure 6 cannot occur.

It follows that in the case there exist $q' \neq q''$ on c_1 such that $\phi(q') = \phi(q'')$ then $\phi|_{c_1} : c_1 \to e$ is harmonic of degree 2 and $d_q(\phi|_{c_1}) = 1$ for all $q \in c_1$. In this case $\phi(e_2) \cap e = \{\phi(v_1), \phi(v_2)\}$ since $\deg(\phi) = 2$. In the case $\phi(v_1) \neq \phi(v_2)$ then this contradicts $g(\Gamma) = 1$. In the case $\phi(v_1) = \phi(v_2)$ then because of the description of $\phi|_{c_1}$ one has $l(e_1) = l(e_0)$. We assume this is not the case, so we can assume $\phi|_{c_1} : c_1 \to e$ is bijective.

In the case for each edge f on $\tilde{\Gamma}$ with $f \subset c_1$ one has $d_f(\phi) = 2$ then again, since $\phi(v_1) \neq \phi(v_2)$ we have $\phi(e_2) \cap e = \{\phi(v_1), \phi(v_2)\}$, contradicting $g(\Gamma) = 1$. Assume there exists $q \in c_1$ being a vertex of $\tilde{\Gamma}$ and two edges e', e'' of $\tilde{\Gamma}$ contained in c_1 with vertex end point q such that $d_{e'}(\phi) = 1$ and $d_{e''}(\phi) = 2$. In particular it follows that $d_q(\phi) = 2$. Let $v' \in T_q(\tilde{\Gamma})$ correspond to e' then there exists $v \in T_q(\tilde{\Gamma})$ with $v \notin T_q(c_1)$ such that $d_\phi(q)(v) = d_\phi(q)(v')$. Let f be the edge of $\tilde{\Gamma}$ defining v, then $\phi(f) \subset e$. From Lemma 4.1 it follows that q is one of the points q_i and $f \subset \gamma_i$. Since $g(\Gamma) = 1$ and $\deg(\phi) = 2$ it follows that $e \subset \phi(\gamma_i)$, but this is impossible because $d_{e''}(\phi) = 2$ and $\deg(\phi) = 2$. This proves $\phi|_{c_1} : c_1 \to e$ is an isomorphism of metric graphs.

Step 4: Finishing the proof of the theorem.
It follows that $\phi(v_1)$ and $\phi(v_2)$ do split e into two parts e' and e'' of lengths $l(e_1)$ and $l(e_0)$. Since $g(\Gamma) = 1$ it follows $\phi(e_2)$ contains e' or e'', we assume it contains e'. In the case $\phi(e_2)$ would contain $\tilde{q} \in e'' \setminus \{\phi(v_1), \phi(v_2)\}$ then because of $g(\Gamma) = 1$ it follows $\phi(e_2)$ contains one of the connected components of $e'' \setminus \{\tilde{q}\}$. On that connected component we get a contradiction to $\deg(\phi) = 2$. Indeed, for a point on that connected component the inverse image under ϕ contains one point of c_1 and two different points of e_2.

So we obtain different points $r_1, \cdots, r_t$ on e' and trees $T_1, \cdots, T_t$ with $T_i \cap e' = r_i$ for $1 \leq i \leq t$ and $T_i \cap T_j = \emptyset$ for $i \neq j$ such that $\phi(e_2) = e' \cup T_1 \cup \cdots \cup T_t$. It is possible (and we are going to prove) that $t = 0$, hence $e' = \phi(e_2)$. In the case $r_i \notin \{\phi(v_1), \phi(v_2)\}$ then there exist two different points r', r'' on e_2 such that $\phi(r') = \phi(r'') = r_i$. Since r_i is also the image of a point on c_1 we get a contradiction to $\deg(\phi) = 2$. Hence $t \leq 2$ and $r_i \in \{\phi(v_1), \phi(v_2)\}$.

Assume $r_i = \phi(v_1)$. We obtain $q \in e_2$ with $q \notin \{v_1, v_2\}$ and $\phi(q) = \phi(v_1)$. There exists $v \in T_q(\tilde{\Gamma})$ such that $d_\phi(q)(v)$ is the element of $T_{\phi(q)}(\Gamma)$ defined by e''. Let f be the edge of $\tilde{\Gamma}$ defining v. From Lemma 4.1 it follows q is one of the points q_i and $f \subset \gamma_i$. Since $g(\Gamma) = 1$ and $\deg(\phi) = 2$ we obtain $\phi(\gamma_i)$ contains e. This implies that for $P \in e'$ there are at least 3 points contained in $\phi^{-1}(P)$, a contradiction. This proves $t = 0$, hence $\phi(e_2) = e'$. Since $\deg(\phi) = 2$ it also implies $d_f(\phi) = 1$ for each edge f of $\tilde{\Gamma}$ contained in e_2, hence $l(e') = l(e_2)$. Since $l(e_2) \notin \{l(e_0), l(e_1)\}$ we obtain a contradiction, finishing the proof of the theorem. □

As a corollary of the theorem we obtain the goal of this paper.

Corollary 4.3. *For each genus $g \geq 5$ there is metric graph G of genus g satisfying $w_4^1 \geq 1$ that has no divisor of Clifford index at most 1 and is not tropically equivalent to a metric graph $\tilde{\Gamma}$ such that there exists a finite harmonic morphism $\pi : \tilde{\Gamma} \to \Gamma$ of degree 2 with $g(\Gamma) = 1$. In particular in the case $g \geq 10$ the graph G cannot be lifted to a curve X of genus g satisfying* $\dim(W_4^1) = 1$.

Appendix

We give a very easy proof of a statement implying Mumford's Theorem in [11, Appendix] in the case of $\dim(W_4^1) = 1$ and $g \geq 10$ not using any assumption on the characteristic. This case corresponds to the situation considered in the paper.

Proposition 4.4. *Assume C is a smooth non-hyperelliptic, non-trigonal irreducible complete curve defined over an algebraically closed field k of any characteristic. In the case $g(C) \geq 10$ and C has at least two different linear systems g_4^1 then C is bi-elliptic.*

Proof. Let g_1 and g_2 be two linear systems g_4^1 on C. Since C has no g_3^1 both linear systems are base point free and g_i defines a morphism $\phi_i : C \to \mathbb{P}^1$. Those morphisms give rise to a morphism $\phi = (\phi_1, \phi_2) : C \to \mathbb{P}^1 \times \mathbb{P}^1$. The projections to the factors induce ϕ_1 and ϕ_2.

The Picard group of $\mathbb{P}^1 \times \mathbb{P}^1$ is equal to $\mathbb{Z} \times \mathbb{Z}$ with (a, b) being represented by the divisor $a(P \times \mathbb{P}^1) + b(\mathbb{P}^1 \times P)$ (with $P \in \mathbb{P}^1$). Using $(1, 1)$ one gets an embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric Q in $\mathbb{P}^3$. By composition we have a morphism $\phi : C \to Q \subset \mathbb{P}^3$ defined by a linear subsystem of degree 8 of $|g_1 + g_2|$.

In the case the linear system does not have dimension 3 then $\phi(C)$ is contained in a hyperplane section of Q. In the case this hyperplane section is a union of two lines on Q then $\phi(C)$ is one of those lines implying some ϕ_i is constant, a contradiction. Otherwise this hyperplane section is a smooth conic γ on Q and $\phi : C \to \gamma \cong \mathbb{P}^1$ has degree 4. This case implies both ϕ_1 and ϕ_2 are projectively equivalent to ϕ, therefore $g_1 = g_2$ and again we obtain a contradiction.

It follows that $\phi : C \to Q \subset \mathbb{P}^3$ is non-degenerated (defined by some g_8^3). In particular $\phi(C)$ is not contained in a plane and therefore $\deg(\phi(C)) \geq 3$. Also $\deg(\phi(C))$ divides 8, therefore $\deg(\phi(C)) = 4$ or $\deg(\phi(C)) = 8$. Then either $\phi : C \to \phi(C)$ has degree 2 or degree 1. In the case the degree is 1 then C is birationally equivalent to a curve on Q belonging to the linear system $(4, 4)$. Because of the adjunction formula this implies $g(C) \leq 9$, a contradiction to our assumptions. Therefore $\phi : C \to \phi(C)$ has degree 2 and $\phi(C)$ is an irreducible curve on Q belonging to the linear system $(2, 2)$. The adjuction formula implies $g(\phi(C)) \leq 1$. Since C is not hyperelliptic we obtain $\phi(C)$ is a smooth elliptic curve E on Q and we obtain a double covering $\phi : C \to E$. □

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