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Complete and sufficient statistics and perfect families in orthogonal and error orthogonal normal models

Abstract: We will discuss orthogonal models and error orthogonal models and their algebraic structure, using as background, commutative Jordan algebras. The role of perfect families of symmetric matrices will be emphasized, since they will play an important part in the construction of the estimators for the relevant parameters.

Perfect families of symmetric matrices form a basis for the commutative Jordan algebra they generate. When normality is assumed, these perfect families of symmetric matrices will ensure that the models have complete and sufficient statistics. This will lead to uniformly minimum variance unbiased estimators for the relevant parameters.

Keywords: Orthogonal models, Perfect families, Commutative Jordan algebras

MSC: 62H05, 62H12

DOI 10.1515/math-2015-0009

Received December 9, 2013; accepted April 3, 2014.

1 Introduction

Orthogonal (ORT) and Error-Orthogonal (EO) models, are two important classes of mixed models, used in various fields of science. See [13] and [14] for further reading.

In this work, we will discuss the algebraic structure of these models in the framework of commutative Jordan Algebras (CJA). The methods herein will be applied to least squares estimators (LSE) that, for both classes of models, are uniformly the best linear unbiased estimator (UBLUE), *i.e.*, they are best linear unbiased estimators (BLUE) regardless of the variance components, see [13].

Moreover, we will show the relevant factor part played by perfect families (PF) of symmetrical matrices ([3]), in the algebraic structure of both classes of the models that we consider: ORT and EO. These perfect families are constituted by symmetric matrices that commute and are a basis for the CJA they generate. We will use these families to show that, when normality is assumed, we obtain uniformly minimum variance unbiased estimators (UMVUE).

In the next section we will focus on CJA, the foundations of the algebraic characterization for the models that we will consider and lay down the necessary results for the proposed algebraic characterization, with special emphasis on the existence of a principal basis.

In Section 3 we will consider UBLUE estimators in the context of mixed models where we will present the conditions for the existence of UBLUE in mixed models with LSE, followed by a section in which we will address

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the issue of variance-covariance matrices and their relation with orthogonal models, error orthogonal models and the role played by PF.

2 Commutative Jordan algebras

Commutative Jordan Algebras were introduced by Jordan in 1934 ([5]) in a reformulation of Quantum Mechanics. They were rediscovered by Seely in 1970, who used them in linear statistical inference, see [8–11] and [12]. The CJA of symmetric matrices are linear spaces defined by such matrices that commute and contain the squares of their matrices. Seely called them quadratic spaces, [8], but for simplicity's sake we will retain the previous designation. This first use of quadratic subspaces in linear statistical inference led to the dissemination of several works where its use plays a central part, see *e.g.*, [2, 4, 15] and [1].

The matrices [7], of a family

$$\mathcal{U} = \{\mathbf{U}_1, \dots, \mathbf{U}_h\}$$

of symmetric matrices commute if and only if they are diagonalizable by the same orthogonal matrix $\mathbf{P}$. Then $\mathcal{U}$ is contained in the family $\mathcal{V}(\mathbf{P})$ of symmetric matrices diagonalized by $\mathbf{P}$. It is easy to see that $\mathcal{V}(\mathbf{P})$ is a CJA, thus the matrices of a family of symmetric matrices belong to a CJA, if and only if they commute.

Since the intersection of a CJA is also a CJA given a family $\mathcal{U}$ of symmetric matrices that commute, the intersection $\mathcal{A}(\mathcal{U})$ of all CJA that contains $\mathcal{U}$ will be the minimum CJA that contains $\mathcal{U}$. We call it the CJA generated by $\mathcal{U}$. Any CJA has a unique basis [10], the principal basis of $\mathcal{A}$, $pb(\mathcal{A})$, constituted by pairwise orthogonal orthogonal projection matrices. If the set $\mathcal{Q} = \{\mathbf{Q}_1, \dots, \mathbf{Q}_m\} = pb(\mathcal{A}(\mathcal{U}))$, we have that:

$$\mathbf{U}_i = \sum_{j=1}^h b_{i,j} \mathbf{Q}_j, \quad i = 1, \dots, h$$

and $\mathbf{B} = [b_{i,j}]$ will be the transition matrix $\mathcal{U}|\mathcal{Q}$.

A family $\mathcal{U}$ of symmetric matrices that commute is perfect [3], if it is a basis of $\mathcal{A}(\mathcal{U})$ and the transition matrix $\mathcal{U}|\mathcal{Q}$, with $\mathcal{Q} = pb(\mathcal{A}(\mathcal{U}))$ is invertible.

As mentioned above, if $\mathbf{U}_1, \dots, \mathbf{U}_h$ commute, they are diagonalizable by the same orthogonal matrix $\mathbf{P}$. The row vectors $\alpha_1, \dots, \alpha_n$ of $\mathbf{P}$ are eigenvectors of the matrices $\mathbf{U}_1, \dots, \mathbf{U}_h$. Writing

$$\alpha_i \tau \alpha_\ell$$

when α_i and α_ℓ are associated to identical eigenvalues for all matrices $\mathbf{U}_1, \dots, \mathbf{U}_h$, we establish an equivalence relation between the row vectors of $\mathbf{P}$.

A τ equivalence class will be of first type if its vectors are associated to non null eigenvalues for at least one of the matrices of $\mathcal{U} = \{\mathbf{U}_1, \dots, \mathbf{U}_h\}$.

Besides type τ equivalence classes, there may be a second type of equivalence class whose vectors are associated to null eigenvalues for all matrices of $\mathcal{U}$. The eigenindex of $\mathcal{U}$ will be the number type τ equivalence classes.

Let $\xi_1, \dots, \xi_d$ be the sets of indices of the $\alpha_1, \dots, \alpha_n$ belonging to these classes and the matrices

$$\mathbf{K}_\ell = \sum_{j \in \xi_\ell} \alpha_j \alpha_j^\top, \quad \ell = 1, \dots, d$$

will be pairwise orthogonal orthogonal projection matrices, and

$$\mathbf{U}_i = \sum_{\ell=1}^d v_{i,\ell} \mathbf{K}_\ell, \quad i = 1, \dots, h,$$

where $v_{i,\ell}$ are the eigenvalues of α_j , $j \in \xi_\ell$, $\ell = 1, \dots, d$, for $\mathbf{U}_i$, $i = 1, \dots, h$.

Now, see [3], the $\mathbf{K}_1, \dots, \mathbf{K}_d$ constitute the $pb(\mathcal{A})$ with $\mathcal{A} = \mathcal{A}(\mathcal{U})$. Then $\mathcal{U}$ is a PF if and only if $h = d$, *i.e.*, if the cardinal of $\mathcal{U}$ is equal to its eigenindex.

We now consider the following proposition:

Proposition 2.1. *If the symmetric matrix $\mathbf{U}_0$ commutes with the commuting symmetric matrices $\mathbf{U}_1, \dots, \mathbf{U}_h$, it will commute with all matrices of $\mathcal{A} = \mathcal{A}(\mathcal{U})$, with $\mathcal{U} = \{\mathbf{U}_1, \dots, \mathbf{U}_h\}$.*

Proof. The matrices of $\mathcal{U}^0 = \{\mathbf{U}_0, \mathbf{U}_1, \dots, \mathbf{U}_m\}$ commute if and only there is a CJA, $\mathcal{U}^0$, that contains both $\mathbf{U}_0$ and the matrices of $\mathcal{A}$ and this completes the proof. $\square$

3 Uniformly best linear unbiased estimators

The mixed model

$$\mathbf{y} = \sum_{i=0}^w \mathbf{X}_i \boldsymbol{\beta}_i, \quad (1)$$

where $\boldsymbol{\beta}_0$ is fixed and the $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \dots, \boldsymbol{\beta}_w$ are independent with null mean vectors and variance-covariance matrices $\theta_1 \mathbf{I}_{c_1}, \dots, \theta_w \mathbf{I}_{c_w}$ will have mean vector

$$\boldsymbol{\mu} = \mathbf{X}_0 \boldsymbol{\beta}_0$$

and the family of its variance-covariance matrices will be

$$\Psi = \left\{ \mathbf{V}(\boldsymbol{\theta}) = \sum_{i=1}^w \theta_i \mathbf{M}_i : \boldsymbol{\theta} > \mathbf{0} \right\},$$

where $\mathbf{M}_i = \mathbf{X}_i \mathbf{X}_i^\top$, $i = 1, \dots, w$.

Now a necessary and sufficient condition [13] for the LSE of the model in (1) to be UBLUE, is that

$$\forall \boldsymbol{\theta} > \mathbf{0} : \left(\mathbf{X}_0^\top \mathbf{X}_0 \right)^- \mathbf{X}_0^\top \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}^c = \mathbf{0}_{k \times n} \quad (2)$$

where $(-)$ indicates a generalized inverse and $\mathbf{T}^c = \mathbf{I}_n - \mathbf{T}$ with

$$\mathbf{T} = \mathbf{X}_0 \left(\mathbf{X}_0^\top \mathbf{X}_0 \right)^- \mathbf{X}_0^\top$$

is the orthogonal projection matrix on the space spanned by the mean vector, $\boldsymbol{\mu}$.

We now consider another proposition:

Proposition 3.1. *The mixed model has LSE that are UBLUE if any if the following equivalent conditions holds:*

- (a) $\forall \boldsymbol{\theta} > \mathbf{0} : \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}^c = \mathbf{0}_{n \times n}$;
- (b) $\forall \boldsymbol{\theta} > \mathbf{0} : \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) = \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}$;
- (c) $\mathbf{T} \mathbf{M}_i = \mathbf{M}_i \mathbf{T}$, $i = 1, \dots, m$;
- (d) $\mathbf{T} \mathbf{Q}_j = \mathbf{Q}_j \mathbf{T}$, $j = 1, \dots, m$, with $\mathcal{Q} = pb(\mathcal{A}(\mathbf{M}))$.

Proof. Since $\mathbf{T} \mathbf{X}_0 = \mathbf{X}_0$, the expression (2) may be rewritten as

$$\forall \boldsymbol{\theta} > \mathbf{0} : \left(\mathbf{X}_0^\top \mathbf{X}_0 \right)^- \mathbf{X}_0^\top \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}^c = \mathbf{0}_{k \times n}.$$

When this conditions holds, we will have

$$\forall \boldsymbol{\theta} > \mathbf{0} : \mathbf{X}_0 \left(\mathbf{X}_0^\top \mathbf{X}_0 \right)^- \mathbf{X}_0^\top \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}^c = \mathbf{T} \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}^c = \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}^c = \mathbf{0}_{n \times n},$$

which implies expression (2). Thus conditions (2) and (a) are equivalent.

Now, $\mathbf{T}^c = \mathbf{I}_n - \mathbf{T}$ so we may rewrite (2) as

$$\forall \boldsymbol{\theta} > \mathbf{0} : \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) - \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) \mathbf{T} = \mathbf{0}_{n \times n}$$

and also

$$\forall \boldsymbol{\theta} > \mathbf{0} : \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) = \mathbf{T} \mathbf{V}(\boldsymbol{\theta}) \mathbf{T}$$

and $\mathbf{TV}(\boldsymbol{\theta})\mathbf{T}$ is symmetric, we will have

$$\forall \boldsymbol{\theta} > \mathbf{0} : \mathbf{TV}(\boldsymbol{\theta}) = \mathbf{V}(\boldsymbol{\theta})\mathbf{T},$$

so (a) implies (b). Now if (b) holds, since $\mathbf{T}$ is idempotent, we will have

$$\forall \boldsymbol{\theta} > \mathbf{0} : \mathbf{TV}(\boldsymbol{\theta}) = \mathbf{TV}(\boldsymbol{\theta})\mathbf{T},$$

thus (b) implies (a) and so according to (2), (a) and (b) are equivalent.

Moreover, $\boldsymbol{\delta}_i$ is a vector with all components null except the i -th which is 1, so we have

$$\mathbf{M}_i = \mathbf{V}(\boldsymbol{\theta} + \boldsymbol{\delta}_i) - \mathbf{V}(\boldsymbol{\theta}), \quad i = 1, \dots, m,$$

thus $\mathbf{T}$, since it commutes with $\mathbf{V}(\boldsymbol{\theta} + \boldsymbol{\delta}_i)$ and $\mathbf{V}(\boldsymbol{\theta})$ also commutes with $\mathbf{M}_i$, $i = 1, \dots, w$. Conversely, if $\mathbf{T}$ commutes with $\mathbf{M}_i$, $i = 1, \dots, w$, it commutes with $\mathbf{V}(\boldsymbol{\theta})$, whatever $\boldsymbol{\theta} > \mathbf{0}$, thus (c) is equivalent to (b) and to (2) and (a).

Lastly if (c) holds, $\mathbf{T}$ commutes, according to Proposition 1, with all matrices in $\mathcal{A}(\mathcal{M})$, and therefore also $\mathbf{Q}_1, \dots, \mathbf{Q}_m$. Conversely if $\mathbf{T}$ commutes with the $\mathbf{Q}_1, \dots, \mathbf{Q}_m$, it commutes with the $\mathbf{M}_\ell = \sum_{j=1}^m b_{\ell,j} \mathbf{Q}_j$, $\ell = 1, \dots, m$, so (c) and (d) are equivalent and the proof is complete. $\square$

Let us put

$$\begin{cases} \mathbf{M}_i^\circ = \mathbf{T}\mathbf{M}_i\mathbf{T}, & i = 1, \dots, w \\ \mathbf{M}_i^c = \mathbf{T}^c\mathbf{M}_i\mathbf{T}^c, & i = 1, \dots, w, \end{cases}$$

when the model is UBLUE we have $\mathbf{T}\mathbf{M}_i = \mathbf{M}_i\mathbf{T}$ and also $\mathbf{T}^c\mathbf{M}_i = \mathbf{M}_i\mathbf{T}^c$ so, it follows that

$$\mathbf{M}_i = \mathbf{M}_i^\circ + \mathbf{M}_i^c, \quad i = 1, \dots, w,$$

since

$$\mathbf{M}_i^\circ\mathbf{M}_\ell^c = \mathbf{M}_\ell^c\mathbf{M}_i^\circ = \mathbf{0}_{n \times n}, \quad i \neq \ell.$$

We may now establish the following proposition,

Proposition 3.2. *If the model has LSE that are UBLUE, we have $\mathbf{M}_i\mathbf{M}_\ell = \mathbf{M}_\ell\mathbf{M}_i$, if and only if $\mathbf{M}_i^\circ\mathbf{M}_\ell^\circ = \mathbf{M}_\ell^\circ\mathbf{M}_i^\circ$ and $\mathbf{M}_i^c\mathbf{M}_\ell^c = \mathbf{M}_\ell^c\mathbf{M}_i^c$. Thus the matrices in $\mathcal{M} = \{\mathbf{M}_1, \dots, \mathbf{M}_w\}$ commute if and only if the matrices in $\mathcal{M}^\circ = \{\mathbf{M}_1^\circ, \dots, \mathbf{M}_w^\circ\}$ and $\mathcal{M}^c = \{\mathbf{M}_1^c, \dots, \mathbf{M}_w^c\}$ commute.*

Proof. Since

$$\mathbf{M}_i\mathbf{M}_\ell = (\mathbf{M}_i^\circ + \mathbf{M}_i^c)(\mathbf{M}_\ell^\circ + \mathbf{M}_\ell^c) = (\mathbf{M}_i^\circ\mathbf{M}_\ell^\circ) + (\mathbf{M}_i^c\mathbf{M}_\ell^c)$$

so if $\mathbf{M}_i^\circ\mathbf{M}_\ell^\circ = \mathbf{M}_\ell^\circ\mathbf{M}_i^\circ$ and $\mathbf{M}_i^c\mathbf{M}_\ell^c = \mathbf{M}_\ell^c\mathbf{M}_i^c$ we have that $\mathbf{M}_i\mathbf{M}_\ell = \mathbf{M}_\ell\mathbf{M}_i$.

Now if $\mathbf{M}_i\mathbf{M}_\ell = \mathbf{M}_\ell\mathbf{M}_i$ we would have

$$\mathbf{M}_i^\circ\mathbf{M}_\ell^\circ + \mathbf{M}_i^c\mathbf{M}_\ell^c = \mathbf{M}_\ell^\circ\mathbf{M}_i^\circ + \mathbf{M}_\ell^c\mathbf{M}_i^c$$

so that

$$\mathbf{M}_i^\circ\mathbf{M}_\ell^\circ - \mathbf{M}_\ell^\circ\mathbf{M}_i^\circ = \mathbf{M}_\ell^c\mathbf{M}_i^c - \mathbf{M}_i^c\mathbf{M}_\ell^c$$

and we have only to point out that the range space of the left [right] side of this matrices equation lies in the range space of $\mathbf{T}[\mathbf{T}^c]$. Thus this equality holds only if

$$\mathbf{M}_i^\circ\mathbf{M}_\ell^\circ - \mathbf{M}_\ell^c\mathbf{M}_i^c = \mathbf{M}_\ell^\circ\mathbf{M}_i^\circ - \mathbf{M}_\ell^c\mathbf{M}_i^c = \mathbf{0}_{n \times n},$$

i.e., if $\mathbf{M}_i^\circ\mathbf{M}_\ell^\circ = \mathbf{M}_\ell^\circ\mathbf{M}_i^\circ$ and $\mathbf{M}_i^c\mathbf{M}_\ell^c = \mathbf{M}_\ell^c\mathbf{M}_i^c$. The rest of the proof is straightforward. $\square$

4 Variance-covariance matrices

We start by assuming that $\mathcal{M}$ is perfect. Then $w = m$ and

$$\mathbf{M}_i = \sum_{j=1}^m b_{i,j} \mathbf{Q}_j, \quad i = 1, \dots, w,$$

thus

$$\mathbf{V}(\boldsymbol{\theta}) = \sum_{i=1}^m \theta_i \mathbf{M}_i = \sum_{j=1}^m \gamma_j \mathbf{Q}_j = \mathbf{V}(\boldsymbol{\gamma}),$$

with

$$\boldsymbol{\gamma} = \mathbf{B}^\top \boldsymbol{\theta} \in \Gamma$$

where

$$\Gamma = \mathbf{B}^\top \Theta$$

contains non-empty open sets since $\mathbf{B}^\top$ is invertible. Thus ([13]) the model is ORT whenever its LSE are UBLUE.

In addition, the matrices in $\mathcal{M}$ must be linearly independent so that $\mathbf{V}(\boldsymbol{\theta}_1) = \mathbf{V}(\boldsymbol{\theta}_2)$ implies $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ and $\boldsymbol{\theta}$ identifies $\mathbf{V}(\boldsymbol{\theta})$, then $w \leq m$, since $\mathbf{M}_1, \dots, \mathbf{M}_w$ belong to $\mathcal{A}(\mathcal{M})$ which is a linear space with dimension m . Moreover if $w < m$, Γ will be contained in the range space of $\mathbf{B}^\top$, $R(\mathbf{B}^\top)$, which will have dimension $w < n$ and so Γ cannot contain non-empty open sets.

Since $\mathcal{M}$ is PF if and only if $w = m$, we turn to the following proposition

Proposition 4.1. *The mixed model in (1) is ORT if and only if $\mathbf{T}$ commutes with $\mathcal{M}$, that is a PF.*

Moreover, if normality is assumed, since Γ contains open sets, the ORT models will have completely sufficient statistics, see [6]. Thus the estimators we obtain for estimable vectors and variance-covariance components will be UMVUE.

In the case of EO models, the requirements for these models are ([13]):

- their LSE are UBLUE
- the variance-covariance matrix of $\mathbf{y}^c = \mathbf{T}^c \mathbf{y}$ can be written as

$$\mathbf{V}^c(\boldsymbol{\gamma}) = \sum_{j=1}^{m^c} \gamma_j^c \mathbf{Q}_j^c$$

with $\mathbf{Q}_j^c = \mathbf{T}^c \mathbf{Q}_j \mathbf{T}^c$, $j = 1, \dots, m$, where the $\mathbf{Q}_1^c, \dots, \mathbf{Q}_{m^c}^c$ are known pairwise orthogonal orthogonal projection matrices and $\boldsymbol{\gamma}^c$ belongs to Γ^c that contains non void open sets.

Putting

$$\mathbf{M}_i^c = \mathbf{T}^c \mathbf{M}_i \mathbf{T}^c, \quad i = 1, \dots, w,$$

we have for $\mathbf{y}^c$, the variance-covariance matrices

$$\mathbf{V}^c(\boldsymbol{\theta}) = \sum_{\ell=1}^w \theta_\ell \mathbf{M}_\ell^c, \quad \boldsymbol{\theta} \in \Theta.$$

We can reason as above to establish the following proposition:

Proposition 4.2. *The model in (1) is EO if and only if $\mathbf{T}$ commutes with $\mathcal{M}$ and $\mathcal{M}^c$ is perfect.*

If one considers ORT models in the case of normality, the estimators for the variance components are UMVUE in the family of the estimators obtained from $\mathbf{y}^c$.

5 Final comments

In order not to have linear restrictions for $\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_m$, the matrix $\mathbf{B}$ must be invertible, and this is a condition which holds if and only if $\mathcal{M}$ is a perfect family.

Thus, $\mathcal{M}$ being a perfect family is a necessary and sufficient condition, once normality is assumed, for having complete and sufficient statistics.

Acknowledgement: This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project PEst-OE/MAT/UI0297/2014 (Centro de Matemática e Aplicações).

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