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Beyond content: exploring the neglected dimensions of mathematics literacy

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Abstract: Becoming mathematically literate means not only knowledge of the content of mathematics but also understanding of the nature of mathematics and the literate practices involved in the creation, communication, and consumption of its content. This case study examined one mathematician's view of the nature of mathematics and his literate practices. Data collected include semi-structured interviews with the mathematician, observations of his daily work routines, and his think-alouds during the reading of a disciplinary text. These data were analyzed qualitatively through an iterative process involving multiple readings and identification and refinement of codes. The analysis revealed that the mathematician (a) viewed mathematics as rigorous and demanding, both theoretical and practical, relatively stable but highly rewarding, interconnected with other disciplines, and involving discipline-legitimated discursive practices; (b) engaged in extensive reading/viewing and writing, valued learning from repeated trials and errors, and collaborated with an international network of peers in research; (c) used a range of reading strategies (e.g. close reading, summarizing, questioning, storying, evaluating, annotating) to help him make sense of the disciplinary text; and (d) marshaled both verbal and visual resources to create specialized knowledge, engage in rigorous reasoning, develop logical argument, and construct professional identity. These findings provide important insights that can help teachers design activities that are authentic to mathematics practices and effective for promoting mathematics literacy.

Keywords: disciplinary literacy; mathematics education; mathematics literacy; mathematics reading; mathematics register; nature of mathematics

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1 Introduction: conceptual and empirical background

Learning in any academic discipline, or developing disciplinary literacy, involves much more than learning its content, that is, its big ideas, core concepts, and key relationships; it also involves learning about how this content is created, communicated, critiqued, renovated, taught, and learned, as well as the nature of the discipline (Fang 2024). According to Moje (2008: 100), developing disciplinary literacy means "coming to understand the norms of practice for producing and communicating knowledge in the disciplines", "examining how disciplinary norms for practice are similar to or different from the everyday norms for practice", and "understanding deeply held assumptions or themes of the discipline". Johnson et al. (2011: 107) stated that to be literate in a discipline means "not just accumulating knowledge about the discipline, but understanding […] what questions are important to the discipline and how to seek answers to those questions [and] being able to read and write successfully within the discipline". Fang and Schleppegrell (2008) argued that disciplinary learning involves learning the lexicogrammatical patterns that experts use to construe and communicate knowledge and ideologies across different disciplines. Thus, as Christie and Maton (2011: 4) noted, induction into a disciplinary community involves learning "the organization of knowledge and of intellectual and educational practices for its creation, teaching and learning".

In the context of mathematics education, then, becoming mathematically literate is now understood to mean knowledge of not only the content of mathematics (i.e. mathematical concepts, processes, operations, and relationships) but also of the nature of mathematics (e.g. the essence and worth of mathematics) and the ways in which mathematics content is produced, structured, communicated, consumed, and renovated (de Lange 2006; Sibert and Draper 2012). Although much is now known about the specific mathematics content to be taught and learned in K-12 schooling (see, for example, NCTM 2000; NGA and CCSSO 2010), much less is known about the nature of mathematics and the literate practices of mathematicians that contribute to the production, communication, and consumption of the mathematics content.

Three empirical studies provide some insights into the nature of mathematics and mathematicians' literate practices. Hoffman and Even (2018) interviewed five mathematicians who taught in a teacher preparation program in Israel regarding what they wished to teach teachers about the nature of mathematics. They also conducted participant observations of the courses taught by these mathematicians. Qualitative analysis of the two data sources (interviews and observations) revealed that the mathematicians sought to expand and deepen teachers' knowledge about what mathematics is. They contended that teachers should know the essence of mathematics, which includes the notions that (a) mathematics is a wide and varied discipline with many domains and many facets, (b) mathematics is rich in connections, and familiarity with these connections is key to properly understand mathematics, and (c) mathematics is a consistent science based on the laws of logic. They also believed that teachers should know that doing mathematics involves asking questions, thinking, understanding the purposes and meanings of what one does, and using intuition and formal representation in problem solving. They emphasized the need for teachers to know that mathematics is a tool for solving real life problems across many disciplines and that engagement with mathematics is a challenging but creative activity that develops rational and logical thought.

Shanahan et al. (2011) studied how two mathematicians, two mathematics teacher educators, and two high school mathematics teachers interacted with texts in their discipline. Specifically, they examined the ways these disciplinary experts used sourcing (i.e. identifying where the text was published, who wrote it, and what its intended purpose is), contextualization (i.e. placing the text in the historical context of its creation), and corroboration (i.e. comparing the text to other texts on the same topic) in their considerations of disciplinary texts. They also examined how text structure and graphic elements influenced the ways experts interacted with the text, the role of interest in their reading, and how they engaged in critiquing the text. By analyzing the data collected from individual interviews, reading think-alouds, and focus group meetings, the researchers found that the mathematicians paid little attention to the author of text (i.e. sourcing) and the time period of its publication (i.e. contextualization); however, they used corroboration (i.e. checking one source against another) as a strategy to help them make sense of and interpret text. The mathematicians used text structure to help them determine where problems and solutions were located. They placed a strong emphasis on accuracy (but not much on credibility), weighing nearly every word and evaluating each piece of information to ensure understanding. They treated both equations and prose equally and as inseparable, interpreting them in a unified manner.

Burton and Morgan (2000) analyzed 53 published research papers in mathematics and interview data involving 70 research mathematicians to explore the identities mathematicians present to the world in their writing and the ways in which they represent the nature of mathematics and mathematical activity. They found that mathematicians foregrounded impersonality through the use of passive voice, asserted authority through the use of boosters (e.g. *it is obvious that*, *clearly*) and the indeterminate use of first person plural pronoun (*we*), moderated claims through the use of hedges (e.g. *could*, *possibly*), engaged and persuaded readers through a narrative style that uses interactive personal pronouns (*we* or *I*) and timeless present tense, indicated membership within the mathematics community through co-authorship and references to the work of other mathematicians, claimed status for and value of their work through demarcation of territory and of knowledge, projected different images of the nature of mathematical activity (e.g. pure vs. applied), presented the source of new mathematics knowledge as primarily from within the system of mathematics itself (i.e. mathematical objects rather than humans), and occasionally indicated the human processes of doing mathematics through expressions of opinions or feelings, conjectures, questions, or explanations for decision. These findings reflect the authors' epistemological stances on mathematics and reveal some of the linguistic/semiotic tools that these mathematicians value and deploy in their meaning making.

Building on and extending this small body of work, we explored, in this study, one mathematician's view on the nature of mathematics and his literate practices in disciplinary inquiry and meaning making. Specifically, we addressed the following two research questions: (a) what is the nature of mathematics from the mathematician's perspective? and (b) what sorts of literate practice did the mathematician undertake in disciplinary inquiry ad meaning-making? Answers to these questions can enrich teachers' professional knowledge base about mathematics and inform their efforts to promote mathematics literacy among students. Although the focus on one participant limits the generalizability of the study's findings, it can be argued that each expert, as a member of his/her own disciplinary community, tends to draw on the social, cultural, and discursive understandings of the community in his/her professional practice (Hyland 2004). This means that each mathematician's work, however idiosyncratic, reflects in some way the norms, standards, worldviews, habits of mind, and practices that his/her community considers legitimate and acceptable.

2 Methods

2.1 Participant and setting

The participant for the study was a male mathematician, whom we shall refer to as Kang, from a large, public, research-intensive university in the United States. He was selected because he was the only mathematics professor responding positively to campus-wide recruitment efforts for our disciplinary literacy research project. Kang was born in China and earned his bachelor's degree in mathematics from one of the top mathematics programs in that country. He earned his doctoral degree in mathematics from a flagship state university in the U.S. He specialized in nonlinear partial differential equations and geometric analysis and had numerous publications in top tier journals in his field. At the time of the study, he was an associate professor of mathematics and had worked at the university for 15 years. He was fluent and articulate in both Chinese and English, regularly volunteering as a translator in local community events.

2.2 Data collection

We collected three kinds of data for the study: interviews, reading think-alouds, and observations. We conducted two “semi-structured” interviews (Seidman 2006) with Kang, one at the beginning of data collection (initial interview) and the other at the end of data collection (final interview). The interview questions were adapted from Chapman (2015), and they probed Kang’s view on the nature of mathematics and his literate practices in disciplinary inquiry and meaning-making. Both interviews took place in Kang’s workplace, and they closely mirrored natural conversation.

Another data source was reading think-alouds (Bereiter and Bird 1985). Kang was asked to select and read an unfamiliar text in his area of expertise with the goal of understanding the text to inform his own research. The text he selected was an article by Caffarelli et al. (2002) from *Annals of Mathematics*, a top tier journal in mathematics. He was instructed to verbalize his thought processes while reading the article. The think-aloud session was videotaped and later reviewed by Kang, who explained or clarified to us some of his own reading behaviors involving the text.

The third data source was observation. We conducted three observations over a two-month period primarily to collect data on Kang’s literate practices during his work hours. Each observation occurred on a typical day as Kang engaged in his professional work on campus. Our role during these observations was that of “observer as participant” (Glesne 1999). In this role, we acted primarily as observers, but had some interaction with the participant. We took field notes throughout the observation sessions. These notes included descriptions of people, places, and activities. They were as descriptive as possible in order to document what was going on during the observed periods. We also included thoughts or questions that arose as the observation was taking place. During the observations, we collected artifacts (e.g. articles, reading notes, photos) that were related to Kang’s literate practices.

2.3 Data analysis

Data were coded by a team of researchers consisting of two faculty members and two doctoral students. Interview transcripts and observation notes were coded for themes related to the nature of mathematics or Kang’s literate practices. They were coded in two stages: initial coding and focused coding (Charmaz 2006). After individually becoming familiar with the data through multiple readings, the research team began initial coding. Together, we coded the data incident-by-incident to identify provisional themes that described what mathematics is or what literate practices Kang engaged in. Once all the data had gone through initial coding, we moved to focused coding, where we adjusted and refined the provisional themes to create more conceptual or selective codes that represented the data.

Transcripts of reading think-alouds and of discussions of these think-alouds were coded for strategies Kang used to help him comprehend and interpret the text (Ericsson and Simon 1993). Each member of the research team individually read the transcripts multiple times in order to create tentative categories of strategies Kang used to help him make sense of the text. We then met as a team to discuss the categories each of us had created, hoping to develop more refined categories. This is a process that involved clustering, enriching, reconciling, decomposing, or expanding the initial categories generated by each team member. Finally, we shared a draft of our preliminary research report with Kang for his verification and approval.

3 Findings

Our study explored one mathematician's view on the nature of mathematics and his literate practices as a member of the mathematics discourse community. We identify and describe the recurring themes under each of these two areas below, using data from initial and final interviews, reading think-alouds (RTA), discussion of reading think-alouds (DRTA), and observations.

3.1 Kang's view on the nature of mathematics

Four themes emerged from our analysis of Kang's discussion of the essence, characteristics, value, and practice of mathematics. These are described below.

3.1.1 Mathematics is simultaneously theoretical (pure) and practical (applied)

Mathematics is often perceived to be theoretical and abstract because it is considered a science of structure, order, and relations. However, according to Kang, mathematics is both theoretical (pure) and practical (applied). Some mathematicians focus on exploring the possible relationships among abstractions (e.g. strings of numbers, geometric figures, sets of equations) that have no obvious perceptual correlates in the physical world. Other mathematicians focus their efforts on solving problems that originate in the world of human experience. Although the works of theoretical and applied mathematicians may seem unrelated, they often influence each other in ways that ultimately lead to advances in both. In fact, Kang noted that the line between theoretical and practical is difficult to draw because sometimes what was once considered very theoretical could later become very practical. He mentioned calculus as a telling case in point, noting

> The whole theory of calculus, years ago it was considered incredibly theoretical. It seemed to have absolutely no use at all. But nowadays, calculus is pretty much the language for all scientists in natural sciences. (Final Interview, L106–111)

3.1.2 Mathematics is not an island

According to Kang, mathematics is not an island unto itself; it is a discipline interconnected with many other disciplines. The development of mathematics is informed by advances in other disciplines such as physics, philosophy, art, and even religion; at the same time, it is also widely used to help solve problems in other disciplines such as biology, statistics, astronomy, engineering, economics, linguistics, law, psychology, and architecture. The interconnectedness of mathematics with other disciplines makes it widely applicable to many disciplines and facilitates its intrusion into almost all spheres of the human life. As Kang explained,

> Sometimes, mathematics is intertwined with physics, other fields. For me, I work on partial differential equations. It has a lot of ties with geometry and physics. So, sometimes you get motivated by problems with physics or geometry when they are translating to the language of differential equations. So, sometimes you are understanding how to understand those problems […] other problems in those fields. So, this is how it goes. (Final Interview, L90–96)

3.1.3 Mathematics is relatively stable but immensely rewarding

Another feature of mathematics is that it is less dynamic than other disciplines. Mathematics advances slowly, with each advancement typically taking decades to achieve but often involving significant ideas. This means that unlike texts in other disciplines, a text in the field of mathematics may remain relevant for a very long period of time. Compared to fast developing disciplines such as chemical biology or computer engineering, new breakthroughs in mathematics are slow and limited. Kang noted that mathematicians are unlike theoretical physicists in that theoretical physicists "have so many ideas everyday but usually none of them works", whereas mathematicians "usually have fewer ideas but try longer and harder to solve or prove them, and the ones that get solved are often significant" (Initial Interview, L234–384). This emphasis on significant ideas is also evident in the everyday work that Kang did, such as conducting peer reviews of articles for academic journals. He said that he was usually "very reluctant" to do the evaluation work if he felt like he could not learn anything from those papers because that is "a waste of time". He preferred reviewing articles for top journals in his field because they publish "very high standard, top notch mathematical work with very significant consequence" (DRTA, L379–384).

Because of the intellectual challenge and scholarly significance involved in doing mathematics, Kang felt that however much time or effort spent working on a problem is well worth it. He took pride in his own work and considered it "very rewarding", especially if he could discover something: "If I establish something. That is very satisfying. That make[s] me really, really happy. If you discover something substantial, that is, I would say, the most rewarding time of your career" (Final Interview, L373–376). His admiration for hard work and success can be seen when he proudly discussed one of his colleagues who studied group theory:

> And he works like twenty-four hours a day, almost. Even he works when he is sleeping. No wonder he achieved so much. He won the National Medal of Science, like a Nobel Prize, in Mathematics, the Abel Prize. (Final Interview, L404–408)

3.1.4 Mathematics is rigorous and demanding

A further feature of mathematics is that it is rigorous. Mathematics is not about running simulations or creating computer models; rather, it is about proving mathematical models using the tools of mathematics discourse (e.g. words, symbols, image, technology), a task that demands both accuracy and precision. Accuracy and precision relate to both presentation of information and development of logical arguments. Kang noted that mathematical writing must be "completely rigorous in logic"; otherwise, it will be devoid of scientific value. Mathematicians are typically very careful in their wording and rely heavily on axioms, definitions, theorems, and proofs when reasoning with the deductive structure of mathematics. Because of its rigor, mathematics is exceptionally challenging and demands careful attention and constant logical inferences. Kang noted that unlike stories that usually do not take much effort to comprehend, a mathematics paper is demanding to read because "if you don't make a reasonable effort, you won't understand" (Final Interview, L186–188).

3.1.5 Mathematics is multisemiotic and multimodal

Mathematics is a science of patterns and relationships among abstract entities. These patterns and relationships cannot be accurately and efficiently expressed by natural language alone. On one hand, language is a powerful tool for classification and conceptualization; on the other hand, it is not very effective in describing the quantitative covariation of continuously variable phenomena such as shapes, angle, and relationships, for which visual and symbolic representations are generally better suited (Lemke 2003). This explains why mathematicians like Kang draw on not just language, but also mathematical symbols and visual displays to make meaning. It is obvious from our examination of the artifacts collected (e.g. published articles, reading notes), language is woefully inadequate in representing mathematical principles, processes, and reasoning. Kang relied extensively on mathematics symbols, equations, and drawings to construct and communicate his understandings. In these representations, mathematical symbols depict the pattern of the relationships among abstract entities in a very efficient manner; the visual displays connect human physiological perception to this virtual reality; and the linguistic description provides contextual information and explanation for the situation represented symbolically and visually (O'Halloran 2000). For Kang, who specialized in nonlinear partial differential equations, mathematics formulas and equations seemed more important than language because they more accurately, concisely, and efficiently present important information (e.g. processes, relationships, operations) that is central to his meaning making (see, for example, Figure 1). When reading disciplinary texts in his field, Kang paid great attention to the visuals in the text, working diligently to interpret formulas and equations and to verify their accuracy and logic as well as their coherence with respect to the explanations offered in the prose. He explained,

> [In mathematics papers], the equations are more important because I hope to have my own judgment, then sometimes I verify my own judgment, very immature judgment of what they said in the text. Sometimes they care enough to explain, sometimes they don't. So, I need to, I need to compare sometimes. Here, they didn't explain anything because they expected to explain this one later. These authors, they didn't expect we all first to immediately understand what he says at this part. (DRTA, L215–222)

3.2 Kang's literate practice

We divide Kang's literate practice into three categories that characterize his daily work as a mathematician: inquiring and problem solving, making meaning with text, and writing to communicate.

3.2.1 Inquiring and problem solving

The bulk of Kang's work involves conducting research and making discoveries. This work requires that he read/view widely, learn from trial and error, and work collaboratively.

Since we study blowup solutions, the maximum of $\tilde{u}_k$ tends to infinity: Let

$$M_k = \max_i \max_{x \in B_1} \frac{|\tilde{u}_{k,i}(x)|}{1+\gamma_i} \to \infty, \quad \varepsilon_k = e^{-\frac{1}{2}M_k},$$

and

$$\tilde{v}_{k,i}(y) = \tilde{u}_{k,i}(\varepsilon_k y) + 2(1+\gamma_i)\log \varepsilon_k, \quad i = 1, \ldots, n. \tag{1.9}$$

Then direct computation shows

$$\Delta \tilde{v}_{k,i}(y) + \sum_j a_{ij} h_{k,j}(\varepsilon_k y)|y|^{2\gamma_j} e^{\tilde{v}_{k,j}(y)} = 0, \quad |y| \le \varepsilon_k^{-1}. \tag{1.10}$$

It is well known that for systems, if the whole system is scaled according to the maximum of all components, it is possible to have some components tending to minus infinity over any fixed compact subsets, which means these components do not appear in the limiting system. Such a situation is called partial blowup phenomenon. If no component is lost in the limit system, such a blowup sequence is called fully bubbling. The main assumption in this article is that $\tilde{v}_k = (\tilde{v}_{k,1}, \ldots, \tilde{v}_{k,n})$ is fully bubbling :

$$\tilde{v}_k = (\tilde{v}_{k,1}, \ldots, \tilde{v}_{k,n}) \to \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n) \text{ in } C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2) \quad \alpha \in (0,1) \tag{1.11}$$

where $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n)$ satisfies

$$\begin{cases} \Delta \tilde{v}_i + \sum_{j=1}^n a_{ij}|y|^{2\gamma_j} e^{\tilde{v}_j} = 0, & \text{in} \quad \mathbb{R}^2, \quad i = 1,..,n \\ \int_{\mathbb{R}^2} |y|^{2\gamma_i} e^{\tilde{v}_i} dy < \infty, & i = 1,..,n. \end{cases} \tag{1.12}$$

Figure 1: Excerpt from Kang's recent writing.

Reading/viewing widely. Kang indicated that he drew inspiration for his work by reading widely and voluminously the work of other scholars, noting that this habit helped him learn new information, gain new understandings, reexamine his own work, explore answers to the unknowns, become familiar with different ways of solving problems, and build motivation for further research. He emphasized that reading recent publications is important because it helped him stay abreast of current trends and issues in his field. However, he was selective in what he chose to read, saying

> To be perfectly honest, I'm very picky about the things. For example, when I read books, I'm very reluctant to read mediocre mathematicians' books because I will be very bored. Then they wouldn't mention something that inspires me. You can read books from a top mathematician and if I read exciting math, I will be excited because it really depends on the quality of the work when I read. (DRTA, L368–374)

Besides reading books and peer reviewed articles, Kang used the resources on the Internet as another way to stay current on recent advances in the field and to tackle the challenges he was facing in his own work. Because journals and books take time to publish, Kang regularly visited a website where scholars shared pre-print manuscripts for up-to-date information related to his research. He also searched for information on the Internet that "relate to the difficulties I'm having in my research works" (Initial Interview, L21), hoping to find needed information or gain new insights for problem solving. Sometimes, Kang listened to lectures on YouTube so that he could better and more efficiently understand topics of interest without having to labor through dense sentences when reading the same information in print.

In addition to surfing the Internet for quick access to new or needed information, Kang listened to presentations by other scholars at professional conferences. He considered these conferences as a congregation of a community of learners, noting that attending conferences also helped him learn new information and gain new insights for problem solving and generate ideas for future research. He shared,

> […] and then sometimes when I go to a conference, I will see that the problems that they work on and then what I can do and then what is the major difficulty to learn how to do, what is more interesting, and what seems to be more interesting for me and accessible to me. (Initial Interview, L93–96)

Learning through trial and error. When reflecting on his own development as a mathematics scholar, Kang emphasized the importance of learning from past mistakes. He admitted that he gained some of the most important insights for inquiring and problem solving from his own past failures and mistakes:

> With research we always make mistakes and then learning from mistakes is really valuable. For all mathematicians, they know a lot of mistakes; so in the long run, when you get into a project, your intuition will tell you, you wouldn't go that way because your past experience teaches you it won't work. (Initial Interview, L212–216)

These past experiences of trial and error served as an important reference for Kang as he considered ways to work through problems. In this sense, repeated failed attempts became a navigational point of reference for leading him in the direction of correctly solving a problem. He described this research process as follows:

> I can tell you my research experience. It usually fails. So, it fails many, many times, but every time it fails, I understand it better, and this is how it works in mathematics. So, after failing many times, one day I could work it out. But sometimes, even the paper […] when the paper is already published, the theory is still flawed. So, this is what happens. (Final Interview, L29–34)

This process of learning from trial and error is an ongoing one and continues even after the point of publishing the work in an academic journal. In some cases, revisiting the work and continued reflection on the trial-and-error process can reveal that work that was once thought to be accurate is actually flawed. Kang shared his own experiences with this, stating

> Yes, but those trials and errors are really part of the understanding, because when you understand it is not there, you always think […] because after working for a long time, you always wish very much it is true. Someday, when the false alarm comes and you thought it was true. But after carrying it out, sometimes you figure out that it is still wrong. But sometimes you cannot. Then, after checking many times, you still think it is true, but it is still wrong. Sometimes even when the paper is already published in top journals, it is still wrong. (Final Interview, L38–46)

Although experiencing failure or making errors in research would likely be frustrating for many, Kang remarked that these are some of the most important lessons novice mathematicians will learn. When asked what he believed was the most important thing for students to understand about the work of mathematicians, Kang responded this way:

> I would say they have to learn from failure. Because failure is a very good thing. Very few people understand this part. When you fail, that is a good way that you gain experience, you gain a lot of maturity; and that makes your success very rewarding afterwards. You probably need to fail hundreds of times before you can really have something substantial, something true and correct. (Final Interview, L387–392)

Experiencing failure is only beneficial if those who experienced it apply what they learned in their mistakes and continue to think and persevere in their quest to find a solution to the problem at hand. In the following quote, Kang shared his experience with persevering despite (or even in light of) his failures:

> I need to spend a lot of time thinking about the difficulty I run into in my research. Usually, it takes a lot of time and produces no result, but I keep trying and keep trying. Then, as I keep trying, I have a better understanding, and I have better evaluation judgment of whether the result may be true or false. (Final Interview, L361–366)

Working collaboratively. In his research work, Kang spent a tremendous amount of time reading, searching the Internet for forthcoming publications, listening to YouTube videos and conference presentations that informed his work, and thinking long and hard about his previous failed attempts at problems in order to come up with different ways of approaching a problem. However, not all the research Kang engaged in is independent work. In fact, he worked very closely with a network of colleagues at home and abroad to tackle important problems in his field. His collaborators included colleagues at his own institution as well as others around the world. While most of the times communication between Kang and his collaborators was done via emails, there were times when he needed to travel in order to talk face to face with his collaborators. These travels took him to places within and outside the U.S. He shared:

> I have a collaborator, one of the major collaborators, in Taiwan of China, so I travel to Taiwan of China 10 times already. Then I travel to Brazil, maybe six or seven times already. I travel to the mainland of China quite a few times, and then I travel to Canada and then other places in the United States. (Initial Interview, L122–126)

Whether meeting face-to-face or communicating via email, one common thread linking all collaborations is a shared interest in tackling a particular problem. Sometimes, collaboration developed through joint research. In other cases, collaboration was solicited by individuals seeking insights from other professionals working on similar problems. According to Kang,

> They [collaborations] all started from our common interest; so we would—one of us would suggest [inaudible]. Usually, the problems would come up naturally from our joint research. When we have something, one of them may say "can we work on such and such a problem", and then we share insights, and then sometimes when we face a difficulty and then we will try to solve the problems all together. Sometimes just one of them works on a special part and then we put our different parts together. (Initial Interview, L53–58)

In each of the collaborations Kang described, participants worked in a symbiotic fashion, each benefiting from the other's insights and work. In one of his collaborations, Kang described a relationship in which both benefited, but he felt that this collaborator also provided him with a form of mentorship that was long lasting and impactful. In this "mentoring" collaboration, Kang and his partner would sometimes work through different parts of a problem; at other times, Kang would attempt to solve the work by himself until he got stuck on a challenging portion and then had to reach out to his partner for guidance. All the while, however, the two were learning from each other and continuing to inspire each other's work. He confided:

> I recently worked closely with a professor in Taiwan of China who is such a big shot in my field. Many times we will meet a common difficulty and then we have different experiences. So, we discuss. And when we discuss, many times we can inspire each other. But sometimes we just work different parts. Since he is like my teacher, so I undertake most of the work. So, when I really run into a very substantial difficulty, sometimes this is how it works. […] I am not sure all collaborators are like that. But, when I really run into a very difficult point, then he may know it. Then, I will describe to him very carefully. Sometimes, he can solve it; sometimes, he cannot. (Final Interview, L440–450)

It is clear that Kang had a solid circle of collaborators with whom he had developed close working relationships. These relationships were formed as a result of mutual interest in a particular topic/area of research or cultivated through the family tree of mentorship. Kang shared how his alma mater provided him with opportunities for collaboration, as he worked with former classmates and others who were trained through an instructional legacy of former advisors. He described his collaboration with his mentor in Taiwan of China as one such relationship that stemmed from his training institution:

> For me, the reason that I collaborated with the professor in Taiwan of China is because he is very famous in our field […] and then he's also from the same school that I came from. Which means his advisor is the advisor of my advisor, so they are from the same person. They were working in fields very close to each other. Then, there is another collaborator in Canada who is also from the same school, and his advisor is also the advisor of my advisor. The same family tree. (Initial Interview, L130–140)

This collaborative "family tree" has provided Kang with opportunities to work with individuals in Taiwan of China and Canada. It is likely that training under the same advisor or in the same institution helped in establishing a common thread in research interest or agenda. In addition to those who had the same advisor, Kang also collaborated with former college classmates. He said,

> There are other collaborators. Some of them are my college classmates and then we used to know each other. We used to be roommates, and even though right now we are working in different institutions, but we still have a common interest where we actively collaborate. (Initial Interview, L144–148)

The collaborative relationships that Kang forged at his alma mater were still strong at the time of this study, as common research interests continued to be the driving force in maintaining and enhancing these bonds.

3.2.2 Making meaning with text

As indicated earlier, Kang engaged in extensive reading as part of his literate practice as a mathematician. When reading the self-selected article from *Annals of Mathematics* (i.e. Caffarelli et al. 2002), he employed a range of strategies to help him make sense of and interpret the text. These strategies were distilled from the think-aloud protocols and confirmed or clarified through discussion of the think-aloud video as well as the two interviews. They include close reading, summarizing, questioning, evaluating, storying, and annotating.

Close reading. When reading to make sense of the self-selected article, Kang paid close attention to every detail in the text. He read slowly to make sure that he understood what he was reading before proceeding further. He said that "if I don't understand, it doesn't matter that much if I read very fast […]" (Final Interview, L155–157) and “I need to have a solid understanding before I feel comfortable to proceed” (Final Interview, L223–226). To ensure understanding, he often had to reread portions of the text. He commented that as a mathematician, he was trained to “fully understand everything you read” and that is “why if I see something I don’t understand, I have to read it again and again” (Final Interview, L125–130). Sometimes, he flipped the pages back and forth during reading because he saw connections between different parts of the text, and what he read in later sections helped him clarify confusion and better understand what he had read earlier, as the reflective discussion below shows:

> Obviously, there were a lot of ties between the introduction part and the abstract; so, I realized the ties as I went back, and I feel like I understand better of what the author wanted to do. (DRTA, L605–607)

> When I was reading the second part, I still remember there was a question I raised in introduction. When I came to this part, I realized that they were explaining the doubts I had, and so I went back. I went back to verify. This is the answer […] to the question I raised in the introduction. (DRTA, L245–249)

When probed about the motive behind the observed behavior of recursive reading, Kang explained,

> I hope that as I read more, my understanding of the previous part can be deepened, because many times when I am not familiar, I hope to get some information to substantiate what I read before. (Final Interview, L165–167)

Summarizing. Kang regularly summarized what he was reading to monitor or clarify his own understanding. For example, he was heard verbalizing, in his own words, the gist of a section of the text he had just read:

> This part they said why they want to work on this new theorem because it was motivated by a few mathematicians in their previous work. They did have an adequate [inaudible] to back up their research. Here, they mentioned the difficulty of this research by a previous author; they couldn’t do it. Here, they said their new theorem proved in this article cannot be improved; so it’s the optimal result. They said even though the theorem they are going to present don’t seem to be optimal at first, but with a little extra assumption that people can recognize that it’s as good as it gets. (RTA, L85–93)

Distilling, rephrasing, and commenting on what was read from time to time seemed to help him cope with the comprehension challenges posed by the dense, formal style of mathematics writing, enabling him to better understand and retain the gist of what he had just read. As Kang noted, the only way to understand deeper is to “put the meaning in a different style of understanding” (DRTA, L470).

Questioning. As he read, Kang asked many questions, some rhetorical in nature, as a way of monitoring, clarifying, or deepening his own understanding. This is evident in his think-alouds below:

> Both functions, wait, why is this quantity fine? If one derivative equals, is it true? It's true but I feel the whole […] this is not. Strange! I'm not sure this is correct. They use […] to prove this and this […], so this one controlled by the information in the boundary, but the boundary […], this is easier. They have the big point. Maybe it's okay, but I still don't fully understand; so, all of these will all be fine? (RTA, L223–229)

> I now see […] this 2.4 is really, really important. Does it mean when this is established, the theorem is proved? Is it true or is it true? (RTA, L264–275)

Sometimes, Kang voiced wonderings and then tried to answer them himself by rereading, logical reasoning, or simply reading ahead. This helped him clarify, confirm, or assess his own understanding while at the same time keeping him engaged with the text, as the following think-aloud excerpt illustrates:

> They must say even with all this information it is useful; so, let's see if they have used, if they have any applications. They mentioned some are variants and [inaudible]. Do we have the applications? We have. They can still prove the same thing that the original math in this […] can be used to prove. They have important applications. (RTA, L317–321)

Evaluating. During reading, Kang engaged in constant evaluation of the information – linguistic statements, mathematical equations/formulas, visual displays – he encountered in the text. For example, in the following think-aloud excerpts, Kang evaluated the text by verifying the accuracy of calculation:

> They control A+ by B+, they control B+, suppose this can be done. I agree. B+ and they also can control, equals to the [inaudible]. Here they use a spectrum, here they use a spectrum, they have to use a spectrum. (RTA, L244–249)

> There is a graph, so let me draw a graph here. So, what they have over here is over B1, is this quantity alpha, which is omega [inaudible], U over X, and that is cool. This one is 590 first and then it is very large, and then they said if you [inaudible], so this part is greater than 256 and then there is a constant, omega is the – this is over here. The territory that you began zero occupies is bigger than universal [inaudible]. This is really good. (RTA, L152–159)

He also evaluated the logic of argument to be sure there is a logical link from one step of reasoning to the next, as can be seen below in his think-alouds and his comments during the discussion of the think-alouds:

> I don't believe it's a mistake, but they didn't write it very well; so it looks like they need some more assumption, but they didn't mention. Maybe it's okay. Obviously, it's not well written. It appears to be a mistake, but it can probably, it can be fixed. (RTA, L139–141)

> We now prove that the CA is large and then meets, what's the graph? So, here on the boundary of B1, there is a U plus, this is U minus. So, they have to have […] one of them cannot occupy the whole thing. (RTA, L229–232)

> Here, because they mentioned something I don't immediately see, I just went to a page to carry out […] do some computation to verify what they said is true then move on. I feel that it is – this is obviously an important quality that they want to use in an argument that follows, so I verify by myself before I move on. (DRTA, L265–270)

At other times, Kang evaluated the credibility of sources because knowing the quality of a publication outlet or the reputation of the author influenced the way he interacted with the text. For "very outstanding mathematicians who are very careful in their research" and "who also have very good reputations", he did not waste time evaluating their calculation or reasoning because he "fully believe them" (Final Interview, L631–636); but for others, he may "have a lot of doubts" about and "wouldn't believe" what they say, especially "if they say something audacious or too aggressive" (DRTA, L197–206). He confided:

> If the author is someone I know who has a very good reputation, then I will see certain parts that I believe is right, but I don't have the energy to verify. If I need to verify, I need to read every sentence. If it is someone […] a nobody, I even look at the public record. If the record is not so good, and I don't believe what he did is right. I never read an article below a certain level, because I just don't trust. (Final Interview, L200–215)

The content of an article is also a factor that impacted Kang's approach to reading. He valued articles whose content is ground-breaking or inspirational. As he professed,

> If I see a mathematics article that appears to be very important, then that is a great motivation for me that I need to make an effort to read into it and hope to learn good mathematics from those articles. (Final Interview, L342–345)

When encountering sections of the text that he believed contain important or insightful information, Kang would become excited and work hard to understand them, as can be seen in the discussion of his reading think aloud below:

> The things they mentioned and the theorem they proved that seemed to be so important, that's so powerful, and I hope to know as much as possible about how to use this important tool. (DRTA, L92–94)

Storying. During reading, Kang was seen using a storyline with made-up characters to make sense of what he was reading. This strategy, called "storying" (Wells 1986), helped Kang transform what he saw on page, verbal (prose) or visual (equation), into something more concrete, familiar, and personal, enabling him to better grapple with technical concepts, unpack abstract relationships, and follow logical reasoning. For example, Kang used a war metaphor involving soldiers from two opposing sides fighting for territorial control to help him visualize and make sense of a mathematical equation presented in the text:

> This is like reading a novel; so, these two functions, they are fighting like two soldiers for territory. If this one is large, they are both large. This one is controlled by this. (RTA, L300–302)

The use of the strategy seemed to help him interpret and better understand the text at the same time he was savoring the reading experience, as can be seen in the think-alouds below:

> That is why they are – if one guy is too much stronger, the other one has to be weaker, then their product is out of control. This is wonderful. If this one is strong, the territory they occupy is large, that is 2.1. And then, when this one is large, this one occupies a small territory, this one becomes small. When this one becomes small, okay, I got it. One more is proved; my point is proved. That's it. As one, it's this; as two is this. Our goal is to find this. They are hoping that they will do that, as three is the compliment, as three is under control. I pretty much see it. (RTA, L292–300)

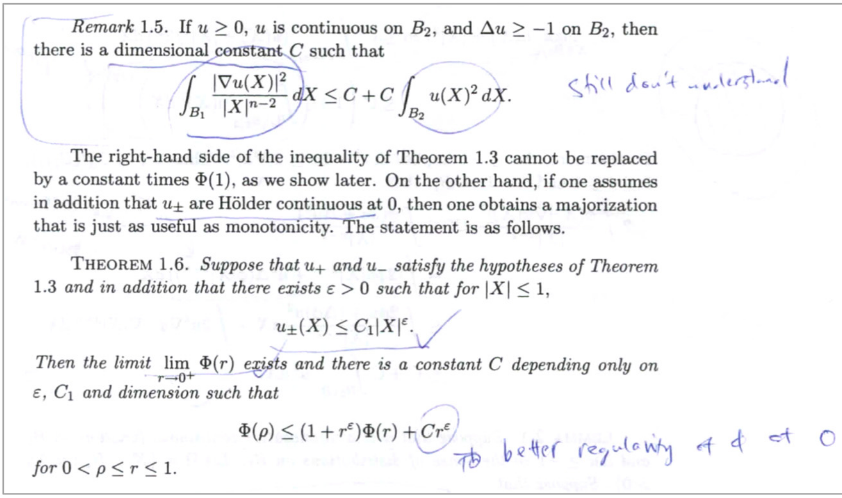
When probed further about his use of the strategy, Kang explained,

> […] when you read into mathematics, they always put it in a very formal way, but you can put it in a very informal manner; so, it's like two guys, it's like watching a fight. This guy is getting stronger […] then they must prove that guy is weaker in order for them to have the balance. I was waiting with anticipation if this guy is large, then the other guy has to be small […] how do we reason that out. (DRTA, L416–424)

Annotating. While reading, Kang underlined, checked, circled, drew, and wrote in the text (or a separate piece of paper) to record his thoughts/reactions and perform calculations. Annotation allowed him to slow down the reading process, giving him time to ponder, highlight, visualize, digest, and verify the densely packed information in the text, as Figure 2 shows.

Kang valued the strategy of annotating because it seemed to help him visualize and evaluate text information, clarify his thinking, and improve his understanding. For example, as he was trying to make sense of a part of the text, he was heard thinking aloud while writing and drawing in an attempt to make sense of what he was reading and to address any doubts he had in mind about the information presented in the text:

> This one, super harmonica, so this one it says, let me draw a graph. This is B2 over here and then they control the interior of this one. [Inaudible] function controlled by the indigo form, then the indigo form is controlled by this, of course. Then, they can see they are the more general case. it true [inaudible] […] if I take the function. (RTA L127–131)



Remark 1.5. If $u \geq 0$, u is continuous on B_2, and $\Delta u \geq -1$ on B_2, then there is a dimensional constant C such that

$$\int_{B_1} \frac{|\nabla u(X)|^2}{|X|^{n-2}}\, dX \leq C + C \int_{B_2} u(X)^2\, dX.$$

The right-hand side of the inequality of Theorem 1.3 cannot be replaced by a constant times $\Phi(1)$, as we show later. On the other hand, if one assumes in addition that $u_\pm$ are Hölder continuous at 0, then one obtains a majorization that is just as useful as monotonicity. The statement is as follows.

THEOREM 1.6. *Suppose that u_+ and u_- satisfy the hypotheses of Theorem 1.3 and in addition that there exists $\varepsilon > 0$ such that for $|X| \leq 1$,*

$$u_\pm(X) \leq C_1 |X|^\varepsilon.$$

Then the limit $\lim_{r \to 0^+} \Phi(r)$ exists and there is a constant C depending only on ε, C_1 and dimension such that

$$\Phi(\rho) \leq (1 + r^\varepsilon)\Phi(r) + Cr^\varepsilon$$

for $0 < \rho \leq r \leq 1$.

Figure 2: Kang's annotation during reading.

> Let me write down 5 or 4 minus K is 4 to the 4K AK+ AK– and then you have got both big of course. (RTA L282–284)

> Wait, I'm confused okay, greater 22 equals 2U greater than U2 equals to greater into U2 plus [inaudible] okay, greater than 2, greater than U2 minus. There is two and then [inaudible]. I have this. (RTA L132–135)

When asked later about his use of the strategy, Kang explained that annotating while reading enhanced his comprehension. He noted that when he encountered unfamiliar expressions in the text, he would "need to write down and then take a look" (DRTA, L158). This gave him time to think about, visualize, and interpret the information presented, resulting in deeper understanding of what he was reading.

3.2.3 Writing to communicate

Kang wrote regularly to communicate his research findings, share his ideas with peers, or seek funding for his research projects. Typical genres of his writing include journal articles, grant proposals, and conference presentations. His writing has several discursive features that are indicative of his epistemological stances on mathematics, conform to discipline-legitimated norms, reflect mathematical habits of mind, and facilitate presentation of information and development of argument.

For Kang, mathematics is a discipline that demands rigor. Therefore, he placed a high priority in ensuring that his writing was precise and clear in presenting information, logical in developing arguments, and responsible in crediting sources. He shared that "in mathematics if you make a mistake in one step, then the whole thing could be completely destroyed. You have absolutely no scientific value at all" (DRTA, L274–278). He acknowledged that although mathematicians do strive to make their writing clear and accessible, they cannot sacrifice rigor in the process. He explained,

> [Mathematicians] usually make great efforts to make their mathematic reasoning easy to read. On one hand, they make the text easy to read so the [manuscript] evaluation process doesn't take two years. On the other hand, they want to make sure, themselves, it is correct. So, they always bring the whole reasoning part into a lot of [...] so they can be sure what they did is right. (Final Interview, L258–264)

Kang emphasized that because mathematics theories are often very profound and not obvious, mathematicians' main job is not to make sentences look artistic but to make the theories and explanations accessible and easy to follow for the reader.

> When they [mathematicians] write, they have to make people understand, because usually mathematic theory is so profound and so involved. It doesn't help your readers if you make the sentence very artistic. It doesn't help that much. (Final Interview, L248–254)

However, he seemed to value rigor even more because that is how he gained credibility among his peers.

> So, what I do is [...] um [...] first, make sure I need to gain my credibility among my peers. So, I need to make sure every step is correct. Then, I need to read it quite a few times to make sure it is correct. Then, after that, I need to rewrite those sentences to make the article look good. (Final Interview, L305–309)

Kang said that even though he liked well written sentences, he would have to simplify it in his mind when reading to ensure understanding. On the other hand, he would not recommend people to write "very plain, very simple sentences" because that "doesn't make the article look good" (Final Interview, L240–243).

To make writing both rigorous and accessible at the same time, Kang recommended that writers pay careful attention to wording through multiple revisions. He indicated that mathematicians "spend a lot of time to make sure that every sentence we put into the final version of the paper is logically sound" (Initial Interview, L173–176) and that the final paper submitted is "completely rigorous in logic" and "completely intact" (Initial Interview, L169–180). This involves rereading what he initially wrote to make sure every step of calculation or reasoning is correct and rewriting the sentences to make the final version look polished. He was especially diligent in crafting a good introduction to his paper. He did so by "making the language look good and also make sure to give credit to people who have a contribution to my project" (Final Interview, L309–312). He admitted that it took him a lot of efforts to write a good introduction, but he felt it was well worth it. He was especially diligent in making sure he did not miss any work that needed be referenced.

As a result of his diligence in striving for rigor, as well as his training as a mathematician and his extensive experience communicating mathematics through writing, Kang's writing has many discursive features typical of what is commonly found in mathematics discourse. A case in point is an excerpt from one of Kang's recent publications, presented earlier in Figure 1. The writing is multimodal/multisemiotic, and in fact predominantly so, with roughly four-fifths of the entire article filled with mathematics symbols, formulas, and equations. It is technical, dense, and rigorous. It uses (a) highly technical terminology (e.g. *blowup solutions, minus infinity, fixed compact subsets, fully bubbling, the limiting system*) and mathematics symbolisms to construct specialized knowledge; (b) mathematics equations to encode mathematical processes, operations, and relationships; (c) logical connectives with precise mathematical meanings to engage in logical reasoning (e.g. *if, let* […] *then, since*); (d) relational verbs for defining, identifying, and attributing (e.g. *which* <u>*means*</u> […]; *Such a situation* <u>*is called*</u> *partial blowup phenomenon.*); (e) passive voice (e.g. *is scaled, is called*) to foreground concepts/ideas and to facilitate informational flow; (f) epistemic hedges (e.g. *it is possible*) to moderate claims and boosters (e.g. *it is well known*) to show confidence or certainty in claims; (g) first person plural pronoun (e.g. *we*) to establish an egalitarian relationship with readers so as to engage and persuade them; (h) nominalization to distill information and facilitate discursive flow (e.g. *such a situation* is a nominalization that refers to and summarizes the information presented in the sentence preceding it); and (i) complex sentences with multiple modifications (e.g. *It is well known* […] *in the limiting system*) to pack a heavy load of information for rigorous reasoning. These devices contribute to the construction of the "mathematics register" (Halliday 1978), a type of disciplinary discourse unique to mathematics meaning making, that enables Kang to engage in rigorous reasoning, provide clear explanations, and advance his argument.

4 Discussion

Our case study explored two neglected dimensions of mathematics literacy – the nature of mathematics and a mathematician's literate practices. The mathematician in our study viewed mathematics as rigorous and demanding, both theoretical and practical, relatively stable but highly rewarding, interconnected with other disciplines, and involving discipline-legitimated discursive practices. He engaged in extensive reading/viewing and writing, valued learning from trial and error, and collaborated with an international network of peers in research. He used an array of reading strategies – including close reading, summarizing, questioning, storying, evaluating, and annotating – to help him make sense of and interpret disciplinary texts, and he marshaled a range of verbal and visual resources to create specialized knowledge, engage in rigorous reasoning, develop logical argument, and construct professional identity. Because our study involved only one participant, it is possible that some important mathematical insights, understandings, practices, or strategies common to the community of mathematicians were not demonstrated by Kang. It is, thus, likely that our findings would have differed somewhat from what is reported in this paper had more mathematicians or another mathematician representing a different branch of mathematics been involved. At the same time, it is important to acknowledge that the mathematics discourse community is likely not homogeneous, with variation among mathematicians in at least some of their literate practices (Burton and Morgan 2000; Weber et al. 2020); and as a scholarly member of the community, Kang demonstrated at least some of the attributes, insights, practices, and habits of mind that are discipline legitimated and likely not uncommon among practicing mathematicians.

This limitation notwithstanding, we did identify some important insights into the nature of mathematics and the literate practices of mathematicians that are relevant to educators with an interest in promoting mathematics literacy. According to Kang, for example, mathematics is both theoretical and practical, and its development, though slow, informs and is informed by that in other disciplines. This finding suggests that students need opportunities to explore and forge connections between mathematics and other disciplines (e.g. science, engineering, architecture) and between mathematics and the real world so that they can see that mathematics is not just an exercise of abstract symbolic operations but is intertwined with other disciplines to address issues and problems of both theoretical and practical significance. With this in mind, educators can then design curricula and activities that are authentic to the nature of mathematics and promote the development of mathematical habits of mind.

Our study also shows that Kang attached great importance to logic and precision in his work. This is understandable because mathematics is a science of abstract objects involving numbers and algorithms, and mathematical statements typically consist of a series of logical arguments applied to certain accepted rules. As such, mathematics relies heavily on logic as its standard of truth. To be logical in reasoning (e.g. induction, deduction) requires precision (i.e. accuracy/exactness) in thinking and conceptualization, and precision leaves no room for ambiguity or doubt. Bass (2011) found a remarkable degree of shared aesthetic sensibility associated with words like elegance, precision, lucidity, and coherence that affects not only how mathematicians appreciate but even how they do mathematics. Thus, learning mathematics entails learning the habits of mind and discursive practices that mathematicians value in their meaning making, including striving to be precise, clear, coherent, logical, and rigorous in presenting information, offering explanation, and developing argument. These are aspects of mathematical literacy that merit serious attention and greater emphasis in the teaching and learning of mathematics.

At the same time it emphasizes precision, mathematics necessarily involves error. Due to the demands of mathematics work, Kang emphasized the importance of learning from trial and error, of seeing failure as "the mother of success", and of being willing to learn from past experiences. Burton and Morgan (2000) found that mathematicians understand that any given instantiation of mathematical concepts will involve error, provided the concepts in play are sufficiently complex. This means that students should be encouraged to take risks and not be afraid of making mistakes because it is through failed attempts that students gain new insights that will enable them to take fewer detours and become more efficient in future problem-solving endeavors. The challenging nature of mathematics work requires that students be patient, diligent, and persistent when working on problems in mathematics. It also demands that students have a firm commitment to learning (because knowledge and success are built on many trials and errors) and to not giving up easily in the face of repeated failures (because the reward of discovery is immense). On the other hand, teachers must embrace and value the mistakes students make, recognizing that mistakes are a natural part of the learning process as students explore and make conjectures. They need to be able to use these mistakes as a window into students' problem-solving process and engage students in analyzing and learning from these mistakes. They also need to know what strategies to recommend so that students can avoid making similar mistakes in comparable situations and be empowered to reach and justify conclusions based on their own mathematics knowledge and reasoning.

Our study found that Kang engaged in disciplinary inquiries through a range of literate practices, such as wide reading of the professional literature in print and online, listening to lectures and talks on YouTube, and writing to disseminate research findings. Similar to experts in other disciplines such as science and history (e.g. Bazerman 1985; Wineburg 1998), Kang engaged in a variety of social practices involving reading, writing, viewing, listening, and talking, not only to keep himself abreast of the field and gain ideas and motivation for his own research, but also to delve into his wonderings and discoveries and to demonstrate his professional competence among peers. These findings suggest that reading, writing, viewing, listening, and talking should be made an integral part of the classroom activities that students engage in while learning mathematics. In other words, to develop competence in mathematics, students must have the opportunity to read and discuss rich, relevant texts in the discipline, write to construct and communicate understanding, engage in multimodal learning, and interact with peers and more knowledgeable others on a regular basis. Furthermore, Kang's explanation of the meticulous and time-consuming nature of his reading of "worthy" texts – echoed by the mathematicians in Shanahan et al. (2011) – is a reminder that the amount of time reserved for students to read mathematics texts should be considerable.

Teachers should address mathematical reading and writing explicitly to make successful participation in mathematical practices more accessible for all students. One way to promote reading and writing in mathematics classrooms is to encourage students to read stories about mathematics topics and biography of mathematicians, as well as to write mathematical stories, on a regular basis. According to mathematics and literacy education scholars (e.g. Borasi et al. 1990; Newkirk 2014), stories are a powerful tool for engaging students in learning new or abstract information, developing their conceptual understanding, stimulating and maintaining their interest in mathematics, deepening their understanding of the nature of mathematics, and enhancing their problem-solving skills.

Another way to support students' mathematics reading and writing is to help them develop skills and strategies for coping with the mathematics register. According to Kang, mathematics is rigorous and demanding, and part of the challenge of learning and doing mathematics has to do with its technical, dense, abstract, and multimodal/multisemiotic discourse. Thus, developing control over the mathematics register can help students better comprehend and solve mathematical problems, as well as communicate mathematical ideas (Wilkinson 2018). A productive classroom routine for teaching the mathematics register is to provide opportunities for students to switch between everyday language and mathematics language as they discuss mathematics concepts and ideas using multiple representations (Herbel-Eisenmann 2002). If students receive more "at bats" with mathematics language, they will not only be better able to understand and use mathematical genres, but also to break from conventions in a productive way, as mathematicians do (Burton and Morgan 2000).

The study of specific instantiations/representations and the study of mathematical abstractions are complementary, but too often the conceptual understanding of students of mathematics was overlooked when teachers emphasized an understanding of specific representations and explanations of mundane (though not unimportant) topics (Lee and Spratley 2010). A corrective shift in instruction should be to move students toward an emphasis on mathematics language and concepts as they progress through school. As Kang's interviews indicate, experts consider mathematics as ultimately dealing with relationships between abstract concepts, and not about any single instantiation or application of those concepts. Moreover, Kang's verbalization of his efforts at comprehension reveals that reading a dense mathematical text involves reperforming much of the work of the author. Mathematics is not about understanding any given example, but about relating mathematical concepts to one another (Burton and Morgan 2000). Therefore, another approach to support students is to conduct modeling and explicit instruction of the strategies that experts like Kang use (e.g. close reading, summarizing, evaluating, questioning, storying, annotating, sourcing) to make sense of and interpret mathematics texts. Encouraging this sort of metacognitive behavior will bring students closer to the literate practices of mathematicians (Weber et al. 2020).

While Kang did much of his learning and inquiry independently, he also worked collaboratively with his mentors, peers, and students to solve problems. This finding is consistent with that reported in Burton (1998). Specifically, Burton interviewed 70 mathematicians and found that they almost unanimously claimed a preference for collaborating on work. Given the demanding nature of mathematics work and the time it usually takes to solve a significant problem in the field, it is understandable that mathematicians value collaboration. Some of the advantages of collaboration reported by Burton's participants include benefiting from the experience of others, an increased quantity and quality of work, opportunity to share ideas, and a lessened feeling of isolation. The participants also noted that they had work that they did individually and work that was better suited for groups. For example, some of the participants indicated that "learning" work or work that resulted in acquiring a new understanding was better suited for individual work. This appears to be true with Kang as well.

If collaboration is needed and prized in the professional community of mathematicians, it is worth asking why school mathematics classrooms continue to promote "an individualistic teaching/learning model", which "locates responsibility within the learner and supports a teaching style which is content-based and fragmented" (Burton 1998: 138)? We need to rethink the transmission model of teaching, which views mathematics as a fixed body of knowledge to be transmitted through lectures and which privileges independent work. This model is didactic and discourages collaboration; it is likely to not only fail to support learners in problem solving but also deprive them of the very pleasure Kang spoke of – the pleasure of connecting with colleagues and working together on significant problems. A collaborative model that embraces peer contributions and emphasizes meaning negotiation is more authentic to and productive in mathematics problem solving and will likely be valued and enjoyed by students. Toward this end, teachers can encourage students to share their work, pose questions to one another, clarify or critique peers' reasoning, compare/contrast different solution strategies, and celebrate peers' accomplishments (Thompson et al. 2008).

5 Conclusions

Although there exist many beliefs and claims about the nature of mathematics and mathematicians' literate practices, these beliefs/claims rarely result from empirical research (Osterholm 2006). Our study explored one mathematician's view on the nature of mathematics and his literate practices. It generated valuable information on traditionally neglected dimensions of mathematics literacy. This information has important implications for literacy and mathematics educators. Specifically, a better understanding of the mathematician's perspectives, experiences, and behaviors related to disciplinary inquiry allows us to design curriculum and pedagogy that are more authentic to mathematics and to develop habits of mind and skills/strategies that are integral to mathematics meaning making. Future studies can extend our work by including more participants from different branches of mathematics and examining their literate practices both within and outside their areas of specialization.

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