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Two Models of Kantian Construction

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Abstract: According to Kant, we gain mathematical knowledge by constructing objects in pure intuition. This is true not only of geometry but arithmetic and algebra as well. Construction has prominent place in scholarly accounts of Kant's views of mathematics. But did Kant have a clear vision of what construction is? The paper argues that Kant employed two different, even conflicting models of construction, depending on the philosophical issue he was dealing with. In the equivalence model, Kant claims that the object constructed in intuition is equivalent to the properties included in the conceptual rule. In the overstepping model of construction, Kant argues that construction goes beyond the concept which is a "mere definition". What is more, both models of construction can be found in the Doctrine of Method in the first Critique. The paper examines reasons that have led Kant to adopt the two models of construction, and proposes a reading that alleviates the apparent contradiction between the two models.

Keywords: construction; geometry; Kant

1 Introduction

According to Kant, we gain mathematical knowledge by constructing objects in pure intuition. This is true not only of geometry but arithmetic and algebra as well (Engelhard and Mittelstaedt 2008). Construction has prominent place in scholarly accounts of Kant's views of mathematics (Friedman 1998; Hintikka 1999; Kim 2006; Poincaré 2009, p. 43). Schelling (2008) suggested it is one of the central discoveries made by critical philosophy. But did Kant have a clear vision of what construction is? In the *Critique of Pure Reason*, he offered a definition: "to construct a concept means to exhibit *a priori* the intuition corresponding to it" (KrV, A713/B741). He developed this account in a later essay:

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In a general sense one may call construction all *exhibition* of a concept through the (spontaneous) production of a corresponding intuition. If it occurs through mere imagination in accordance with an *a priori* concept, it is called *pure* construction. (AA VIII, p. 191)¹

This is clear enough. To construct means to create an object in pure intuition (such as a triangle) that corresponds to a concept (such as that of triangle). The paper will argue, however, that Kant's usage of the term shows less clarity. Kant employs the idea of construction in diverse contexts that do not chiefly concern mathematics. For example, Kant evokes construction to explain why (real) definitions have no place in philosophy (KrV, A727/B755f.), to help explain teleology (AA II, p. 94), to argue that conceptual non-contradiction does not amount to real possibility (KrV, A220/B267f.), and to describe intellectual beauty (AA V, p. 363f.). According to Van Cleve (1999, p. 36), the idea of geometrical construction was instrumental even in establishing the ideality of space. This is a diverse set of philosophical issues. I will argue that, across these problems, Kant employs the idea of mathematical construction in two different and seemingly conflicting ways. Specifically, the paper will claim that Kant employs two distinct models of construction, and that both models can be found in the Doctrine of Method of the first *Critique*.

I will first establish there are (or seem to be) two models of construction in Kant's texts. Consider a statement from the On Definitions segment of the Doctrine of Method. Kant writes:

Thus there remains no other concepts that are fit for being defined than those containing an arbitrary synthesis which could be constructed *a priori*, and thus only mathematics has definitions. For the object that it thinks it also exhibits, and this can surely contain neither more nor less than the concept. (KrV, A729/B757f.)

Kant claims that the mathematical object (an object drawn in pure intuition) can contain only what was present in the concept of that object. The earlier *Inquiry* makes a similar point:

mathematics arrives at its concepts synthetically; it can say with certainty that what it did not intent to represent in the object by means of the definition is not contained in that object. (AA II, p. 291)

Again, the mathematical object contains only what is included in, or intended by, its definition. In view of its content, the object is coextensive with its definition. I will call this the *equivalence model* of construction.

¹ When referring to Kant's work, »AA« stands for *Akademieausgabe* (Kant 1900–). When citing *Critique of Pure Reason*, I indicate the pagination of the first and the second edition. Whenever possible, I quote the English translations from the Cambridge Edition of Kant's works (Kant 1992–).

Now consider another statement made in the Doctrine of Method. Here Kant appears to be saying something completely different:

For I am not to see what I actually think in my concept of a triangle (this is nothing further than its mere definition), rather I am to go beyond it to properties that do not lie in this concept but still belong to it. ... [This] procedure is that of mathematical [...] construction. (KrV, A718/B746.)

Here Kant states that to construct a figure means to go beyond the properties included in its concept (or its “mere definition”). The constructed object has more content than the corresponding concept. Accordingly, I will call this the *overstepping model* of construction, as the object oversteps or exceeds the concept/definition. Kant offers another instance of this model in the *Critique of Judgment*, while discussing beauty in mathematics:

The many rules, the unity of which (from a principle) arouses this admiration, are one and all synthetic, and do not follow from a *concept* of the object, e.g., from that of a circle, but need this object to be given in intuition. (AA V, p. 364)

Again, Kant states that some of properties displayed by the object made in pure intuition do not lie in its concept. Incidentally, Kant thinks the delight of serendipitous discovery of these properties (or rules) accounts for the aesthetically pleasing effect of mathematical research.

There is an apparent conflict between the two models. According to the equivalence model, the object cannot “contain neither more nor less than the concept” (KrV, A730/B758). According to the overstepping model, however, constructing an object involves going beyond the “concept ... to properties that do not lie in this concept but still belong to it” (KrV, A718/B746). In the equivalence model, the object is the intuitive equivalent of the conceptual content; in the overstepping model, the object surpasses that content.

Scholarly accounts of Kant’s theory of construction also appear to differ on this issue. Robert Butts seems to subscribe to the equivalence model:

Once again we remark the recurring Kantian theme: we can know *a priori* things (in this case, can define things) only that which we put into them. Mathematics *per se* involves the free creation of constructed concepts; except for the required appeal to non-contradiction, there are no constraints on such constructions or inventions. And once again we must also observe that in requiring these inventions to present their objects in *a priori* intuitions, without “borrowing” (*borgen*) anything from experience, nothing more is involved than revealing the content of rules that we ourselves introduce as bearers of the *meanings* of mathematical concepts. (1981, p. 275)

Angela Breitenbach, on the other hand, appears closer to what we call the overstepping model:

According to Kant it is through *a priori* intuitive representation that we gain insight into the properties of mathematical objects that could not be analytically inferred from their concepts. This process begins with the construction of an object according to a concept and ends with the cognitive judgment that the object has such-and-such properties. It is in between these two acts, I suggest, that the imagination makes a contribution that is free from conceptual determination. (2013, p. 966)

It is not easy to reconcile these two views. Following Butts, mathematical objects reveal nothing more than the content we introduce via arbitrarily assumed concepts. For Breitenbach, imagination also plays a crucial part, adding to the object properties not included in the “conceptual determination.” Both Butts and Breitenbach appear to be partially correct in their assessments of Kant’s views. Butts describes the equivalence model and Breitenbach argues for the overstepping model. Their views diverge, I submit, because it was Kant himself who presented the idea of construction in seemingly conflicting ways.

In the following two sections, I summarize the reasons that led Kant to adopt the equivalence model and overstepping model. The third section examines the conflict between the two models. In the fourth section, I examine possible ways of defending Kant from the charge of inconsistency. I argue that the preferable solution is to claim that the two models provide two different accounts of the same epistemic process, but from two different standpoints. According to this defence, the alleged conflict is more an ambiguity than a substantial contradiction in Kant’s thought.

2 The Equivalence Model

I will first examine the equivalence model. Kant employs the equivalence model of construction when debating the possibility of definitions in philosophy. Kant first discussed this issue in the *Inquiry* (1764) and later developed it in the Doctrine of Method (“On Definitions”, KrV, A727/B755f.) of the first *Critique*. In both texts Kant argues that philosophy cannot use definitions in the proper sense of the term, as any alleged definition employed in philosophy inevitably lacks certainty, i.e., the philosopher cannot be sure whether her definitions fully grasp the thing she wants to define. Mathematical definitions, on the other hand, do have such certainty, and the main reason for that is that they lead to a construction of the corresponding object.

The detail of the argument runs as follows. First, let us consider the role of definitions in mathematics. A mathematician starts by assuming a definition, say “a figure consisting of points equidistant from a point C.” It is through this definition that the concept of a circle is first made. In this sense, mathematical definitions are arbitrary, as they arbitrarily make corresponding concepts by joining two or more characteristics. Alternatively, mathematical definitions are “declarations”,

(KrV, A729/B757.) as they declare into existence a concept that is being thought by them, rather than simply describe it. But this is not the only sense in which mathematical definitions are productive. In accordance with the definition, the mathematician constructs a corresponding figure, which is in our example a circle imagined in pure space or drawn (as a proxy) in empirical intuition. The definition is “genetic” in relation to the corresponding object (see Reflection 3001, AA XVI, p. 609). It serves not only as an account of a thing that is being defined, but also as an instruction of how to produce it.

Here, we should be careful to get Kant’s account right. On the one hand, mathematical definition is genetic in relation to the object (a figure in geometry, a number in arithmetic, and a symbolic sequence in algebra). In this respect it appears that the object is dependent on the definition. But we should keep in mind that the converse dependence holds as well. Since mathematical definition is a rule telling us how to produce a corresponding object, its sole meaning is that object itself. Without construction, there is nothing thought in mathematical definition. This is stressed by Michael Friedman:

The activity of geometrical construction is therefore necessary for even the thought of [geometrical] ideas ... Schemata of geometrical concepts, in other words, not only serve to contribute towards the objective reality of such concepts, but are also essential to our rigorous representation of the concepts themselves. (Friedman 1998, p. 123; cf. Petitot 2009, p. 95; Callanan 2014, p. 587)

This is in keeping with Kant’s famous statement that “we cannot think of a line without drawing it in thought, we cannot think of a circle without describing it” (B 154). In a letter to Reinhold (May 19th 1789) we find another affirmation of this view:

Here as always in mathematics, the definition is at the same time the construction of the concept ... If a circle is defined as a curve all of whose points are equidistant from a center, is not this concept given in intuition? (AA XI, p. 41)

In a nutshell, assuming a mathematical definition, understanding the corresponding concept that was made by that definition, and constructing an instance of that concept are for Kant strictly coextensive.

Crucially for Kant’s view on definitions in the “Doctrine of Method”, this coextension gives mathematical definitions an epistemic edge. The mathematician knows that the definition she uses is trivially adequate to what she wants to define, for it is only through this definition that the object comes into being. Since the mathematician creates a thing following a definition of the concept of that thing, she knows for sure that the thing she creates cannot contain more than the concept declared by that definition.

The philosopher, on the other hand, has no such certainty in *her* use of definitions. Again, her epistemic lack is twofold. First, the concepts she is dealing with are

“given”, meaning she did not make them via an arbitrarily assumed definition. And second, she cannot generate (*a priori* in pure intuition or otherwise) an instance of the concept given to her. For example, take the concept of time that Kant evokes already in *Inquiry* (AA II, p. 284). Aristotle, Augustine, and other philosophers thinking about time did not make that concept with the help of an arbitrarily assumed definition. Rather, the concept of time was given to them, meaning it was formed through *a priori* cognition, experience, and language. So, any definition of time they proposed, e.g., a measure of change in respect to before and later, was merely an attempt to clarify that concept, rather than a declaration that would produce the concept of time. Similarly, the definition of time the philosopher arrives at has no genetic power comparable to that of the mathematician. The definition “measure of movement” cannot guide her in the construction of a corresponding intuition. That is why the philosopher cannot be sure her definition fully grasps the concept of the thing she tries to define, e.g., time. For there is always a possibility, warns Kant, that her definition has missed a key conceptual component of what time is. Aristotle and Augustine would have no trouble defining time with certainty if have constructed an instance of time in accordance with their definition (*per impossibile*), for in that case they would know for sure that their definition would (trivially) encompass the thing that is being defined; and since they were not able to do this, the best they could offer were partial “expositions” of time (see also Beck 1956, p. 187).

This argument is summed up in the following passage from the section “On Definitions” of the Doctrine of Method:

thus only mathematics has definitions. For the object that it thinks it also exhibits *a priori* in intuition, and this can surely contain neither more nor less than the concept, since through the explanation of the concept the object is originally given, i.e., without the explanation being derived from anywhere else. (KrV, A729/B757f.)

For since the concept is first given through the definition, it contains just that which definition would think through it. (KrV, A731/B759)

A similar conclusion was reached already in the *Inquiry*:

mathematics arrives at its concepts synthetically; it can say with certainty that what it did not intent to represent in the object by means of the definition is not contained in that object. (AA II, p. 291)

Lectures on Logic offer a similar insight:

None of the marks of a thing can escape or slip away from the mathematician, however, nothing can evade his sharp eye. For he has himself arbitrarily assumed them, if not invented them, and has arbitrarily ascribed them to it; consequently he must of course be perfectly conscious of them all. (AA XXIV, p. 153)

The crucial thing for us is that Kant's argument seems to entail a particular account of construction. It appears that for the argument from the On Definitions segment to work, the intuition we get with construction must involve nothing more than what is contained in the concept and definition thereof. It is only in this way that the mathematician can be certain the definition completely accounts for the corresponding intuition. Accordingly, the properties (or marks, *Merkmale*, in Kant's jargon) pertaining to the object are trivially equivalent to the ones intended in the conceptual rule used for its construction. Kant understands construction here in terms of the equivalence model.

Before we move to the next section, I will briefly comment on the background of the argument Kant uses here, i.e., of the idea that we can fully define only what we are able to create. Interestingly, Kant sometimes says this argument applies not only to mathematics. In a Reflection, he extends it to the type of technical knowledge conducive to empirical products: "One can define (explicate) empirical concepts when one can make the object. For example: plaster, cinnabar, sulphur. The complete exposition of a thing" (Reflection 2964; AA XVI, p. 588; note, however, that here Kant limits definitions to mere "explications"). Apparently, a similar argument was current among German intellectuals of the 18th century. This is how Lewis White Beck describes one contemporary instance of this reasoning:

But if I cannot make something blue, I do not know what blue really is. Conceivability, definability, constructability, and communicability all imply one another. In a laughable example, Tschirnhaus says that anyone who can define laughter will know how to make people laugh. (1969, p. 192)

Giorgio Tonelli (1959) has shown that the argument was well-known in contemporary theories of mathematical knowledge. More surprisingly, this style of argument seems to have been anticipated by Giovanni Batista Vico. At one point in his philosophical development, Vico held that "in mathematical thinking the finite human mind creates a quasi-world of entities and truths analogous to the real world of nature created by the infinite mind of God. Thus, only in mathematical knowledge can man attain maximum clarity, exactness, and certainty." (Morrison 1978, p. 585). According to Vico, a mind can know as true (*verum*) only what is made (*factum*) by that very same mind; and mathematics is the principal locus of such knowledge. An even earlier instance of this reasoning can be found in Lactantius: *solus potest scire qui fecit*, "Only he can know who makes (the thing)", wrote the Church father (Gilson 1940, p. 32). Kant's view that we can define with certainty only what we can create is then part of what Jaakko Hintikka has called the "tradition of genuine knowledge as maker's knowledge" (1972, p. 274). In his employment of the equivalence model, Kant appears to draw from a rich intellectual tradition linking knowledge to intellectual creation.

3 The Overstepping Model

We move on the overstepping model. The fundamental understanding of construction is here the same as in the equivalence model. Kant again understands construction as an exhibition of a concept in a corresponding intuition. There is one major difference though. The equivalence model claimed the constructed object involves nothing more than what is intended in the concept. The overstepping model, on the other hand, claims the intuition conveys more information than the concept. According to this model, to construct means to go beyond the concept. The main thrust of this way of thinking about mathematical construction is nicely expressed by the question posed by Emily Carson: “So the question now is this: what is added in the geometrical construction in pure intuition that is not and could not be contained in the concept?” (Carson 1997, p. 501)

As the first example of the overstepping model of construction, we will consider the “Postulates of empirical thinking”. Famously, the first postulate claims that “whatever agrees with the formal conditions of experience (in accordance with intuition and concepts) is possible” (KrV, A218/B265). If we want to establish that an object is possible, it is not enough to check whether its concept is non-contradictory. The object must agree with the conditions of experience as well. Kant illustrates this point with an example drawn from geometry:

Thus in the concept of a figure that is enclosed between two straight lines there is no contradiction, for the concepts of two straight lines and their intersection contains no negation of a figure; rather the impossibility rests not on the concept in itself, but on its construction in space, i.e., on the conditions of space and its determination ... (KrV, A220/B268f.)

As a concept, straight two-sided polygon (also digon or biangle) involves no contradiction. But when we try to construct it in intuition (e.g., imagine it or draw it on a surface), we discover it is impossible. Kant’s explanation is that the two-sided polygon fails to agree with formal conditions of experience, that is, with (Euclidian) space. Hence, the example helps us see that conceptual non-contradiction is not enough to establish real possibility of an object; we must also check whether the object agrees with conditions of experience.

We will return to the details of this argument in the final section of the paper. For now, what interests us for is only the process of construction implied in this argument. First, the mathematician (arbitrarily) assumes the concept of a figure enclosed between two straight lines. Then she attempts to construct it in (Euclidian) space, only to discover that what she gets are two superimposed straight lines, rather than a figure. By attempting to construct a figure, she learns new information not involved in its concept, i.e., that the figure is impossible. The concept of the two-sided polygon gave her no indication that its object is impossible; she discovers that only when she attempts to construct it in intuition. She must overstep the concept to find out

whether it can refer to anything at all. Accordingly, we are dealing here with the overstepping model of construction.

Moving on to the next instance of the overstepping model, we now return to the Doctrine of Method. We will consider the starting section of the first chapter of “The discipline of pure reason in dogmatic use” (KrV, A712/B740f). Here Kant argues that the philosopher, contrary to the mathematician, cannot make her concepts intuitive. Kant begins by noting that the philosopher and the mathematician approach the same problem, namely, finding the sum of all angles in a triangle, in fundamentally different ways. The philosopher, being limited to conceptual analysis, merely reflects on “the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles.”

Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyse and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts. (KrV, A716/B744.)

Concept analysis fails to bring the philosopher any closer to finding the sum of all angles of a triangle. The mathematician, on the other hand, immediately proceeds to construct the proof:

Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. [Kant goes on to recap the proof from the Book 1 of Euclid’s *Elements*]. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question. (KrV, A716/B744f.)

According to Hintikka (1967, p. 368), Kant’s key point here is that the mathematician employs a constructive proof—*ekthesis* or “setting forth” in Euclid’s terms—which in this case amounts to extending a side and drawing a parallel to another side. It is in this sense that geometry is constructive, and not in the (less plausible) sense of imagining the object in mind’s eye. What is crucial for us is that constructing the object (or the proof) in intuition allows the mathematician to demonstrate the sum of all angles, while the philosopher is stuck analysing the concept, trying in vain to get any further than the initial definition of triangle. And it is on this point, I submit, that the overstepping model comes to play. Kant continues:

What might be the cause of the very different situations in which these two reasoners find themselves, one of whom makes his way in accordance with concepts, the other in accordance with intuitions that he exhibits *a priori* for the concepts? ... At issue here are not analytic propositions, which can be generated through mere analysis of concepts (here the philosopher would without doubt have the advantage over his rival), but synthetic ones, and indeed ones that are to be cognized *a priori*. For I am not to see what I actually think in my concept of a triangle (this is nothing further than its mere definition), rather I am to go beyond it to properties that do not lie in this concept but still belong to it. (KrV, A717/B745f.)

To find the sum of all angles, I must go beyond the “mere definition” of triangle, and I do that by operating a constructive proof on a figure drawn in pure or empirical intuition (and this is just what the philosopher fails to achieve, being limited to mere conceptual analysis). We find here the same fundamental idea as in the regular two-sided polygon example. Analysing the concept of the figure failed to establish its real impossibility which became apparent only when we tried to construct it in intuition; similarly, reflecting on the concept of triangle fails to bring us any closer to the sum of its angles, so we must use constructive proofs performed in intuition. But this entails a specific understanding of construction. Namely, constructing a figure reveals properties that were not contained in the concept of that figure. Hence, to construct means to go beyond, or to overstep, the concept.

4 The Apparent Conflict Between the Two Models

The overstepping model of construction, I submit, is *prima facie* incompatible with the equivalence model that Kant uses in the discussion of definitions in mathematics nad philosophy. Kant argued there that

the object that [mathematics] thinks it also exhibits *a priori* in intuition, and this can surely contain neither more nor less than the concept. (KrV, A729/B757f.)

Following this idea, to construct meant to translate a concept into intuition, both having equivalent content in terms of properties and characteristics. But when using the overstepping model—only few pages away and still in the Doctrine of Method—Kant claims:

For I am not to see what I actually think in my concept of a triangle (this is nothing further than its mere definition), rather I am to go beyond it to properties that do not lie in this concept but still belong to it. (KrV, A718/B746.)

According to this model, to construct means to go beyond the concept. So, in the equivalence model, concept and intuition trivially involve the same information; in the overstepping model, however, intuition conveys more information (real possibility, properties such as the sum of angles, etc.) than the corresponding concept. There is, I claim, a *prima facie* conflict between the two models. The figure either involves nothing more than its concept, or some of its properties are reachable only via construction. It is unclear how both can be true.

To better appreciate the clash between the two models, I will now shuffle the examples used in each model. Say we want to establish (like the *Inquiry* and the On Definitions segment of the Doctrine of Method) that only mathematics can use

definitions with certainty. Kant's argument for this was that mathematician can create the object she wants to define, and that this ability makes her definitions certain. In the *Inquiry*, Kant suggested we take the definition of a circle or a triangle (AA II, p. 283), proceeding to construct it in intuition. Using this innocuous example, he had little problems making his point that mathematical definitions are coextensive to construction in intuition, since the definition of circle serve as an instruction of how to construct a circle. But instead of that, let us now use an example from the overstepping model. Specifically, let us replace the definition of circle with the definition of a straight two-sided polygon (alternatively, we could take the regular decahedron example Kant mentions in his review of the Kästner treatises; see AA XX, p. 414). Now, it appears that the mathematician cannot take the definition and summon a corresponding object into existence, for it is impossible to construct a straight two-sided polygon (at least according to Kant's understanding of mathematics). It is difficult then to claim that the definitions of the two-sided polygon or regular decahedron are "genetic". They may be arbitrary in the sense that they make the concept by arbitrarily putting together two or more characteristics; but they are neither genetic nor constructive, since they fail to produce a corresponding intuition. But Kant's argument from the subsection "On Definitions" relied precisely on the assumption that the mathematician creates the object she defines at will. The often-quoted *Reflexion* 2930 states: "Der Mathematicus in seiner Denition sagt: *sic volo, sic iubeo*" (AA XVI, p. 579) ("The mathematician says in his definition: I want it so, and so I ordain.") The mathematician orders things into existence (or intuition), according to this line of thought. "Because anyone who grasps a mathematical concept knows its definition, and because the definition allows one to construct instances of it, it follows that one cannot possess a mathematical concept and still doubt whether it has any instances" (Heis 2014, p. 609). Such certainty is just what the mathematician from the digon and regular decahedron examples lacks, however. The constraints imposed by the formal conditions of experience prevents her from producing the object of her definition. And this means she fares no better than the metaphysician (e.g., Leibniz) who arbitrarily assumes the definition of a simple substance without being able to create instance of it (AA II, p. 277). She fails to create the object she wants to define as well; hence, she no longer has grounds for thinking her definitions have an epistemic edge of the ones used by the philosopher.

The idea Kant is arguing for in the equivalence model (i.e., that the mathematician can use definitions because she creates the objects of her study at will) collapses when we try to back it up with the examples Kant evokes in the overstepping model (i.e., regular two-sided polygon, regular decahedron). The notion that the mathematician exercises the "creator's knowledge" (Hintikka 1972, p. 274) does not sit well with definitions whose objects we cannot construct. On the contrary, the regular decahedron example and the digon example seem to show the limits of

construction, and hence of the mathematician's power to use definitions with certainty.

We get the same results if we turn to other examples from the overstepping model, such as the task of finding the sum of all angles (KrV, A716/B744). According to Kant, we discover the sum of interior angles of triangle by using constructive proofs (*ekthesis*), meaning we must go beyond the concept of triangle to "properties that do not lie in this concept but still belong to it" (KrV, A718/B746). I take this to mean that constructive proofs operated on a figure in intuition reveal properties that were not included in the concept of a triangle as established by its initial definition. Taken as an example of a mathematical procedure, this operation (trivially) fails to back up the statement Kant makes in the equivalence model, namely that the constructed figure "can surely contain neither more nor less than the concept" (KrV, A729/B757). The fact what we need to use constructive proofs to find the sum appears to outright refute the equivalence model. Turning to yet another example used in the overstepping model, a letter to A. W. Rehberg emphatically states that "from the mere concept of a number, we cannot tell whether the root of that number will be rational or irrational. We have to *try it out*" (AA XI, p. 209)—meaning the property of »being irrational« is not discernible via conceptual analysis of a number (say $\sqrt{2}$). But surely, if a mathematical object such as a number involved nothing that was not already included in its concept, we should be able to tell the fact that $\sqrt{2}$ is irrational from the concept of $\sqrt{2}$.

The two models of construction appear to clash. This is shown by the fact that the examples used in one model undermine the goals Kant tried to achieve when using another model of construction.

We can think about the conflict in even simpler terms. The two models rely on two plausible, but not entirely compatible ideas. In the equivalence model, Kant evokes the idea that, among the objects given in intuition, the mind can truly know only the objects created by the mind itself. *Solus potest scire qui fecit*, to recall Lacantius' maxim. The overstepping model, on the other hand, relies on the idea that creating an object helps the mind see that object better. A quip attributed to Graham Wallace illustrates this idea nicely: "How can I know what I think till I hear what I say?" Hence, the mind must make its mental objects sensible to know them better. The two ideas agree that the power to create the object of our knowledge is somehow linked to a superior form of cognition, but they diverge on the reasons why that is so. The first idea claims that we know a created object better just because it is created by us (unlike other sensible objects that are simply given). The second idea claims that we know a created object better because it is present in intuition (and is no longer just an mental object). Their emphasis is different. And their difference comes forth when we consider the following question. Does the object contain more once we create it intuition than what was thought in its initial definition? According to the first idea, the answer is no, for the object is coextensive with its concept/definition.

According to the second idea, the answer is yes, for creating the object in intuition means overstepping the initial definition. The two ideas appear to clash, and so do the two models of construction in Kant.

5 Saving Construction

Can we salvage Kant's idea of mathematical construction? In the concluding section I will suggest a reading that can help us reconcile the two models of construction. In the end, I will suggest that the appearance of a conflict between the two models stems from Kant's ambiguous use of the term "definition". But before I proceed to present this reading, I will sketch two less optimistic approaches to the apparent contradiction between the two models.

First, recall that Kant employed the idea of construction in various contexts not concerned primarily with the nature of mathematical knowledge. He invoked the impossibility of a straight digon to highlight the difference between conceptual non-contradiction and real possibility; he used geometry to illustrate formal purposiveness; he saw in the *a priori* nature of geometrical construction an argument for the sensible nature of space; finally, Kant used the idea of construction to argue that definitions can be used only in mathematics. This is a diverse set of philosophical issues. They are concerned primarily with more general ontological and epistemological questions ("Does logical non-contradiction entail real possibility?" "Does space pertain to sensibility?"); they deal with mathematical knowledge only incidentally. So, one could claim that here construction plays the role of a philosophical device. It is not the theme of Kant's thinking, but rather a tool he employs to think through issues that are important to him. It is little wonder, then, if Kant has failed to pay enough attention to precise workings of mathematical construction. Even worse, it might have happened that he employed two different models of construction, depending on the philosophical argument he was making at the time. So, when discussing the difference between conceptual non-contradiction and real possibility, he used the overstepping model as it shows nicely the point that by analysing the concept of a figure without making it sensible, we cannot discern the real possibility of that figure. And likewise, Kant employed the equivalence model when he wanted to argue that the mathematicians, in contrast to the philosopher, uses definitions without the risk of missing any relevant characteristic of the *definiendum*. One might conclude, therefore, that Kant tailored the idea of construction so that it suited the argument he was trying to make. According to this reading, Kant had no intrinsic interest in mathematical construction but had rather shaped it in accordance with his current philosophical preoccupations. While this reading is uncharitable to the extreme, it shows that there is a need for a reading that would close the gap between

the two models. If we fail to reconcile the two models, it appears that mathematical construction was for Kant indeed no more than a philosophical device.

There is another, only slightly less pessimist approach to Kant's treatment of mathematical construction. This approach suggests that one of the two models of construction does not sit well with Kant's critical philosophy and that we should therefore reject it while keeping the other one as the preferable (Kantian) account of mathematical construction. A paper by Gerold Prauss (1994) on the straight digon problem can be read as implying just that. While Prauss does not distinguish between the two models, he appears to discover an important conceptual flaw in what we called the overstepping model of construction. Concentrating on the "The Postulates of Empirical Thinking in General" (see especially KrV, A220/B267f.) Prauss claims that there is major inconsistency in Kant's treatment of the straight digon. As seen in Section 2 of this paper, Kant suggests that it is in vain that we look for a contradiction in the concept of the straight digon, as there is no logical repugnance between its characteristics or *Merkmale*. More specifically, there is no contradiction between a figure having two straight sides and having a positive area. The fact that the figure having both characteristics is incompatible with the general conditions of sensibility, and hence impossible, is established only when we try to construct it in intuition. Hence, conceptual non-contradiction does not amount to real possibility. Once again, this point is even better made by the regular decahedron example Kant evokes in the Kästner review (AA XX, p. 414): there is surely no conceptual contradiction between a solid having ten faces and the same solid being regular. And yet the regular decahedron proves to be impossible when we try to construct it in intuition. To recall, we enlisted this argument to the overstepping model as the intuition grants us knowledge that was impossible to infer from concept alone.

Now, Prauss suggests this argument has a disastrous consequence. For when Kant states that we cannot find a contradiction in the mere concept of straight digon, and that we may discover its impossibility only by attempting to draw it in intuition, he is implying that a geometrical concept can have meaning independently of intuition (for otherwise we could not have analysed it before attempting to draw it).

For in saying this he tacitly assumes something that is, not only for his critical philosophy, simply disastrous, namely, that a concept like that of a straight line can be formed without intuition and therefore possess definiteness as meaning or sense without intuition. It is precisely this assumption that Kant is making when he says in that passage that a straight biangle can, despite its synthetic impossibility, be possible analytically—as though a concept such as that of a straight biangle can possess its meaning even *without* such an intuition as corresponds to the subconcepts "biangle" and "straight". (Prauss 1994, 229).

If Prauss is right, then that what we have called the overstepping model rests on a clearly erroneous or at any rate un-Kantian supposition, i.e., that a mathematical

concept can have meaning independently of any reference to intuition. This critical suggestion would apply not only to the straight digon and the regular decahedron examples. One could extend it to the passage from the “Doctrine of Method” in which Kant suggests that the philosopher, being limited to a mere analysis of the concept of triangle, searches in vain for the sum of inner angles (KrV, A718/B746f.). Again, this argument appears to suppose that triangle has meaning independently of construction, for otherwise there would be no concept of triangle the philosopher would be able to analyse. Other instances of the overstepping model are exposed to the same objection, since they all rely on the idea that one could compare the concept of the figure and the figure itself as two separate bearers of meaning (or characteristics, *Merkmale*). This supposition is highly suspicious in view of Kant’s insistence on intuitive and synthetic nature of mathematical knowledge. The equivalence model, on the other hand, has no such problem. According to it, having a mathematical concept, knowing its definition, and being able to construct an instance thereof are always coextensive (see especially Heis 2014, p. 608). Following this critique, then, the equivalence model should be chosen as the preferable Kantian account of mathematical construction.²

The two readings suggested above both accept that there is something fundamentally wrong with Kant’s account of mathematical construction. The first one proposes that Kant has neglected to pay enough attention to the notion of construction and have rather tailored it to fit various arguments not necessarily concerned with mathematics per se; hence the conflict between the two models. The second reading takes the cue from Prauss and claims that the overstepping model is based on a mistaken, or at least un-Kantian, supposition and that we should therefore

2 This reading would still have to reply to the challenge (mentioned in Section 4 of the paper) posed by the straight digon, regular decahedron, and other figures that are impossible to construct. Namely, if (1) having a mathematical concept, (2) having a definition of that concept, and (3) being able to produce an entity falling under that concept are coextensive, then it is not clear how we can account for geometrical concepts without a corresponding entity in intuition. For we have (1) a concept of a regular decahedron and are (2) able to define it as a regular solid with ten faces; yet we are (3) unable to produce an instance of it. It appears once again that the equivalence model cannot account for such entities. To this challenge it could be answered that the alleged definition of regular decahedron is not a mathematical definition in the Kantian sense of the word as it is not genetic. According to Kant, mathematical definitions are just those arbitrarily assumed definitions that do suffice to generate a corresponding object in pure intuition. We can indeed make a concept of regular decahedron by arbitrarily defining it as a regular solid with ten faces; but using this definition we *cannot* generate a corresponding object and so it does not amount to a mathematical definition. Kant developed a similar position vis-à-vis the problem of Euclidian parallels: “A subject could fully grasp the marks: *straight line*, *coplanar* and *intersecting*, know full well how to construct straight lines, planes and intersecting lines, understand fully how these marks are put together to form the concept *parallel lines*, but still not know how to construct them. The definition is thus not genetic. Indeed, since it provides no method for constructing parallels, the definition does not present the possibility of parallels and so is not even a real definition (Heis 2014, p. 613).”

accept the equivalence model as the preferable Kantian account of mathematical construction. But there is one less pessimist and more charitable approach available. According to this approach, the alleged conflict between the two models does not result from any substantial contradiction in his thought, but rather from an ambiguity in Kant's terminology. To defend Kant from the charge of inconsistency, we should therefore modify our reading of the key passages dealing with mathematical construction, namely those that pertain to the overstepping model.

Consider once again two key statements from the "Discipline of pure reason in dogmatic use" (belonging to the overstepping model):

For I am not to see what I actually think in my concept of a triangle (this is nothing further than its mere definition), rather I am to go beyond it to properties that do not lie in this concept but still belong to it ...

In vain, therefore, would I reflect on the triangle philosophically, i.e., discursively, without thereby getting any further than the mere definition with which, however, I had to begin. (KrV, A718/B746f.)

In the Section 3 of the paper, we have taken these statements at their face value. Namely, we have understood Kant as claiming that the mathematician, using the method of construction (e.g., by employing constructive proofs, by exhibiting the figure itself), introduces new content that was not present in the initial definition he has started with. This appeared to contradict his previous statements from the "On Definitions" subsection belonging to the equivalence model. There, Kant put forward the claim that the concept that is being defined, as well as the mathematical object itself, must involve nothing more than what is thought in the definition, for it is just by using the definition that we have first made the concept and generated the figure. At face value, then, the two treatments of construction appear to clash.

But one could read the statements quoted above in a more nuanced way. The fact that Kant here qualifies definition as "mere" ("bloße Definition") should give us pause. Recall that for Kant, mathematical definitions are productive twice. They make a concept they define and generate a particular instance of that concept. They are arbitrary and genetic. Kant describes the mathematician as ordaining things into existence: "The mathematician says in his definition: I want it so, and so I ordain", to recall the oft-quoted Reflection 2930 (AA XVI, p. 579). In *Lectures on Logic*, Kant insists on the epistemic edge of mathematical definitions: "Synthetic definitions, insofar as they have a fabricated object, can never err, because I say, in accordance with my faculty of choice, that I want to think this" (AA XXIV, p. 915).

In this light, it is peculiar that the "Doctrine of Method" describes the mathematician as "going beyond a concept which is a *mere* definition" (KrV, A718/B746). The way Kant describes mathematical definitions elsewhere in his work gives us no

ground for thinking that there should something epistemically deficient about them. Hence, the adjective “mere” seems out of place. But perhaps the qualifier “mere”—which appears twice in the key passage quoted above—should be read as signalling that Kant is not using the term “definition” in the strict sense of the term set up in the subsection “On Definitions”. Perhaps he is using it in a looser sense. Namely, Kant could be read as describing mathematical construction as seen from the perspective of a philosophical method. What do we mean by that?

Recall that, in the discussion preceding the passage quoted above, Kant has just given an account of the way a philosopher handles the task of finding the sum of inner angles. She approaches the problem discursively, i.e., by analysing the concept of triangle. And she fails to find the solution, precisely because the task should be approached with the help of constructive proofs (KrV, A716/B744). If we now compare her procedure with that of a mathematician, one difference comes to front. The mathematician proceeds intuitively, using constructive proofs or Euclidian *ekthesis*. While the philosopher stays limited to conceptual analysis, the mathematician appears to do much more: she draws a triangle, extends one of its sides, and performs other constructive steps needed for determining the sum of inner angles (KrV, A716/B744). And it is, I argue, precisely in comparison with the philosopher that the mathematician can be said to “go beyond the concept which is a mere definition”. From the perspective of the philosopher, the mathematician oversteps what appears like a mere definition of the triangle and proceeds to create an intuitive, visible, concrete instance of that figure. *But this description of the mathematician’s procedure makes sense only when compared to that of a philosopher. In her own terms, the mathematician does not overstep the definition, for the definition she uses is coextensive with constructing a figure.* Mathematical definition is genetic, meaning it is a rule for generating an instance that falls under the concept that is being defined. Having an intuition of the object is not extraneous to having a mathematical definition, but an integral part of it. If we take it as such, it makes then no sense to see in definition something the mathematician oversteps by reaching into intuition. From the perspective of the mathematician herself, the conceptual and the intuitive form a unity. To define is to construct, and to perceive a mathematical entity is to realize the conceptual rule that served as the ground for its construction.³ After all, the reason

3 Kant offers a rare and interesting account of how we perceive an mathematical entity constructed by another agent in the *Critique of the Power Judgment*: »If someone were to perceive a geometrical figure, for instance a regular hexagon, drawn in the sand in an apparently uninhabited land, his reflection, working with a concept of it, would become aware of the unity of the principle of its generation by means of reason, even if only obscurely, and thus, in accordance with this, would not be able to judge as a ground of the possibility of such a shape the sand, the nearby sea, the wind, the footprints of any known animals, or any other nonrational cause, because the contingency of coinciding with such a concept, which is possible only in reason« (AA 5, p. 370).

why Kantian idea of construction left such an impression on the young Schelling is because he saw in it a speculative unity between the two stems of Kant's epistemology, i.e., conceptual cognition and intuition (Schelling 2008).

But again: when compared with philosophical cognition, the mathematician appears as if he was doing something more than the philosopher, and hence something *over and above* mere conceptual inquiry. For the mathematician, mathematical definition simply is construction; for the philosopher, construction appears as an extraneous addition to the mathematical concept that somewhat still pertain to it. This is, I suggest, a reading that may help us explain the most puzzling passage in Kant's discussion of mathematical construction in the "Doctrine of Method". Namely, when Kant writes that in mathematical cognition, "I am to go beyond it (i.e., mere definition) to properties that do not lie in this concept but still belong to it" (KrV, A718/B746), he offers an account of mathematical procedure from the perspective of philosophical method. From that standpoint, the mathematician goes beyond the mere definition to properties that do not lie in it but in way still pertain to the corresponding concept. From the standpoint of mathematics itself, however, generating an object is coextensive with having a genetic definition and understanding a concept, rather than a process that supervenes on having a definition.

If this reading is correct, the apparent clash between the two models becomes less pressing. In the overstepping model, Kant describes mathematical cognition as it appears when compared to a discursive type of knowledge, namely, philosophical analysis. In the equivalence model, mathematical construction is described in its own terms, i.e., as a cognition in which assuming a definition, making a concept, and generating an object are strictly coextensive. Rather than a substantial clash in Kant's thinking about mathematics, the two models would signal the plurality of standpoints from which Kant approached the problem of mathematical construction.

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