Research Article

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On initial data in adjustments of the geometric levelling networks (on the mean of paired observations)

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Abstract: There are many systematic errors in the precise levelling measurements. The most of them we have studied and their impact on the final accuracy of levelling is solved by appropriate corrections. The main objective of the current article is to reveal the greatest systematic error in the processing of levelling data, i.e., the use of only the averages of the fore and the back measurements of the elevations in levelling lines as initial data in the adjustment of the highest order levelling networks. Regardless of the type of distribution, simulations of random paired samples reveal that the averages of each pair only up to 33% of all cases are more closely located to the known theoretical expectation with respect to their parents. This fact implies that the collected data are not processed in the best way. In order not to lose information, we adjusted a reduced network of the Third Precise Levelling of Finland network in all possible combinations by the use of the fore, the back, and the mean of each line elevation. As a result, the final accuracy increases more than 10 times in comparison to an adjustment with the use of the averages only.

Keywords: accuracy, levelling network adjustment, simulations

1 Introduction

In many scientific and practical tasks, we measure different physical quantities to understand natural processes or solve some engineering problems. In the interest of increasing the accuracy of results, we usually measure a single quantity more than once. Suppose that we measured n times a single quantity with equal accuracy. As a result, we can obtain a plausible estimator of this quantity such as the average of these measurements. If $X_1, X_2, \ldots, X_n$ are our independent observation results, we can calculate their average $\bar{X}$ by equation (1).

$$\bar{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}. \tag{1}$$

If the standard deviation of a single measurement is σ, we can express the standard deviation $\sigma_{\bar{X}}$ of the average $\bar{X}$ by equation (2) (Dekking et al. 2005).

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}. \tag{2}$$

According to equation (2), when $n \to \infty$, then the standard deviation $\sigma_{\bar{X}}$ of the average $\bar{X}$ tends to be zero. Therefore, when the number of measurements $n \to \infty$, then the value of the average $\bar{X}$ tends to be the true value of the determined quantity X. Let us illustrate this theory with a simple example.

Suppose that we measured n times a levelling loop with a length of 1 km. The standard deviation of each measurement is $\sigma = 1\ \text{mgpu}/\sqrt{\text{km}}$. Because we start and end at the same point, we should have closing errors equal to 0 mgpu. In fact, we obtain closing errors with different values and signs. These values usually vary in the range from -3σ to $+3\sigma$. However, if we increase the number of measurements n, the average of these n measurements tends to the theoretical value of the closing error, which is 0 mgpu. Figure 1 illustrates this process.

According to Figure 1, some stabilization of the averages in respect to the true value of closing error is possible when $n > 30$ observations. That is to say, we need more than 30 independent measurements of a single quantity, if we want to be sure about the plausibility of the average of these n measurements (Dekking et al. 2005, Montgomery and Runger 2014, Cvetkov 2023c). However, the precise levelling is an expensive and a time-consuming activity. Due to this fact, we usually measure each line elevation in precise levelling lines twice in two opposite directions (Kääriäinen 1966, Saaranen et al. 2021). The basic

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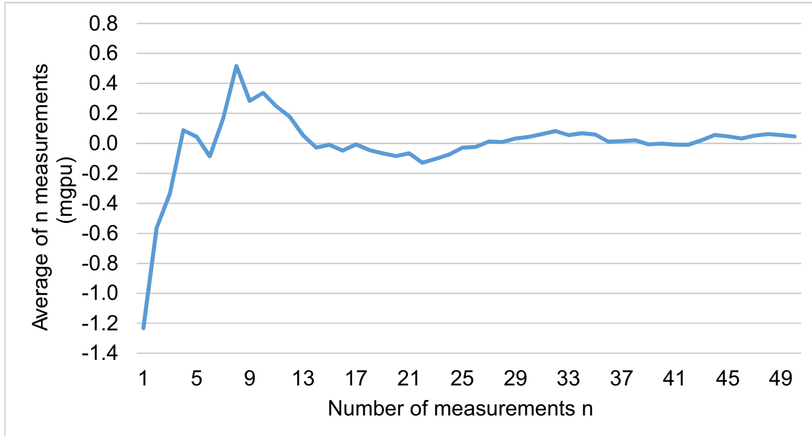

Figure 1: The averages of *n* independent measurements of levelling loop with a length of 1 km. All measurements have standard deviation $\sigma = 1\ \mathrm{mgpu}/\sqrt{\mathrm{km}}$.

idea behind the second measurement of elevations in the precise levelling is to avoid gross errors in observations. In other words, we do the second measurement to control uncertainties in measurements rather than some significant increase in the accuracy of their averages. If the standard deviations of both measurements are σ, then on the basis of equation (2), we can write the standard deviation of their average equation (3).

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{2}} \approx 0.707\sigma. \tag{3}$$

In other words, the standard deviation of an average of two measurements is 1.4 times less than the standard deviation of each measurement. However, the average of only two measurements is quite unstable and may be further from the true value of the measured quantity than either of parent measurements. According to Figure 1, the average of the first and the second closing errors of our example-levelling loop is −0.6 mgpu. This value is far away from the theoretical value of 0 mgpu, if we assume that the standard deviation of our measurements is $\sigma = 1\ \mathrm{mgpu}/\sqrt{\mathrm{km}}$.

Figure 2 shows an additional example about the “remoteness” of the average of two independent observations and their average from a known true value of a quantity.

It is a well-known fact that the closing errors in levelling loops are true errors, whose values are known earlier. The red dots in Figure 2 present the closing errors in mgpu

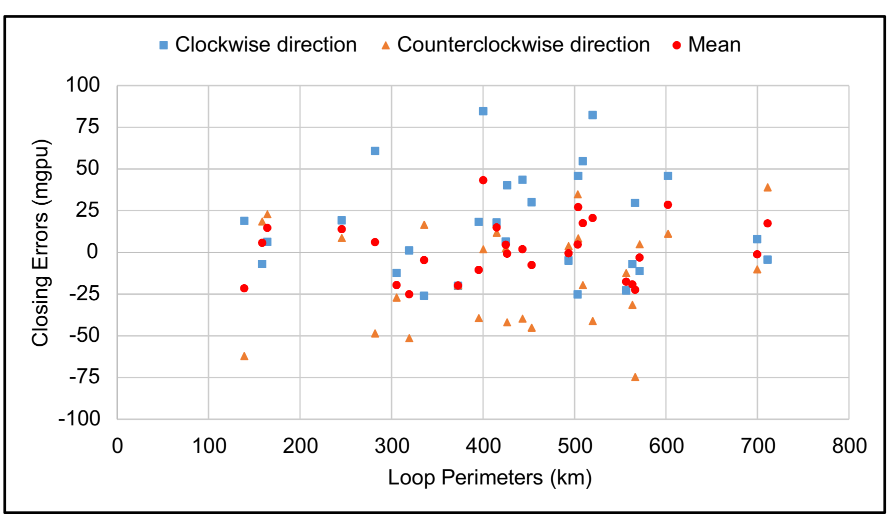

Figure 2: The closing errors in the Third Precise Levelling of Finland in 1978–2006. At the beginning of the Third Levelling, the maximum accepted difference between the back and forth measurements was $1.6\sqrt{L}$ mm. Since the late 1980s, the limit was $2\sqrt{L}$ mm, which is a standard deviation of ± 1.0 mm/$\sqrt{\mathrm{km}}$ (Saaranen et al. 2021, p. 25).

Table 1: Names, notations with parameters, theoretical expectations, and standard deviations of the distributions, which we used in our research

Distribution	Notation with parameters	Expectation	Standard deviation
		usp*	usp*
Normal	N(0, 1)	0.00	1.00
Uniform	U(0, 1)	0.50	0.29
Contaminated normal	CN(0.9, 3)	0.00	Undefined
Laplace	Lp(0, 1)	0.00	1.41
Student's *t*	t(4)	0.00	1.41
Beta	B(3, 2)	0.40	0.20
Snedecor's *F*	F(40, 40)	1.05	0.12
Exponential	E(1)	1.00	1.00

*A unit distribution scale parameter.

of 28 loops in the Third Levelling of Finland network (Saaranen et al. 2021), which we denote by the "mean." The blue square dots and the orange triangle dots are closing errors of the same loops, but calculated in clockwise and counterclockwise directions in correspondence with the heading of the lines in Table 2. We calculated their values based on the information in Appendix C, columns 5–11, as presented in the study by Saaranen et al. (2021). Each "mean" closing error is the average of the "clockwise direction" and the "counterclockwise direction" measurements. Since the "clockwise direction" and the "counterclockwise direction" are closing errors, they are true errors, whose values are known earlier. Thus, we can assess for each loop, and this measurement has the least closing errors among the "clockwise direction," the "counterclockwise direction," or the "mean." It is obvious that in half of the loops, the "mean" closing errors are not the least ones. This fact implies that we do not use the most appropriate data in the adjustment of the precise levelling networks.

All the aforementioned facts provoke the author to search for factual answers to the following questions:

- How often is the average of two random observations from a predefined distribution more closely located to the theoretical expectation in comparison to its parents?
- Do the averages of both measurements of the heights between terminal benchmarks of levelling lines in a precise levelling network minimize the closing errors in this network?
- Could standard errors of the adjusted geopotential numbers of all benchmarks in a national the highest order geometric levelling network to be smaller than 2 mgpu?
- Can we reduce the number of lines in a levelling network without lack of accuracy?

Table 2: Summarized data about levelling lines, their length, differences I and II between geopotential numbers of the start and the end line benchmarks, and the averages of both measurements I and II

Line	Distance (km)	Height differences* I (mgpu)	Height differences* II (mgpu)	Mean (I + II)/2 (mgpu)
Kauklahti-Noormarkku	363.778	35284.13	35231.27	35257.70
Noormarkku-Jyväskylä	257.280	42172.85	42164.91	42168.88
Kauklahti-Vaajakoski	337.837	80998.47	81017.91	81008.19
Jyväskylä-Vaajakoski	4.836	3580.65	3581.17	3580.91
Vaajakoski-Särkisalmi	239.680	−16433.88	−16399.66	−16416.77
Kauklahti-Särkisalmi	397.821	64603.73	64598.05	64600.89
Noormarkku-Ylivieska	490.659	21623.15	21533.81	21578.48
Haapajärvi-Ylivieska	64.685	−40398.99	−40415.51	−40407.25
Jyväskylä-Haapajärvi	201.790	19778.81	19749.75	19764.28
Haapajärvi-Ammänsaari	302.491	94304.50	94322.86	94313.68
Sarkisalmi-Ammänsaari	513.336	126924.65	126920.11	126922.38
Ylivieska-Oulu	129.545	−57206.60	−57204.58	−57205.59
Oulu-Kuusamo	254.097	257863.35	257827.31	257845.33
Ammänsaari-Kuusamo	173.309	65926.36	65930.36	65928.36
Oulu-Muonio	434.562	237884.45	237875.21	237879.83
Kuusamo-Sodankylä	266.298	−91600.12	−91656.04	−91628.08
Sodankylä-Inari	210.199	−54031.64	−54184.14	−54107.89
Muonio-Inari	255.329	−125752.04	−125707.40	−125729.72

*The values of the I and the II measurements we calculated based on the information given in Appendix C, columns 5–11 in the study by Saaranen et al. (2021). All height differences contain rod metre, refraction, and temporal tidal and land uplift corrections.

2 Simulations and a real data test

In this section, we briefly describe the simulations that we executed to investigate how often the averages of two random observations, derived from predefined distributions with known parameters, are more closely located to the known expectation of the distributions in comparison to their parents. Assuming that the true error of either of the parent observations is less than the true error of their mean, we adjusted the part of the Third Precise Levelling of Finland in all possible combinations. In these adjustments, we used not only the average values of the measured height differences between terminal benchmarks in the levelling lines but also the height differences from both measurements in opposite directions.

2.1 Simulations

To investigate how often the true error of an average of two random observations is less than the true errors of its parents, we generated two random samples of size 10,000 from a known distribution. Let us for clarity name these samples as "I" and "II." Taking those observations from samples "I" and "II," which have the same order, we arranged 10,000 random pairs. We also calculated the average of each pair. Thus, we formed 10,000 new random samples. Each new sample includes three values, those of two randomly generated numbers from samples "I" and "II" and their mean. Since we had known the theoretical expectation of the distribution, it was easy to count how often each observation in any pair or their mean was near the expectation. For example, suppose that we have two random numbers from the standard normal distribution where the expectation is equal to 0 and the standard deviation is equal to 1, i.e., N(0, 1). Let their values be 2.1 and 0.5, respectively. The mean of these values is the value 1.3 = (2.1 + 0.5)/2. As a result, we have a sample of size 3, i.e., 2.1, 0.5, and 1.3. Because $|0.5 - 0| < |1.3 - 0| < |2.1 - 0|$, we can conclude that the true error of the random number with value 0.5 is less than the true error of the other simulated random number, namely, 2.1, and less than their average equal to 1.3. Thus, sample "II" scores a point against the sample of averages, which we named "mean," the sample "I." To investigate the impact of the type of the applied distributions on the analysed frequencies, we repeated the aforementioned simulation by applying nine distributions with different scale and shape parameters. Table 1 presents the names, notations with parameters, theoretical expectations, and standard deviations of these distributions (Montgomery and Runger 2014).

2.2 Real data

To investigate whether 3^n independent adjustments of a geometric levelling, based on the results from both measurements of line elevations and their means, will produce better results than the classic approach, we used a part of the data from the study by Saaranen et al. (2021). Table 2 presents summarized data about levelling lines, their length, differences I and II between geopotential numbers of the start and the end line benchmarks, and the averages of both measurements I and II. The symbol I means the levelling in the direction of the heading of lines and the symbol II means the opposite direction.

Figure 3 illustrates the scheme of the analysed network, part of the Third levelling of Finland network. We adjusted this network in four variants. Table 3 presents the initial heights, which we used in each adjustment variant. The datum level in our adjustments was the geopotential number of the benchmark 2,183 in Kauklahti. To reproduce results more closely to the official ones, we performed all adjustments with weights inversely proportional to the length of the levelling lines.

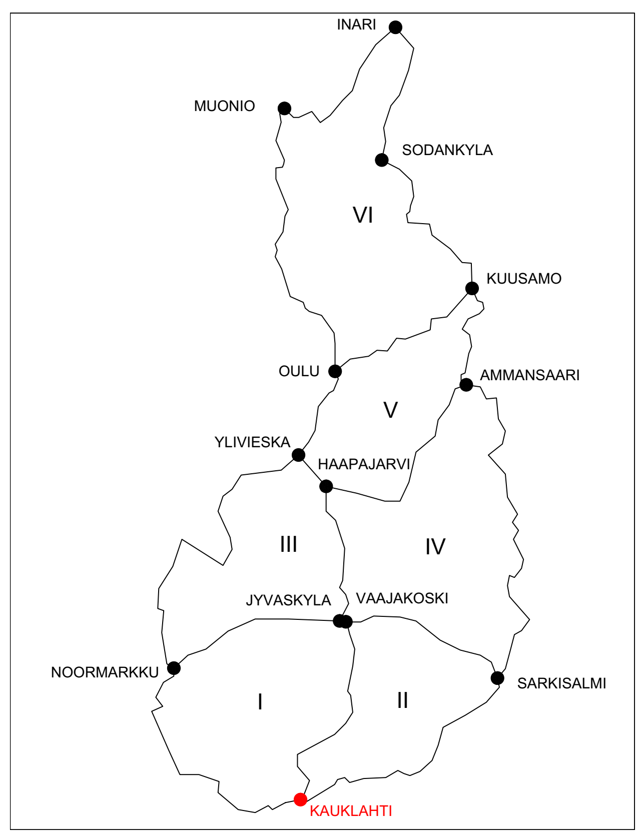

Figure 3: Scheme of the investigated network, part of the Third levelling of Finland network.

Therefore, the initial heights in the adjustment in Variant 1 are the mean values of both measurements of the line elevations. The initial heights in the adjustment in Variant 4 are those values from both measurements of the line elevations or their means, which we selected by 3^{18} = 387,420,489 independent adjustments of the analysed network. The selection criterion was the minimum value of a posteriori standard deviation.

3 Results

In this section, we present the results produced by the simulations, which we described in Section 2.1. We also present the results from adjustments of the precise levelling network shown in Figure 3, performed with the initial data of Variants 1–4.

3.1 Simulation results

Figure 4 presents the frequencies of occurrence of the "I," the "II," and their "mean" nearby the theoretical expectation, based on random values from known distribution. According to Figure 3, the averages of two observations rarely are more closely to the expectation in comparison to their parent observations "I" and "II," regardless of the distribution.

Figure 5 compares the standard deviations of samples "I," "II," "mean," and the sample of the so-called the closest values. "The closest" samples include those observations in each pair or pair means, for which the absolute difference between them and the known theoretical expectation is close to zero. Suppose that we have two random numbers from the standard normal distribution where the expectation is equal to 0 and the standard deviation is equal to 1, i.e., N(0, 1). Let the values of these numbers be 1.1 and 2.5, respectively. Thus, their mean is equal to 1.8 = (1.1 + 2.5)/2. Because $|1.1 - 0| < |1.8 - 0| < |2.5 - 0|$, we will include the value 1.1 in the sample of "the closest," instead of either 1.1 or 1.8.

According to Figure 5, the standard deviations of "the closest" samples are approximately 1.5 times less than the standard deviations of the "mean" samples.

3.2 Adjustment results

Table 4 illustrates the stepwise decreases of the closing errors of the loops in the network, presented in Figure 2. This process is the most obvious in the case of the circumference of the network loop.

Table 3: Height differences between geopotential numbers in lines, which we used in the adjustment variants of the levelling network, presented in Figure 3

	Height differences in lines			
Line	**Variant 1 (mgpu)**	**Variant 2 (mgpu)**	**Variant 3 (mgpu)**	**Variant 4 (mgpu)**
Kauklahti-Noormarkku	35257.70	35257.70	35257.70	35257.70
Noormarkku-Jyvaskyla	42168.88	42168.88	42168.88	**42164.91**
Kauklahti-Vaajakoski	81008.19	81008.19	81008.19	**80998.47**
Jyvaskyla-Vaajakoski	3580.91	3580.91	3580.91	**3580.65**
Vaajakoski-Sarkisalmi	−16416.77	−16416.77	−16416.77	**−16399.66**
Kauklahti-Sarkisalmi	64600.89	64600.89	64600.89	64600.89
Noormarkku-Ylivieska	21578.48	**21533.81**	21533.81	21533.81
Haapajarvi-Ylivieska	−40407.25	−40407.25	−40407.25	−40407.25
Jyvaskyla-Haapajarvi	19764.28	19764.28	19764.28	**19778.81**
Haapajarvi-Ammansaari	94313.68	94313.68	94313.68	**94322.86**
SArkisalmi-Ammansaari	126922.38	126922.38	126922.38	**126920.11**
Ylivieska-Oulu	−57205.59	−57205.59	−57205.59	**−57206.60**
Oulu-Kuusamo	257845.33	257845.33	257845.33	**257863.35**
Ammansaari-Kuusamo	65928.36	65928.36	65928.36	**65926.36**
Oulu-Muonio	237879.83	237879.83	237879.83	237879.83
Kuusamo-Sodankyla	−91628.08	−91628.08	−91628.08	−91628.08
Sodankyla-Inari	−54107.89	−54107.89	−54107.89	−54107.89
Muonio-Inari	−125729.72	−125729.72	**−125752.04**	−125752.04

Note: To facilitate the reader, we have bolded those values of the height differences in each variant, which are different from the previous one.

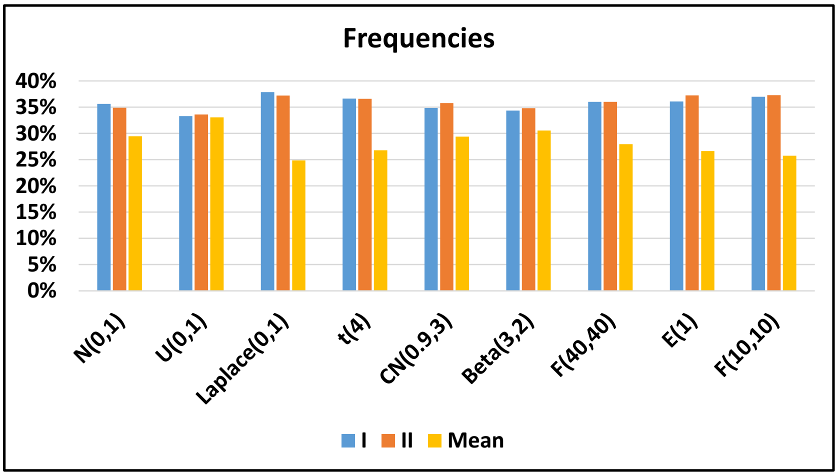

Figure 4: The frequencies of occurrence of the first, the second observation, or their mean most closely to a known expectation of the applied distributions.

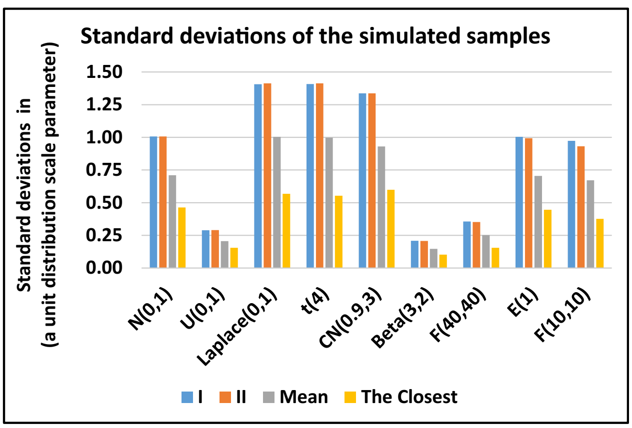

Figure 5: Standard deviations of the samples of the first, the second observations, the means, and the sample of "the closest" values.

Table 4: The closing errors in mgpu of each loop presented by network shown in Figure 2

Loop	Circumference (km)	Closing errors			
		Variant 1 (mgpu)	Variant 2 (mgpu)	Variant 3 (mgpu)	Variant 4 (mgpu)
I	963.731	−0.70	−0.70	−0.70	4.79
II	975.338	−9.47	−9.47	−9.47	−2.08
III	1014.414	52.57	7.90	7.90	−2.66
IV	1262.133	−8.56	−8.56	−8.56	0.57
V	924.127	−9.55	−9.55	−9.55	0.28
VI	1420.485	40.75	40.75	18.43	0.41
The Outer	3234.836	65.04	20.37	−1.95	1.31

Figure 6 illustrates the stepwise increase of the accuracy from Variants 1–4. According to Figure 6, the accuracy of Variant 1, which is based on the established adjustment approach of levelling networks, is approximately 2, 3, and 12 times less than the accuracy of Variant 2, Variant 3, and Variant 4, respectively.

We calculated a priori accuracy μ on the basis of the closing errors in Table 4 by equation (4).

$$\mu = \frac{1}{n}\left(\sum_{i=1}^{n}\frac{\varphi_i^2}{F_i}\right)^{0.5}, \tag{4}$$

where n is the number of loops. In our case, $n = 7$, φ_i is the closing error of loop i in mgpu, and F_i is the circumference of loop i in km.

Figure 7 shows two important facts: (1) the decrease of the standard errors of the adjusted geopotential numbers of the benchmarks in the network (Figure 2) from Variants 1–4 and (2) the strong correlation ($p = 0.99$) between the

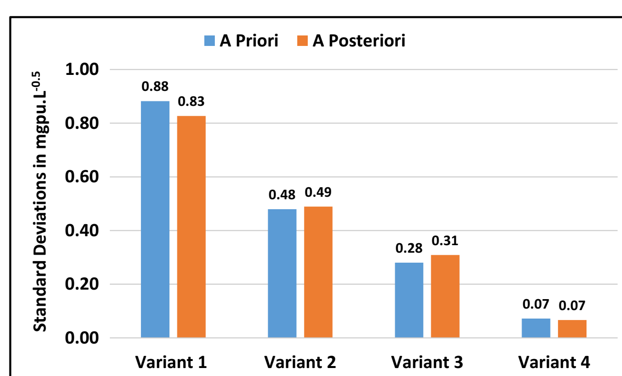

Figure 6: A priori and a posteriori accuracy of each variant in mgpu/$\sqrt{L}$.

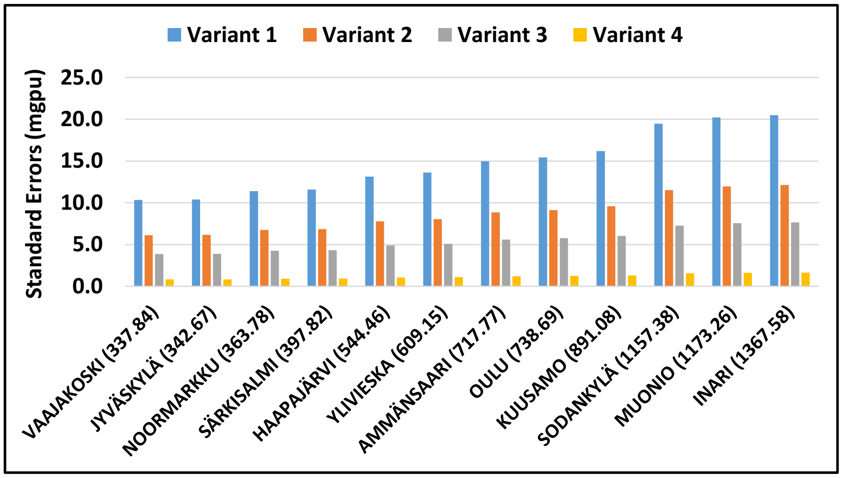

Figure 7: Benchmark standard errors, derived in Variants 1–4. The numbers in the brackets are the distances to the datum point in km, measured along the shortest route.

standard errors of the adjusted geopotential numbers of the benchmarks and their remoteness from the datum benchmark in Kauklahti.

Figure 8 shows differences among the adjusted geopotential numbers of the benchmarks in the analysed network, yielded by Variants 1–4 and their official values (Saaranen et al. 2021). Looking at Figure 8, we can detect the presence of an upward tilt from the south to north.

4 Discussion

The first aim of the current article was to highlight the behaviour of the averages of two random observations of positioning around the theoretical expectation under the assumption that we know their distribution. There are situations, where both observations are greater than the expectation. There are cases when both observations are smaller than the expectation. There are pairs, where either of the observations is very close to the expectation, but the other observation is far from the expectation. All these cases are shown in Figure 2. According to the results presented by Section 3.1, the aforementioned cases are approximately 70% of all possible combinations of grouping of averages and their parents around the expectation of distributions, regardless of the parameters and type of distributions. In all those cases, the average of two random observations is further located to the expectation in comparison to either of its parents. Thus, the true error of such average is greater than the true error of

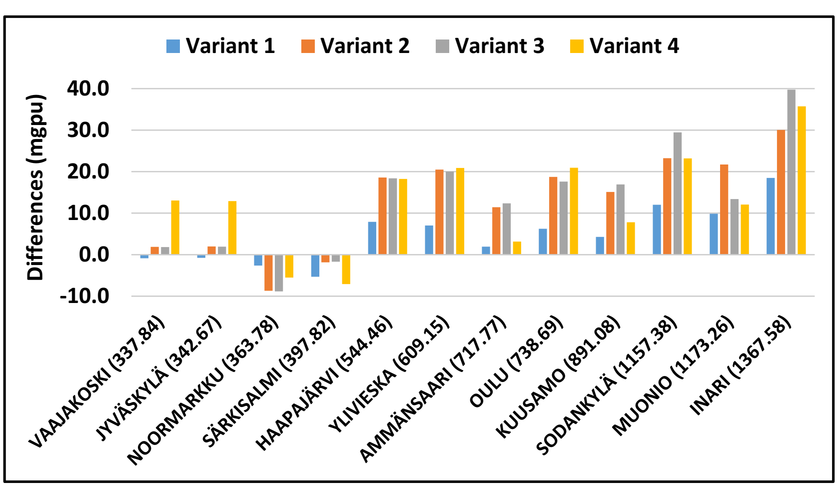

Figure 8: Differences among the official values of the adjusted geopotential numbers of some benchmarks and the geopotential numbers of the same benchmarks, derived in Variants 1–4. The numbers in the brackets are the distances to the datum point in km, measured along the shortest route.

either of its parents. Actually, the true error of the average of a random pair of observations is less than the true errors of each observation in the pair in almost 30% of all cases. All those facts show that there are better approaches for data processing, when we have paired observations.

Let us look at the standard deviations of the simulated samples, presented in Figure 5. The standard deviations of the samples, constructed by the averages, are 1.4 times less than the standard deviations of their parent samples, regardless of the distribution. This fact fully supports equation (3).

Regardless of the fact that distributions included in Table 1 have different standard deviations, which depend on the specific parameters of any distribution, the ratios between the standard deviations of the samples of means and the sample of "The Closes" are approximately equal to 1.5. This fact is valid for each distribution, which we generated in our simulations. In addition, we do not know the real distribution of precise levelling observations neither if all observations in levelling lines come from the same distribution. Therefore, we cannot make any reasonable assumptions about the expectation of the averages of the fore and the back measurements in the precise levelling lines. Thus, if we use all available measurement data, we will obtain higher accuracy and results that are more plausible than with the use of the averages only.

The second question, which we raised in Section 1, was whether the averages of the fore and the back measurements of the heights in a precise levelling network minimize the closing errors in this network. The simple answer is no. Some of the scientific and probabilistic reasons are explained earlier in this article. The adjustment of the reduced network of the Third Levelling of Finland network (Figure 2) fully supported the simulation results.

Replacing the average values of the elevations in the lines Noormarkku – Ylivieska and Muonio – Inary by those values, obtained by their back and fore measurements, respectively, led to a significant increase in the accuracy of the network. According to Table 4, the use of the difference between the geopotential numbers of benchmarks in Noormarkku and Ylivieska, obtained by the back measurement, decreased the closing error of the third loop in the network in Figure 2 from 52.57 to 7.90 mgpu. In addition, the closing error of the outer loop in the network dropped from 65.04 to 20.37 mgpu.

Replacing the average value of the elevation in the line Muonio – Inary by the elevation value, obtained by the fore measurement of this line, reflected on decreasing the closing error of the sixth loop in the network in Figure 2 from 40.75 to 18.43 mgpu. In addition, the closing error of the outer loop in the network fell down from 20.37 mgpu in Variant 2 to −1.95 mgpu in Variant 3.

Because of the only two replaced averages, the a priori accuracy collapsed from 0.88 mgpu/√L in Variant 1 to 0.28 mgpu/$\sqrt{L}$ in Variant 3, which is more than three times increase of the network accuracy. By the way, the official a priori accuracy of the Third Levelling of Finland (Saaranen et al. 2021), calculated by the closing errors of 29 loops, is 0.86 mm/√L.

According to Table 4 and Figures 6 and 7, the process of increasing in the adjustment accuracy from Variant 1 through Variants 2 and 3 to Variant 4 is obvious and clear. A posteriori accuracy dropped from 0.83 mgpu/$\sqrt{L}$ in Variant 1 to 0.07 mgpu/$\sqrt{L}$ in Variant 4. The median value of the standard errors of the adjusted geopotential numbers in Variant 1 is 14.30 mgpu, but 1.15 mgpu in Variant 4. The maximum values of the standard errors of the adjusted geopotential numbers in Variant 1 and Variant 4 are 20.49 and 1.64 mgpu, respectively. Thus, we have given answers to the third and the fourth questions in Section 1. We can reduce the number of lines in a levelling network without any lack of accuracy. In addition, the standard errors of the adjusted geopotential numbers of all nodal benchmarks in a state levelling network, even if some benchmarks are located more than 1,500 km far away from the datum point, are possible to be below 2 mgpu.

The main aim of data proceeding approaches and algorithms is to produce results that are more plausible. Since the standard deviations are criteria for uncertainty, i.e., for plausibility of results, we give more trust to those results with less standard errors. Thus, the results obtained in Variant 4 are more credible than those of Variant 1. Based on this assumption, the adjusted geopotential numbers of the analysed here network reveal a trend of systematic increase of their values from Variant 4 to Variant 1 and official results, presented by the study by Saaranen et al. (2021), from south to north. Figure 8 illustrates this trend. The main reason for tilting of network upward in Variant 1 in comparison to the other variants is the use of averages of both measurements of the line elevations in the adjustment. We can see that in the main part of the network, the differences between the adjusted geopotential numbers of benchmarks from Variant 1 and the official ones are in the range of their standard errors. However, the differences between the adjusted geopotential numbers by Variant 4, Variant 3, and Variant 2 from one side and Variant 1 and the official variant (Saaranen et al. 2021) from the other are more than twice higher than their standard errors produced by Variant 4, Variant 3, and Variant 2, respectively.

Finally, all facts and results, which we discussed here, are completely similar to the results presented by Cvetkov (2023a,b).

5 Conclusions

In this article, we presented the results of simulations of random paired samples from nine distributions to investigate the frequencies of positioning of the averages of random pairs nearby the preliminarily known expectation. On the basis of the results of these simulations, we adjusted a reduced network of the Third Levelling Network of Finland in four variants. Based on the results from simulations and adjustments, the following conclusions we made:

- The average of two random observations from predefined distribution is more closely located to the theoretical expectation in comparison to both its parents between 25 and 33% of all cases. The actual frequency depends on the distribution of the pairs or more precisely on the shape of the distribution. In the case of a peaked distribution like the Laplace (0,1) distribution, we found that the average of two observations is only 25% nearby the theoretical expectation than each of its parent observations. In the case of the Uniform (0,1) distribution, the frequency of occurrence of the average nearby the expectation is approximately 33%.
- Because of the facts, explained in the aforementioned point, the averages of the fore and the back measurements of the heights/geopotential units in a precise levelling network do not minimize the closing errors in this network. Only one or two bad fore or back measurements of the line elevations can deteriorate significantly the adjustment results and the network accuracy (Cvetkov 2023a,b).
- Applying adjustments with all available data is a computational expensive approach. However, the power of modern super computers reveals a new opportunity, unthinkable even a decade ago. Nowadays, we have a greater capacity to analyse data without any simplification and loss of information, especially when we talk about the averages of two observations. Such an approach will be fruitful. Applying adjustments in all combinations, we minimized the standard errors of the adjusted geopotential numbers of all benchmarks in a reduced network of the Third Precise Levelling of Finland network up to 1.64 mgpu. The median of the standard errors is equal to 1.15 mgpu. Thus, we obtained the geometric levelling accuracy of a national vertical network unachievable by other levelling methods (Apollo et al. 2023, Nsiah Ababio and Tenzer 2022, Tanaka and Aoki 2022, Peneva and Georgiev 2010).
- The adjustment of the geometric levelling network in 3^n independent combinations is a computational expensive, but saves field measurements and therefore, is a cheap approach. In this article, we used less than 2/3 of all data in the Third Precise Levelling of Finland, but we received accuracy approximately 10 times greater than the accuracy announced by Saaranen et al. (2021). This fact reveals some important future roles of the geometric levelling of the highest order. The first one is to be a backbone of networks of Global Navigation Satellite Systems permanent stations (Borowski, 2015, Apollo et al. 2023) or networks of atomic clocks. The second role is to be used for verification of the results, yielded by other levelling methods.

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