Research Article

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Estimating the variance in coordinate measuring for binary orbit

Abstract: The variance of a purely random error - "pure error" - in measuring the relative coordinates during the calculation of the orbits of binary stars, (its "unbiased estimate"), is necessary for each test used in the orbit calculation for double stars, such as the adequacy test for the orbit model, gross-error tests and the like. Since this variance is unknown, in this paper we present the robust PEROBEPE1 method which provides an unbiased variance estimate for the pure error in the coordinate measurements concerning the orbits of double stars. This estimate is independent of the model adequacy for a double-star orbit and thus can be used in any test concerning double stars.

Keywords: double star; orbit; pure error; robust estimate

1 Introduction

The trajectory of a star, including double stars, is a complex and random process. In Fig. 1 we present the general physical measurement model which can be described by means of trend, signal, and noise (Moritz 1980, Perović 2005):

$$l_i = (\mathbf{Ax})_i + s_i + \varepsilon_i - \text{measurement,} \qquad (1)$$

where

$\quad$ $\mathbf{Ax}$ – trend,

$\quad$ s_i – signal s at measured points,

$\quad$ ε_i – noise.

In the case of a binary orbit the trend $\mathbf{Ax}$ is a regular curve, a linearised known function of parameters $\mathbf{x}$ – the elliptical apparent orbit in the tangential plane. There are many methods for determining the orbits of binaries developed by various authors (Docobo 1985, Eichhorn and Xu 1990, Pourbaix 1994, Olević and Cvetković 2004, etc.).
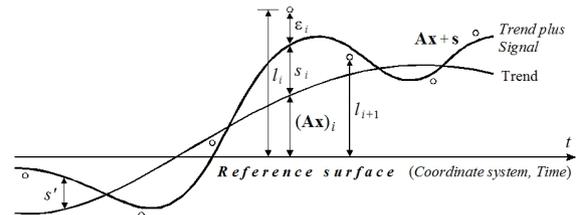

Figure 1: General physical measurement model: $\mathbf{Ax}$ – trend, s – signal (s' – signal s at not measured points, s_i – signal s at measured points), ε_i – noise.

The second function – signal $\mathbf{s}$ – has an irregular variation about zero, is superimposed on $\mathbf{Ax}$, and yields the function $\mathbf{Ax}+\mathbf{s}$. Generally, the task is to determine the continuous curve $\mathbf{Ax} + \mathbf{s}$, but this is not the topic here. In the case of a double-star orbit it is unknown what generates the signal. Irregular variations in time from sources unknown to us about the elliptical orbit combined with cyclical deviations due to cyclical movements of the secondary star could be a critical part of this signal.

The noise ε is a synonym for purely random - "pure" errors of measurements. When a double star is observed, these are random errors in the measurements of the separation ρ (angular distance between components of double star) and position angle θ (angle between the straight line connecting the components and the direction towards the north celestial pole). The transformation to the rectangular coordinates x and y is done by means of $x = \rho \cos\theta$ and $y = \rho \sin\theta$. (The terminology, "signal" and "noise", originates from the use of statistical prediction techniques in communication engineering (Moritz 1980)). In mathematical models, noise is often referred to as a purely random Gaussian error, designated as ε, or, more informally as "pure error" (Searle 1971, Draper and Smith 1981, Perović 2005), and it has a normal distribution

$$\varepsilon \sim N[0, \sigma^2]. \qquad (2)$$

Its standard deviation σ is also called "pure error" (Draper and Smith 1981, Perović 2005). In this text it will be referred to as pure error only (without quotation marks).

The determination of parameters $\mathbf{x}$ is regarded as "adjustment", noise elimination as "filtration", and the sig-

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nal determination **s** at the points, where no measurements take place, as "prediction". Therefore, any given model combines adjustment, filtration and prediction.

The influence of the signal s can be studied in two ways, which differ in principle; the first of these concern the study of cyclical variations, or the least-squares collocation method, whereas the second one concerns the determination of the global signal influence through the variance component. For the determination of the variance components, see Rao and Kleffe (1988) and Perović (2005). Here, however, we are concerned with noise determination, and therefore the signal s will not be the subject.

In all tests dealing with statistical hypotheses, such as tests concerning mathematical models of double-star orbits, or the existence of gross errors, one finds the pure-error variance, i.e. the pure error σ. Thus it is indispensable in all tests, noting that, one uses its unbiased estimate when it is unknown. In cases where the measurements aimed at determining the orbit of a binary star are performed using different techniques, the pure error σ is necessary.

The pure error in the case of the measurements used in the calculation of double-star orbits has not been previously studied. In prior papers the adequacy of the elliptical model for orbits of binaries has been assumed, and any influences variable in time have not been mentioned. In order to test for gross errors in the observations, approaches based on deviations from the model of a binary-star orbit have been used. In other words, they have been based on deviations from the elliptical-orbit model assuming its adequacy *a priori*. Instead, the model adequacy should be the first to be tested (where the pure error σ is also included) and then, if the adequacy is confirmed, it will be justified to use the deviations from the model in the examinations concerning the presence of gross errors in the observations, accuracy estimate, and the like.

Since the noise, by definition, has a Gaussian distribution, its (mathematical) expectation is zero and it is sufficient to examine the noise variance or, i. e. the pure error σ. This is just the topic of the present paper.

2 Motivation

The history of the gross error problem is almost 400 years old, beginning with Galileo who in 1632 used the least absolute sum in order to reduce the effect of observational errors in estimating a measured quantity (in Hald 1986), and Rudjer Boscovich, who in 1757 rejected clearly outlying observations (in Hampel et al. 1986). From this grew

the modern methods of discovering gross errors, such as the robust methods (e. g. Tukey 1960, Huber 1964, 1981) – where the gross errors influence on the estimates is reduced, and others which have various applications in, e.g., statistical medicine/genetics on multiple hypothesis testing (Benjamini and Hochberg 1995), in GPS positioning (Gokalp at al. 2008) using the Fuzzy Logic Method, and in borehole positioning (Nyrnes et al. 2005).

When the sample contains a small number of measurements, as in our example concerning the orbit of a binary star, then the bootstrap method (Efron 1979, Efron and Tibshirani 1993) can be useful. Note that the term "bootstrap method" was introduced by Efron (1979), however it has been used by geodesists for more than 200 years, under the term "additional observations".

The theory of measurement adjustments for such situatiions can be found in Teunissen (2000), and a review of robust methods can be found in Perović (2005).

Benjamini and Hochberg (1995) presented various approaches to the problem of multiple significance testing and specially considered the false discovery rate, finding then that a simple sequential Bonferroni-type procedure is proved to control the false discovery rate. In the same way Wetherill et al. (1986) propose to base the detection of multiple unusual points on sequential methods.

Here we prefer the Legendre principle (Legendre in 1805, in the first publication on least squares (Plackett 1972)): "If among these errors are some which appear too large to be admissible, then those observations which produced these errors will be rejected, as coming from too faulty experiments, and the unknowns will be determined by means of the other observations, which will then give much smaller errors", as well as Perović's principle (Perović 2005): "There is no method enabling to obtain good estimates with erroneous observation". These principles are chosen such that the investigation method can be robust with a high criterion power.

When the coordinates of the points along a binary orbit are measured, the pure error σ remains unknown. It is to be estimated on the basis of the measurements, since it should neither be estimated from the deviations of the model unless this is preceded by accepting the model adequacy, nor should it be ignored in the model adequacy test. We so thus have a tie-breaking problem for the procedure, and it follows that one has to find a method of estimating the pure error σ, independent of the model adequacy. Here

we present a robust method which we call ***PEROBEPE1***[1], which also solves the problem of outliers.

3 The idea of PEROBEPE1 method for noise-variance estimation

For the sake of simplicity the trend of the data is eliminated. This is can be done because **Ax** is a known function. Then we shall have a situation like that in Fig. 2 where we present the total error of the following stochastic process

$$\Delta = s + \varepsilon. \tag{3}$$

Let the measurements of the coordinates for n points along the trajectory (orbit) of a double star be denoted as (x_i, y_i), $i = 1, 2, \cdots, n$.

The idea of the PEROBEPE1 method for estimating the pure error is contained in the following line of reasoning. The time interval covered by the observations is divided into short intervals Δt_j, such that their total number is k, $(j = 1,2,...,k)$. In this way the interval width is sufficiently small so that the stochastic-process curve (in our case the double-star trajectory), presented in Fig. 1 as "trend+signal", can be replaced by the straight line $p - p$ (dashed line in Fig. 2) with an accuracy to a negligibly small difference with respect to the measuring errors and with a number of points as large as possible (Perović 2008). This time it is sufficient that the highest residual of the straight line from the elliptical orbit is less than 1/5 of the average measurement residual from the elliptical orbit. Thus the straight line $p - p$ is assumed to be an adequate model for the realisation of the orbit within the observation interval j, $(j = 1, 2, \cdots, k)$.

For the purpose of adjustment we can use the equation of the straight line

$$aX + bY + 1 = 0, \tag{4}$$

where a and b are unknown parameters of the straight line, and X and Y are the true coordinates of a point along the line.

Let the unknown parameters a and b be given through (known) approximate values a_0 and b_0 and unknown differential increments da and db, i.e.

$$a = a_0 + da, \quad b = b_0 + db, \tag{5}$$

1 The abbreviation PEROBEPE1 comes from the initial letters of the author's family name and method idea: PErović's ROBust method of Estimating Pure Error 1.

and then a mathematical trick is used where corrections v_x and v_y are added to the measurements x and y to obtain the true coordinates X and Y, i.e.

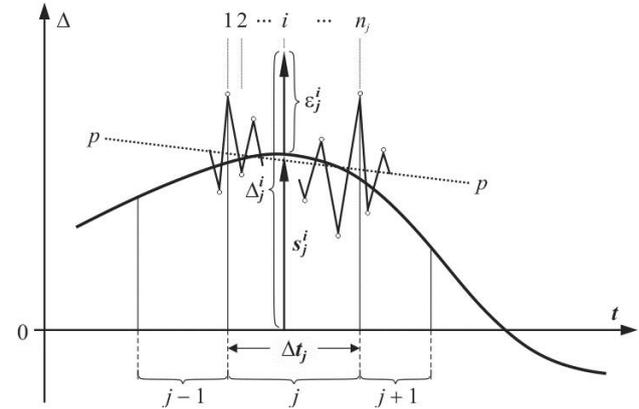$$X = x + v_x, \quad Y = y + v_y. \tag{6}$$

Figure 2: The total error of the following stochastic process.

By substituting Eq. 5 and Eq. 6 into Eq. 4, we obtain the conditional correction equations with unknown parameters or the linear model of conditional adjustment with unknown parameters (Perović 2005)

$$\mathbf{B}v + \mathbf{A}t + \mathbf{w} = \mathbf{0}, \tag{7}$$

where

$$\mathbf{B}_{n_j, 2n_j} = \begin{bmatrix} a_0 & b_0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & a_0 & b_0 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & a_0 & b_0 \end{bmatrix},$$

$$\mathbf{v}_{2n_j, 1} = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ \cdots \\ v_{x_{n_j}} \\ v_{y_{n_j}} \end{bmatrix}, \tag{8}$$

$$\mathbf{A}_{n_j, 2} = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_{n_j} & y_{n_j} \end{bmatrix}, \mathbf{t}_{2,1} = \begin{bmatrix} da \\ db \end{bmatrix}, \tag{9}$$

$$\mathbf{w}_{n_j, 1} = \begin{bmatrix} w_1 \\ \vdots \\ w_{n_j} \end{bmatrix}, \quad w_i = x_i a_0 + y_i b_0 + 1, \quad i = 1, 2, \cdots, n_j. \tag{10}$$

In the case of a general function, a model of type Eq. 7 is referred to as Gauss-Helmert's model (Wolf 1978, Perović 2005).

The stochastic model of observations will then be

$$\mathbf{E}\left[\varepsilon\right] = \mathbf{E}\left[\mathbf{v}\right] = \mathbf{0} \text{ with } \mathbf{K}_\varepsilon = \mathbf{K_l} = \mathbf{K_v} \equiv \mathbf{K} = \sigma^2 \mathbf{I}, \qquad (11)$$

where $\mathbf{E}$ is the mathematical-expectation operator, σ^2 from Eq. 2 is the pure-error variance, and $\mathbf{K}$ is the designation for the variance-covariance matrix.

Applying the least-square method (LSM) one obtains the estimates (Perović 2005)

$$\mathbf{t} = -\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}, \qquad (12)$$

and

$$\hat{\mathbf{v}} = \frac{-1}{c^2}\mathbf{B}^{\mathrm{T}}(\mathbf{At} + \mathbf{w}) = \frac{1}{c^2}\mathbf{B}^{\mathrm{T}}(\mathbf{I} - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}})\mathbf{w}, \qquad (13)$$

$$\hat{v}_{x_i} = \frac{-a_0}{c^2}(x_i a + y_i b + 1), \quad \hat{v}_{y_i} = \frac{-b_0}{c^2}(x_i a + y_i b + 1),$$
$$i = 1, \ 2, \ \cdots, \ n. \qquad (14)$$

(where a and b are calculated following Eq. 5) with the variance-covariance matrix of correction estimates

$$\mathbf{K}_{\hat{\mathbf{v}}} = \sigma^2 \mathbf{Q}_{\hat{\mathbf{v}}}, \text{ and } \mathbf{Q}_{\hat{\mathbf{v}}} = \frac{1}{c^2}\mathbf{B}^{\mathrm{T}}(\mathbf{I} - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}})\mathbf{B}, \qquad (15)$$

where $\mathbf{Q}_{\hat{\mathbf{v}}}$ is the cofactor matrix of correction estimates, and

$$c^2 = a_0^2 + b_0^2. \qquad (16)$$

4 The gross-error test in PEROBEPE1 method

Since we do not know if the measurements contain gross errors, we must carry out the gross-error test and derive the estimate for σ simultaneously.

There exist many methods used for the purpose of testing gross errors, among which the robust ones yield better results than the ordinary least squares method. In the case of measurements aimed at binary orbit determination using modern technologies such as specle interferometry, the popular "bootstraping" method may be used, but the experiment would be very expensive because the measurements must cover at least 3 – 4 years (since the orbital period is equal to 11.326 years), and in the case of classical measuring technologies this method would thus be inapplicable.

The method used here for gross error testing in binary orbit determination is Pope's (1976) method (Perović and

Cvetković 2011). However, since in this method the gross errors within a given group are only tested with the data from this group, its criterion power (ability) is low because of the small number of measurements, which in turn is due to requiring the interval width to be as small as possible.

For this reason we propose here a robust method for testing the gross errors in the measurements, in which an estimate of the variance of the measured values is obtained from all the measurements, but not from those containing gross errors. The number of degrees of freedom in estimating the variance of measurements therefore becomes much larger than in the case of Pope's method (Pope 1976). Consequently, the criterion power for the gross-error test for this method exceeds that concerning the application of Pope's tau-method. Such an estimation of the variance in measurements is now independent of measurements with gross errors, such that for the gross-error test one can use the Student distribution.

The procedure of testing the gross errors should contain global and local tests within each group.

However, since σ (wich is unknown) appears in the global test, the global test cannot be carried out. Therefore, only local tests of gross errors are used. In other words, the gross errors are tested for every individual observation.

If also the general designation l is used for observations of point coordinates (Perović 2005), we should then introduce double subscripts, such that

$$l_{ji}, \ j = 1, \ \cdots, \ d, \ \cdots g, \ \cdots k,$$
$$i = 1, \ \cdots, \ v, \ \cdots, \ \xi, \ \cdots, \ 2n_j; \quad n = \sum_{j=1}^{k} n_j \qquad (17)$$

where j is the ordinal number of the interval or group, i is the ordinal number of a measurement within the group, and n is the total number of measuring points. Then within a group j: $l_{j1} = x_{j1}, l_{j2} = y_{j1}, \cdots, l_{j,(2n_j-1)} = x_{jn_j}, l_{j,2n_j} = y_{jn_j}$; and the total number of measurements (measuring coordinates) is $2n$.

Let G_{ji} be the gross error in the observation l_{ji}. In order to test the gross errors we introduce "conventional" alternative hypotheses where the simultaneous presence of only one outlier within a group j ($j = 1, \ \cdots, \ k$) is assumed. Therefore, if we form these conventional hypotheses for all observations successively, as the result we obtain a set of $2n$ conventional hypotheses $H_{a,ji}$, where each of them is one-dimensional. The procedure of consecutive testing for these $H_{a,ji}$ is known as "a data snooping strategy" (Baarda 1968, Kok 1984, Perović 2005).

We then test the null hypothesis

$$H_{0,ji}: \ G_{ji} = 0 \text{ versus alternative } H_{a,ji}: \ G_{ji} \neq 0, \quad (18)$$

where

$$j = 1, \cdots, d, \cdots g, \cdots k, \ i = 1, \cdots, v, \cdots, \xi, \cdots, 2n_j.$$

The procedure must be iterative as explained below.

Iteration I. The variance estimate for pure error σ^2 is calculated for each interval (group):

$$m_j^2 = \widehat{\sigma_j^2} = \frac{\hat{\mathbf{v}}_j^{\mathrm{T}} \hat{\mathbf{v}}_j}{f_j}, \ f_j = n_j - 2d.f.,$$

$$j = 1, \ \cdots, \ d, \ \cdots g, \ \cdots k, \qquad (19)$$

where $\hat{\mathbf{v}}_j^{\mathrm{T}} = [\hat{v}_{j1} \ \hat{v}_{j2} \ \cdots \ \hat{v}_{j, 2n_j}], (\hat{v}_{j1} = \hat{v}_{x_1}, \hat{v}_{j2} = \hat{v}_{y_1}, \cdots, \hat{v}_{j, (2n_j-1)} = \hat{v}_{x_{n_j}}, \hat{v}_{j, 2n_j} = \hat{v}_{y_{n_j}})$, and its definite estimate is based on measuring the coordinates of all the n points (from all k groups)

$$m^2 = \widehat{\sigma^2} = \frac{1}{f} \sum_{j=1}^{k} f_j m_j^2, \text{ with } f = \sum_{j=1}^{k} f_j = n - 2kd.f. \quad (20)$$

If the measurements contain gross errors, the estimate m^2 from Eq. 20 will be biased.

We then calculate the standardised estimates of the corrections

$$t_{ji}^{\circ} = \frac{\hat{v}_{ji}}{m\sqrt{Q_{\hat{v}_{ji}}}}, \quad j = 1, \ \cdots, \ d, \ \cdots g, \ \cdots k,$$

$$i = 1, \ \cdots, \ v, \ \cdots, \ \xi, \ \cdots, \ 2n_j. \qquad (21)$$

If we wanted to use statistics Eq. 21 for testing of gross errors, where m comes from Eq. 20 and $Q_{\hat{v}_{ji}}$ from the principal diagonal of the cofactor matrix $\mathbf{Q}_{\hat{v}}$ from Eq. 15, we would have to check its distribution because it is unknown. This, however, is an intermediate distribution, between tau, τ_f, and the Student, t_f, one. This is due to the stochastical mutual dependence between $\hat{v}_{ji}$ and m (because of dependence between m_j and $\hat{v}_{ji}$).

Therefore, here a statistic is proposed in which the numerator and denominator are independent. This is achieved by estimating the pure error σ without taking into account k' ($1 \le k' \le k$) outliers with coordinates $l_{dv}, \ \cdots, \ l_{g\xi}$ which in this order belong to the groups $d, \ \cdots, g$. Within each group $d, \ \cdots, g$ the results are adjusted without this single "suspicious" point – in group d without $(x_{d,(v+1)/2}, y_{d,(v+1)/2})$,..., in group g without $(x_{g, \xi/2}, y_{g, \xi/2})$, etc. – and we find the variance estimates

$$m_d'^2 \text{ with } f_d' = f_d - 1, \cdots, m_g'^2 \text{ and } f_g' = f_g - 1, \qquad (22)$$

which now will be independent of the suspicious results $l_{dv}, \ \cdots, \ l_{g\xi} \ (l_{dv} = x_{d,(v+1)/2}, \cdots, l_{g\xi} = y_{g, \xi/2})$ that we examine for gross errors. So the pooled dispersion estimate denoted as

$$m'^2 = m_{(dv, \cdots, g\xi)}^2 = \widehat{\sigma^2}_{(dv, \cdots, g\xi)}, \qquad (23)$$

$$m'^2 = \frac{1}{f'}\left(f_1 m_1^2 + \ \cdots \ + f_d' m_{d,(dv)}^2 + \ \cdots \ + f_g' m_{g,(g\xi)}^2 \right.$$
$$\left. + \cdots + f_k m_k^2\right), \qquad (24)$$

with

$$f' = f_1 + \cdots + f_d' + \cdots + f_g' + \cdots + f_k = f - k' \qquad (25)$$

will be independent of the examined results $l_{dv}, \ \cdots, \ l_{g\xi}$.

In this iteration in groups where

$$\max_i|t_{ji}^{\circ}| \ \ge \ 2, \quad j = 1, \ \cdots, \ d, \ \cdots g, \ \cdots k, \qquad (26)$$

the corresponding results $l_{dv}, \ \cdots, \ l_{g\xi}$ are regarded as possibly containing gross errors and they are subjected to the test for gross errors.

Now in the test statistic

$$t_{ji} = \frac{\hat{v}_{ji}}{m'\sqrt{Q_{\hat{v}_{ji}}}} = \frac{\hat{v}_{ji}}{m_{(dv, \cdots, g\xi)}\sqrt{Q_{\hat{v}_{ji}}}}, \quad ji = dv, \ \cdots, \ g\xi$$

$$(27)$$

the denominator is independent of the numerator so that the statistic under the null hypothesis $H_{0,ji}$ has a central Student distribution with f' degrees of freedom, i.e. a non-central Student distribution under the alternative one with f' degrees of freedom. It follows

(a) $t_{ji}|H_{0,ji} \sim T(f')$ and

(b) $t_{ji}|H_{a,ji} \sim T'\left(f', \sqrt{\lambda_{ji}}\right), ji = dv, \ \cdots, \ g\xi.$ (28)

with non-centrality parameter (Perović 2005)

$$\sqrt{\lambda_{ji}} = \frac{G_{ji}\sqrt{Q_{\hat{v}_{ji}}}}{\sigma}, \quad ji = dv, \ \cdots, \ g\xi. \qquad (29)$$

For this reason statistic Eq. 27 can serve in testing the hypothesis $H_{0,ji}$ against its alternative $H_{a,ji}$ from Eq. 18. So, if

$$\max|t_{ji}| \ \ge \ t_{1-\alpha_0/2; \ f'}, \quad ji = dv, \ \cdots, \ g\xi, \qquad (30)$$

where $t_{1-\alpha_0/2; \ f'}$ is the quantile of the two-tailed $t(f')$-test with confidence probability $1 - \alpha_0$, α_0 significance level for the test, then the corresponding result l_{ji} is rejected as a result which contains a gross error.

For the significance level of the local test we recommend $\alpha_0 = 0.01$.

Iteration II. In the second and other iterations the procedure is formally the same as in Iteration I, with the only difference being the total number of measurements (measured points) (in second iteration it is $n - k'$) and in the groups with results already obtained, (in second iteration it is $n_d - 1, ..., n_g - 1$ points), the results will then be classified as suspicious according to the criterion

$$\max_i|t_{ji}^{\circ}| \ \ge \ 2, 5, \quad j = 1, \ \cdots, \ d, \ \cdots g, \ \cdots k. \qquad (31)$$

The iterations are stopped when the null hypotheses $H_{0,ji}$ are accepted in each group.

The test is robust because in the pure-error estimation m' from Eq. 24 none of the results $l_{dv}, \cdots, l_{g\xi}$, suspected of or confirmed to contain gross errors take part.

Finally, on the basis of the remaining measurements (denoting their number again as n) we calculate the (definite) unbiased estimate of the pure-error variance for the coordinate measurements concerning the points of the orbit of a double star according to Eq. 20.

Note 1. The PEROBEPE1 method yields good results if the blunders are removed, e. g. the results with errors of order of 10σ and more. Such measurement results should therefore be removed prior to the applications of this method.

Note 2. For a large number of measurements (e. g. more than 1000) the distribution of the measurements foreseen to be rejected can be dense. Then the number of iterations can be large when the results with errors, which are not gross, would be rejected, so the pure error estimate m would be reduced artificially, i.e. the obtained estimate would be biased (reduced). In such a case the number of iterations should be limited to 3 or 4.

5 Some Robust Estimations

As previously stated, for the case of measurements containing outliers the robust methods yield better results than the least squares (LS) method. Out of many robust LS methods we shall use here two of them: Huber's Robust LS (Huber 1981) as one of the first such methods, and PEROBLS3 (Perović 2001, 2005) as one of the methods yielding the best results (Perović 2005). All these methods were derived for parameter adjustment, and since here a conditional adjustment is used, the formulae of these methods will be adapted to the conditional adjustment case.

The linear model will be defined by Eq. 7, whereas the stochastic one is generalised in such a way that the variances in the variance-covariance matrix are represented through the observation weights P_i; the stochastic model of observations will therefore be

$$\mathbf{E}\,[\varepsilon] = \mathbf{E}\,[\mathbf{v}] = \mathbf{0} \; with \mathbf{K}_\varepsilon = \mathbf{K}_{\mathbf{l}} = \mathbf{K}_{\mathbf{v}} \equiv \mathbf{K} = \sigma^2\mathbf{P}, \quad (32)$$

where $\mathbf{P} = diag\{P_i\}$ is the observation weight matrix and σ^2 is the variance coefficient.

In the case of iterative procedures the weights $P_{i,v}$ are determined in the v-th iteration, so the LS solutions are:

$$\mathbf{t}_v = -\,(\mathbf{A}^T\mathbf{M}_v^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{M}_v^{-1}\mathbf{w}, \quad (\mathbf{M}_v^{-1} = \mathbf{B}\mathbf{P}_v^{-1}\mathbf{B}^T), \quad (33)$$

$$\hat{\mathbf{v}}_v = -\,\mathbf{P}_v^{-1}\mathbf{B}^T\mathbf{M}_v^{-1}(\mathbf{A}\mathbf{t}_v + \mathbf{w}), \quad (34)$$

$$\mathbf{K}_{\hat{\mathbf{v}},v} = \sigma^2\mathbf{Q}_{\hat{\mathbf{v}},v},$$

with $\mathbf{Q}_{\hat{\mathbf{v}},v} = \mathbf{P}_v^{-1}\mathbf{B}^T(\mathbf{M}_v^{-1} - \mathbf{M}_v^{-1}\mathbf{A}(\mathbf{A}^T\mathbf{M}_v^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{M}_v^{-1})\mathbf{B}\mathbf{P}_v^{-1}.$ (35)

5.1 Huber's robust LS

For the mathematical adjustment model defined by Eq. 7 and Eq. 32, by simultaneously solving Eq. 7 and the following one,

$$\mathbf{v}^T\mathbf{P}\mathbf{v} \;=\; (n - u)b\,\sigma^2, \quad (36)$$

Huber's M-estimators according to Eqs. 33 to 35 and the estimate for σ^2 are obtained in the iterative procedure:

$$\widehat{\sigma_v^2} \;=\; \frac{\hat{\mathbf{v}}_v^T\mathbf{P}_v\hat{\mathbf{v}}_v}{(n - u)\,b} \quad - \text{Huber's proposal 2}, \quad (37)$$

Huber (1964) defined b as

$$b \;=\; \int\limits_{-\infty}^{\infty} \psi^2(z)f(z)\,dz, \quad (38)$$

and function $\psi(z)$ as

$$\psi(z) \;=\; \left\{ \begin{array}{ll} -c, & \text{for } z \le -c \\ z, & \text{for } -c < z < c \\ c, & \text{for } z \ge c \end{array} \right\}, \quad (39)$$

where

$$1 \le c \le 2, \quad (c_{optimal} = 1.345), \quad (40)$$

and

$$f(z) \;=\; \frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}, \quad -\infty \le z \le +\infty \quad (41)$$

is the density function for the standard normal distribution in Z. The weight function is defined as:

$$P_{i,v+1} = \left\{ \begin{array}{ll} P_i & - \text{ for } |z_{i,v}| \le c \\ P_i\frac{c}{|z_{i,v}|} & - \text{ for } |z_{i,v}| > c \end{array} \right\},$$
$$i = 1, 2, \cdots, n; v = 1, 2, \cdots \quad (42)$$

with

$$z_{i,v} \;=\; \frac{\hat{v}_{i,v}}{\sigma\sqrt{Q_{\hat{v}_{i,v}}}}. \quad (43)$$

The coefficients b and c are determined according to the percentage α_c of censored results, as given in Table 1 for two-sided censoring.

Table 1: Coefficients b and c in Huber's method.

Censoring α_c [%]	c	b
5	2	0.92054
10	1.5	0.77846
20	1.345	0.71017
30	1	0.51606

5.2 Robust PEROBLS3 method

In this method it is assumed that the observation weight is proportional to the mean square error which is composed of the variance and gross error square. Thus, for an observation l_i containing gross error G_i, the mean square error is $\sigma_i^2 + G_i^2$, and the weight of the observation l_i will be

$$P_i' = \frac{\sigma^2}{\sigma_i^2 + G_i^2}, \qquad (44)$$

where σ_i^2 is the variance of the observation l_i and σ^2 is the variance coefficient.

Instead of the standard LS, $\mathbf{v}^T \mathbf{P} \mathbf{v} = \min$, we use its robust variant $\mathbf{v}^T \mathbf{P}' \mathbf{v} = \min$.

The method is completely based on the probability theory and mathematical statistics, and the process is iterative and the weights in the $(v+1)$-th iteration are calculated following the formula

$$P_{i,v+1} = \left\{ \begin{array}{ll} P_i & ; \quad |\tau_{i,v}| \le c \\ \frac{P_i}{1 + P_i \widehat{G_{i,v}^2}/m_v^2} & ; \quad |\tau_{i,v}| > c \end{array} \right\}, \quad (45)$$

where:

$$\widehat{G_{i,v}^2} = \frac{\hat{v}_{i,v}^2 - m_v^2 Q_{\hat{v}_{i,v}}}{r_{ii,v}^2} \text{ is the estimate for } G_i^2, \qquad (46)$$

$c = c_{f_v} = \tau_{1-\alpha_c/2}(f_v)$ is the quantile of $\tau(f_v)$ distribution, (47)

$\alpha_c[\%] \in (5, 10, 20, 30)$ is the censoring percentage, (48)

$$\mathbf{R}_v = \{r_{ij,v}\} = \mathbf{Q}_{\hat{v}_v} \mathbf{P}_v \text{ is the redundant matrix}, \qquad (49)$$

$$m_v^2 \equiv \widehat{\sigma_v^2} = \frac{1}{f_v} \hat{\mathbf{v}}_v^T \mathbf{P}_v \hat{\mathbf{v}}_v \text{ is the PERG proposal B for estimate of } \sigma^2, \qquad (50)$$

and

$$f_v = tr(\mathbf{R}_v) \text{ gives the degrees of freedom.} \qquad (51)$$

The other LS estimates are obtained following Eqs. 33 to 35.

6 Results and discussion

6.1 Experiment and analysis

We use the results from the example (Perović and Cvetković 2011). In the case of binary star WDS 04184+2135 = MCA 14Aa,Ab we take 44 speckle-interferometric measurements of the relative coordinates obtained with a telescope of aperture equal to 3.8 m between 1975 and 2005. For this binary the orbit has been calculated; the period is equal to 11.326 years, and therefore the measurements cover an interval of about three orbital periods.

The measuring results for the coordinates θ and ρ are divided into 10 groups ($k = 10$) which are seen in Fig. 3 and in Table 2.

The input data to be adjusted are the rectangular coordinates $x = \rho \cos \theta$ and $y = \rho \sin \theta$.

The subdivision (grouping) is preceded by the determination of the adjusting ellipse in a standard way and it is based on this ellipse (Fig. 3).

The total number of the measured points is 44, but since the data treatment (reduction) within a group requires three or more points, group 8 is rejected and the former groups 9 and 10 become groups 8 and 9, respectively. This leaves us with $n = 43$ measured points and $k = 9$ groups.

Iteration I. The pure-error estimates obtained according to Eq. 19 are given in Table 3. Applying Eq. 20 gives us
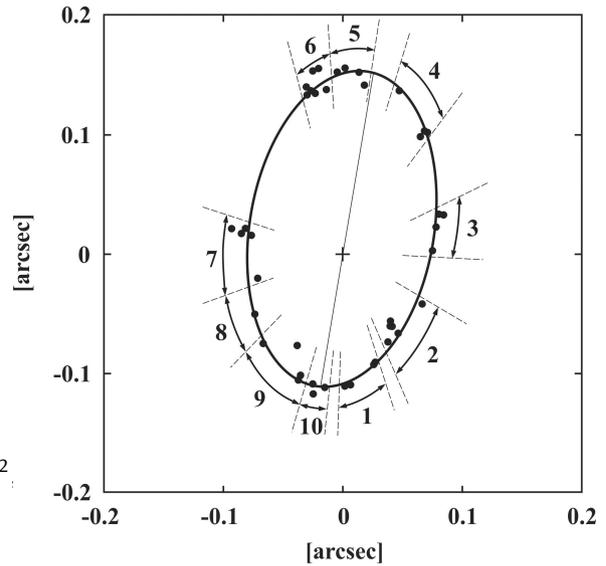
$m = 0,006\,152$ arcsec, with $f = 25$ d.f.

Figure 3: The orbit of binary star MCA 14Aa,Ab

Since in group 6 for the sixth measured point the 11th and 12th results have normalized corrections which exceed two, the same is true for the second point in group 8 (3rd and 4th results), since

$$\max_i |t^\circ_{6,i}| = |t^\circ_{6,11}| = |t^\circ_{6,12}| = 2.176 > 2, \text{ and}$$

$$\max_i |t^\circ_{8,i}| = |t^\circ_{8,3}| = |t^\circ_{8,4}| = 2.589 > 2,$$

(also see Table 3), the measurements of the coordinates for point 6: $l_{6,11} = x_{6,6}$ and $l_{6,12} = y_{6,6}$, point 2: $l_{8,3} = x_{8,2}$ and $l_{8,4} = y_{8,2}$ (according to our designations (27) it is: $d = 6$, $\nu = 11$ and $g = 8$, $\xi = 4$) must be the first to be tested for gross errors.

In groups 6 and 8 the adjustment is carried out without these results and according to Eq. 23 the pure-error estimates are obtained. These are given in Table 3 under Iteration II:

$m'_6 = 0.003\,825$ arcsec and $m'_8 = 0.002\,704$ arcsec, and according to Eq. 24 the estimate $m' = 0,004\,331$ arcsec is written in Table 3 near its bottom under Iteration II.

Since the Student statistics (Eq. 27) for these groups have maximum values exceeding the permitted one $2.803 = t_{0,995;\,23}$ for a significance level $\alpha_0 = 0.01$, i.e. since it is

$$\max_i |t_{6,i}| = |t_{6,11}| = |t_{6,12}| = 3.091 > 2.803,$$

and

$$\max_i |t_{8,i}| = |t_{8,3}| = |t_{8,4}| = 3.677 > 2.803,$$

in both cases the alternative hypotheses $H_{a,ji}$ from Eq. 18:

$H_{a;\,6,11}:\ G_{6,11} \neq 0$ in group 6 and $H_{a;\,8,3}:\ G_{8,3} \neq 0$ in group 8; the same and

$H_{a;\,6,12}:\ G_{6,12} \neq 0$ in group 6 and $H_{a;\,8,4}:\ G_{8,4} \neq 0$ in group 8.

Conclusion: The coordinate measurements for point 6 $(x_{6,6};\, y_{6,6})$ in group 6 and of point 2 $(x_{8,2};\, y_{8,2})$ in group 8 are rejected as results which contain gross errors for a significance level of the individual - local test $\alpha_0 = 0.01$.

Iteration II. Now in groups 6 and 8 the number of observed points are $n_6 = 7$ and $n_8 = 3$, respectively. The pure-error estimates (Table 3, iteration II) are obtained from the adjustment procedure:

$m_6 = 0.003\,825$ arcsec with $f_6 = 5$ and $m_8 = 0.002\,704$ arcsec with $f_8 = 1$, and according to Eq. 24,

$m = 0.004\,331$ arcsec with $f = 23$

which is the (definite) unbiased pure-error estimate σ because

$$\max_i |t^\circ_{j,i}| = |t^\circ_{7,7}| = |t^\circ_{7,8}| = 2.053 < 2.5.$$

According to this criterion (Eq. 30) there are no more results suspicious of gross errors, and no other iterations are needed.

Comparing the pure-error estimate ($m = 0.004\,331$ arcsec) obtained by applying this method with that ($m = 0.006\,152$ arcsec) obtained by applying Pope's gross-error test (Pope 1976) we infer that a significant difference exists.

The effect of group merging is also examined. So:

- the first two points from group 1 are included in group 10,
- the last two points from group 1 are included in group 2,
- groups 7 and 8 are merged,
- gropus 3, 4, 5 and 6 cannot be changed.

This merging results in enlarging the variance estimate m^2 within the extended groups, which is expected because the deviations of the straight line from the curvature of ellipse in these intervals are not negligible. The results are presented in Table 4, and the pooled standard estimate is

$m = 0.04023$ arcsec, with $f = 24$,

which exceeds by an order of magnitude the PEROBEPE1 estimate of 0.00433 arcsec.

Due to the enlarging of the standard estimate, gross errors were not detected.

For more on the influence of interval width on the estimates in the PEROBEPE1 method (see Subsection 6.2 of this paper – Bootstrap method).

The results obtained by applying PEROBEPE1 are compared with the results obtained by applying the robust methods, Huber's and PEROBLS3. For both methods the 10% censoring is used, so that

$b = 0.77846$ for Huber's method, and

$c = c_{i,f} = \tau_{0.95}(5) = 1.640$ for PEROBLS3 (in group 6).

In the first iteration **P** is taken as the unit matrix. The iterations are executed until the difference between the parameters $m^2_{\nu+1}$ and m^2_ν is negligible.

Bonferroni's statistics are also used. With the first type error $\alpha = 0.05$ and the significance level of the local test $\alpha_{0,j} = 1 - (1 - \alpha)^{1/n_j}$ the use of Bonferroni's method does not result in detecting of gross errors in any group. So the pooled standard estimate is $m = 0.00615$ arcsec, with $f = 25$ (Table 5).

The estimates of standard errors by application of the various methods are given in Table 5.

From Table 5 one concludes the following: 1) Bonferroni's statistics do not detect gross errors in any group, 2) Huber's robust method yields enlarged standard estimates compared to the ordinary LS, also noticed by Perović

Table 2: The division of measured points in groups.

Group j	i	θ	ρ	Group j	i	θ	ρ	Group j	i	θ	ρ
		Measurements				Measurements				Measurements	
1. (n_1 = 4)	1	3.1	0.110	4. (n_4 = 4)	1	161.1	0.144	6.	7	187.7	0.156
	2	16.2	0.095		2	146.7	0.117		8	189.6	0.155
	3	15.5	0.096		3	145.5	0.123	7. (n_7 = 5)	1	259.0	0.079
	4	0.9	0.111		4	146.7	0.123		2	255.8	0.085
2. (n_2 = 6)	1	34.9	0.069	5. (n_5 = 5)	1	181.9	0.152		3	259.1	0.087
	2	33.5	0.073		2	179.5	0.155		4	257.6	0.096
	3	32.9	0.072		3	175.1	0.152		5	285.9	0.075
	4	26.7	0.083		4	174.9	0.152	(7).	1	304.3	0.090
	5	57.4	0.078		5	172.9	0.142	8 (8). (n_8 = 4)	1	340.7	0.108
	6	34.4	0.081	6. (n_6 = 8)	1	191.4	0.139		2	333.3	0.086
3. (n_3 = 4)	1	106.0	0.080		2	192.3	0.141		3	340.5	0.112
	2	91.9	0.074		3	190.0	0.136		4	318.3	0.101
	3	111.3	0.089		4	192.5	0.143	9 (9). (n_9 = 3)	1	352.2	0.113
	4	112.5	0.086		5	193.0	0.136		2	347.0	0.112
					6	186.0	0.138		3	348.0	0.120

Table 3: Presentation of estimates m through iterations.

Group j	f_j	m_j [arcsec]	$\max_i|t^\circ_{ji}|$	$\max|t_{ji}|$	f_j	m_j [arcsec]	$\max_i|t^\circ_{ji}|$
		ITERATION I				ITERATION II	
1	2	0.000 929	0.209		2	0.000 929	0.297
2	4	0.005 382	1.123		4	0.005 382	1.595
3	2	0.002 045	0.464		2	0.002 045	0.659
4	2	0.003 479	0.792		2	0.003 479	1.125
5	3	0.004 473	1.192		3	0.004 473	1.693
6	6	0.007 471	2.176 >2	3.091	5	0.003 825	1.622
7	3	0.005 936	1.445		3	0.005 936	2.053 <2.5
8	2	0.011 409	2.589 >2	3.677	1	0.002 704	0.624
9	1	0.005 918	0.962		1	0.005 918	1.367
$f; m$	25	0,006 152		(2.807)	23	0.004 331	

Table 4: The estimates m through merging groups.

Group j	m_j[arcsec]	f_j
1+2	0.00511	6
3	0.00205	2
4	0.00348	2
5	0.00447	3
6	0.00747	6
7+8	0.13815	2
1+10	0.00548	3

(2005), and 3) Perović's robust PEROBLS3 method yields estimates close to the PEROBEPE1 estimates.

6.2 Bootstrap method

A modern alternative to the traditional approach to statistical inference is the bootstrapping method introduced by Efron (1979). In determining the orbit of binary WDS 04184 resampling is possible, but this would take at least 3-4 years since its orbital period is 11.326 years. For this reason the measurements are simulated.

Table 5: The estimates m [arcsec] through methods.

Group	PEROBEPE1	Bonferroni	Huber	PEROBLS3
1	0.00093	0.00093		
2	0.00538	0.00538		
3	0.00205	0.00205		
4	0.00348	0.00348		
5	0.00447	0.00447	0.00511	
6	0.00383	0.00747	0.00836	0.00432
7	0.00594	0.00594		
8	0.00270	0.01141		
9	0.00592	0.00592		
Mean	0.00433	0.00615		

The simulation of measurements is carried out in 360 points distributed uniformly along the arc of the elliptical orbit. Measurement errors following the normal N[0;σ] distribution in the coordinates x and y with a standard σ = 0.005 arcsec are simulated. The obtained mean value and the standard error are

$$\bar{\varepsilon} = 0.00006 \text{ arcsec}, \quad m = 0.00509 \text{ arcsec},$$

and at six points for each of them one gross error is added, as shown in Table 6.

Table 6: The simulated gross errors, in [arcsec].

Epoch (measurement point)	Gross Error
1980.503	$G_Y = -3\sigma = -0.015$
1980.660	$G_Y = 4\sigma = 0.020$
1983.618	$G_X = -5\sigma = -0.025$
1983.712	$G_X = -3\sigma = -0.015$
1987.267	$G_Y = 6\sigma = 0.030$
1988.872	$G_X = 5\sigma = 0.025$

The measurement errors are added to the coordinates of the points calculated along the ellipse and, in this way, the simulated results of measuring the coordinates x and y are obtained.

For the purpose of dividing the measurement results four widths are chosen, with classes of intervals with a "central" angle α (from the ellipse focus) over the interval arc, as shown in Table 7.

The analysis of the PEROBEPE1 method by applying bootstrapping leads to the following results: 1) the number of iterations in each group is 2; 2) all simulated gross errors of measurements, as well as 4 errors with statistics

belonging to the criterion Eq. 30, are detected, and 3) the standard estimates for each group are close to its value from the error simulation (value 0.00509 arcsec). Also the equality tests between each individual variance and the *a priori* variance (0.005 arcsec)^2 are in favour of their equality. These results are given in Table 8.

The influence of interval width on the standard estimates m is also examined. For the 360 points already designated along the elliptic orbit the exact coordinates (without measurement errors) x and y are calculated. Then according to the division, as given in Table 8, in all groups the mean square error of deviations in the elliptical orbit from the adjusting straight line m_{EL}^2 is calculated (Table 9).

Now we are able to calculate the interval width influence on the standard estimate m. The estimate m^2 consists of the measurement variance m_M^2 and the variance m_{EL}^2 due to the elliptic orbit curvature within the intervals, i.e.

$$m^2 = m_M^2 + m_{EL}^2. \tag{52}$$

From here we find the measurement variance estimate which is free of the elliptic curvature influence,

$$m_M^2 = m^2 - m_{EL}^2. \tag{53}$$

These estimates and the relative interval width influence in the estimate m are presented in Table 9.

From Table 9 one concludes that in classes 1, 2 and 3 the influence of the interval width on the estimate m is less than 1%. By interpolating with a three-degree polynomial according to Table 9 we obtain that the relative differences $(m - m_M)/m_M$:

1.1% for interval widths $\alpha = 25°$, and

2.4% for interval widths $\alpha = 30°$.

Therefore, if 5% were adopted as the upper limit for the influence of the interval width on the standard estimate, then the interval widths could have a central angle of 30°.

According to the measurements for the orbit of binary WDS 04184+2135 (Table 1), the widths of intervals 2 and 7 are both 30°, so that their relative differences in the standard estimates are 2.4%, the widths of intervals 3 and 8 are 20.6° and 22.4°so that their differences are 1.1%, and for other intervals the widths are less than 16°.

7 Conclusions

The PEROBEPE1 method of measuring standard estimation (using stochastic noise defined by Eq. 1 uses a statistical test to find outliers in a single measurement with a hard threshold of gross errors. The significance level in the method is limited to 0.01.

Table 7: Classes and groups for 360 simulated measurements.

Interval Class q	Interval width α	Number of groups	Number of points within a group
1	12°	k = 30	12
2	15°	k = 24	15
3	20°	k = 18	20
4	36°	k = 10	36

Table 8: The F-test of homogeneity of two variances, a posteriori and a priori.

Class interval q	Width interval α	Number of groups	Estimate of standard m [arcsec]	D. f.	$F = m^2/0.005^2$		$F_{0.95;f,\infty}$
1	12°	k = 30	0.00518	291	1.071	<	1.140
2	15°	k = 24	0.00528	303	1.113	<	1.138
3	20°	k = 18	0.00531	317	1.128	<	1.134
4	36°	k = 10	0.00522	330	1.091	<	1.132

Table 9: Estimates of the standard in classes and the width interval influence on them.

Class q	α	Number of groups	m [arcsec]	m_{EL} [arcsec]	m_M [arcsec]	$(m\text{-}m_M)/m_M$ [%]
1	12°	k = 30	0.00518	0.000185	0.00517	0.06
2	15°	k = 24	0.00528	0.000285	0.00527	0.15
3	20°	k = 18	0.00531	0.000498	0.00529	0.44
4	36°	k = 10	0.00522	0.001578	0.00498	4.90

The standard estimate is affected by the interval width, but in the case of measuring the coordinates for the purpose of determining binary orbits this width can be determined *a priori* such that its influence becomes negligible. So, if a value of 1% is assumed for the negligibility limit, then in the case of binary WDS 04184+2135 the interval width can have a central angle up to 20°. Geodesists studying similar problems most frequently adopt 5% for this limit, so that the interval width can have a central angle of 36°.

On the basis of the results from Table 5 we can conclude the following: 1) the PEROBEPE1 method yields standard estimates better than those from a sequential Bonferroni-type procedure, and 2) this method yields results concordant with the results of the PEROBLS3 method. Additionally, on the basis of the results from Table 8, we can conclude that this method yields standard estimates close to the exact ones (F-test did not make any distinction).

Overall, we conclude that the PEROBEPE1 method can be used in the estimation of coordinate measurement standards when determining the orbits of binary stars.

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