Xuezhong Wang* and Cuixia Li

A note on the parallel GSAOR method for block diagonally dominant matrices

Abstract: Recently, Liu et al. [Q. B. Liu, G. L. Chen, J. Huang, On the parallel GSAOR method for block diagonally dominant matrices, *Appl. Math. Comput.* **215** (2009), 707–715] study the convergence of the parallel multisplitting generalized SAOR iterative methods based on the generalized AOR iterative method for solving a linear system whose coefficient matrix is a block diagonally dominant matrix or a generalized block diagonal dominant matrix. In this paper, we extend the domain of convergence from $1 \leqslant \omega_i < 2/(1+\mu_1(PMQ))$ and $1 \leqslant \omega_i < 2/(1+\mu_2(PMQ))$ to $0 < \omega_i < 2/(1+\mu_1(PMQ))$ and $0 < \omega_i < 2/(1+\mu_2(PMQ))$ for the parallel multisplitting generalized SAOR iterative methods.

Keywords: convergence, multisplitting, generalized SAOR method, block diagonally dominant matrices, generalized block diagonally dominant matrices

MSC 2010: 65F10, 65F50

DOI: 10.1515/jnma-2014-1001
Received January 15, 2014; accepted May 28, 2014

1 Introduction

For the linear system

$$Ax = b \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular, and $x, b \in \mathbb{R}^n$ are n-dimensional vectors. Let us consider the splitting of the matrix A of linear system (1.1):

$$A = D - L - U$$

where $D = \operatorname{diag}(A)$, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively. James [2] presented a generalized accelerated overrelaxation (GAOR) method given by

$$x^{m+1} = L_1(r, \Omega)x^m + (D - r\Omega L)^{-1}b, \qquad m = 0, 1, 2, \dots$$

or

$$x^{m+1} = U_1(r, \Omega)x^m + (D - r\Omega U)^{-1}b, \qquad m = 0, 1, 2, \dots$$

where

$$L_1(r, \Omega) = (D - r\Omega L)^{-1}\{(I-\Omega)D + (1-r)\Omega L + \Omega U\}$$

and

$$U_1(r, \Omega) = (D - r\Omega U)^{-1}\{(I-\Omega)D + (1-r)\Omega U + \Omega L\}$$

are iterative matrices and $\Omega = \operatorname{diag}(\omega_1, \omega_2 \dots, \omega_n)$ with $\omega_i \in R^+$ and $r \in [0, 1]$.

Then generalized symmetric AOR method (GSAOR) can be defined as follows [8].

$$x^{m+1} = Tx^m + Cb, \qquad m = 0, 1, 2, \dots$$

where $T = U_1(r, \Omega)L_1(r, \Omega)$ and $C = U_1(r, \Omega)(D - r\Omega L)^{-1}\Omega + (D - r\Omega U)^{-1}\Omega$.

In the following, we recall the mathematical descriptions of the block linear system and the BMM introduced in [5, 6].

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Let $s(\leqslant n)$ and $n_i(\leqslant n)$, $i = 1, 2, \dots, s$, be given positive integers satisfying $\sum_{i=1}^{s} n_i = n$, and denote

$$V_n(n_1, \dots, n_s) = \{x \in \mathbb{R}^n | x = (x_1^T, \dots, x_s^T)^T, x_i \in \mathbb{R}^{n_i}\}$$

$$\mathbb{L}_n(n_1, \dots, n_s) = \{A \in \mathbb{R}^{n\times n} | A = (A_{ij})_{s\times s}, A_{ij} \in \mathbb{R}^{n_i\times n_j}\}$$

for the convenience, we will simply use $\mathbb{L}_n$ for $\mathbb{L}_n(n_1, \dots, n_s)$ and V_n for $V_n(n_1, \dots, n_s)$. Then, the block linear system to be solved can be expressed as the form

$$Ax = b, \qquad A \in \mathbb{L}_n, \qquad x, b \in V_n \tag{1.2}$$

where $A \in \mathbb{L}_n$ is nonsingular and $b \in V_n$ are general known coefficient matrix and right-hand vector, respectively, and $x \in V_n$ is the unknown vector.

If block matrices $M_k, N_k, E_k \in \mathbb{L}_n$, $k = 1, 2, \dots, \alpha$, satisfy:

1) $A = M_k - N_k$, M_k nonsingular, $k = 1, 2, \dots, \alpha$;
2) $E_k = \operatorname{diag}(E_{11}^{(k)}, \dots, E_{ss}^{(k)})$, $k = 1, 2, \dots, \alpha$;
3) $\sum_{k=1}^{\alpha} ||E_{ii}^{(k)}|| = 1$, $i = 1, 2, \dots, s$;

then we call the collection of triples (M_k, N_k, E_k), $k = 1, 2, \dots, \alpha$, is a block matrices multisplitting (BMM) of the block matrix $A \in \mathbb{L}_n$, where $||\cdot||$ denotes the consistent matrix norm.

O'Leary and White [4] invented the matrix multisplitting method in 1985 for solving parallely the large sparse linear systems on the multiprocessor systems and was further studied by many authors [1,5–14,17–citenum19].

Suppose that we have a multiprocessor with α processors connected to a host processor, that is, the same number of processors as splittings, and that all processors have the last update vector x^k, then the kth processor only computes those entries of the vector

$$M_k^{-1} N_k x^k + M_k^{-1} b$$

which correspond to the block diagonal entries $E_{ii}^{(k)}$ of the block matrix E_k. The processor then scales these entries so as to be able to deliver the vector

$$E_k(M_k^{-1} N_k x^k + M_k^{-1} b)$$

to the host processor, performing the parallel multisplitting scheme

$$x^{m+1} = \sum_{k=1}^{\alpha} E_k M_k^{-1} N_k x^m + \sum_{k=1}^{\alpha} E_k M_k^{-1} b = Hx^m + Gb, \qquad m = 0, 1, 2, \dots .$$

In this paper, we investigate the domain of convergence of block GSAOR multisplittings methods for solving linear system (1.1). When the coefficient matrix A is a block H-matrix or a block strictly diagonally dominant matrix.

2 Parallel multisplitting GSAOR methods

Given a positive integer α ($\alpha \leqslant s$), we separate the number set $\{1, 2, \dots, s\}$ into a nonempty subsets J_k, $k = 1, 2, \dots, \alpha$, such that

$$J_k \subseteq \{1, 2, \dots, s\}, \qquad \bigcup_{k=1}^{\alpha} J_k = \{1, 2, \dots, s\}.$$

Note that there may be overlappings among the subsets $J_1, J_2, \dots, J_\alpha$. Corresponding to this separation, we introduce matrices

$$
\begin{aligned}
& D = \operatorname{diag}(A_{11}, \dots, A_{ss}) \in \mathbb{L}_n \\
& L_k = (\mathcal{L}_{ij}^{(k)}) \in \mathbb{L}_n, \quad \mathcal{L}_{ij}^{(k)} = \begin{cases} L_{ij}^{(k)}, & i, j \in J_k, \quad i > j \\ 0 & \text{otherwise} \end{cases} \\
& U_k = (\mathcal{U}_{ij}^{(k)}) \in \mathbb{L}_n, \quad \mathcal{U}_{ij}^{(k)} = \begin{cases} U_{ij}^{(k)}, & i \neq j \\ 0 & \text{otherwise} \end{cases} \\
& E_k = \operatorname{diag}(E_{11}^{(k)}, \dots, E_{ss}^{(k)}) \in \mathbb{L}_n, \quad E_{ii}^{(k)} = \begin{cases} E_{ii}^{(k)}, & i \in J_k \\ 0 & \text{otherwise} \end{cases} \\
& \quad i, j = 1, 2, \dots, s, \qquad k = 1, 2, \dots, \alpha.
\end{aligned}
$$

Obviously, D is a block diagonal matrix, $L_k, k = 1, 2, \dots, \alpha$, are block strictly lower triangular matrices, $U_k, k = 1, 2, \dots, \alpha$, are general block matrices, and $E_k, k = 1, 2, \dots, \alpha$, are block diagonal matrices. If they satisfy:

1) D is nonsingular;
2) $A = D - L_k - U_k, k = 1, 2, \dots, \alpha$;
3) $\sum_{k=1}^{\alpha} E_k = I$;

then the collection of triples $(D-U_k, L_k, E_k)$ and $(D-L_k, U_k, E_k)$, $k = 1, 2, \dots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_n$. Here, I denotes the identity matrix of order $n \times n$.

Let (M_k, N_k, E_k), $k = 1, 2, \dots, \alpha$, is a BMM of the block matrix $A \in \mathbb{L}_n$. We will definite local parallel multisplittings blockwise relaxation generalized SAOR method (LMBGSAOR) and global parallel multisplittings blockwise relaxation generalized SAOR method (GMBGSAOR).

Algorithm 2.1 (local parallel multisplittings blockwise relaxation method).

Given the initial vector.
For $m = 0, 1, 2, \dots$ repeat (I) and (II), until convergence.
(I) For $k = 1, 2, \dots, \alpha$, (parallel) solving y_k:

$$M_k y_k = N_k x^m + b.$$

(II) Computing

$$x^{m+1} = \sum_{k=1}^{\alpha} E_k y_k.$$

Algorithm 2.1 associated with LMBGSAOR method can be written as

$$x^{m+1} = H_{\text{LMBGSAOR}} x^m + G_{\text{LMBGSAOR}} b, \qquad m = 0, 1, \dots \tag{2.1}$$

where

$$
\begin{aligned}
H_{\text{LMBGSAOR}} &= \sum_{k=1}^{\alpha} E_k U^{(k)}(A) L^{(k)}(A) \\
U^{(k)}(A) &= (D - r\Omega U_k)^{-1}\{(I - \Omega)D + (1 - r)\Omega U_k + \Omega L_k\} \\
L^{(k)}(A) &= (D - r\Omega L_k)^{-1}\{(I - \Omega)D + (1 - r)\Omega L_k + \Omega U_k\} \\
G_{\text{LMBGSAOR}} &= \sum_{k=1}^{\alpha} E_k\{U^{(k)}(A)(D - r\Omega L_k)^{-1}\Omega + (D - r\Omega U_k)^{-1}\Omega\}.
\end{aligned} \tag{2.2}
$$

By using a suitable positive relaxation parameter β, we will establish global parallel multisplitting blockwise relaxation GSAOR method which is based on Algorithm 2.1.

Algorithm 2.2 (global parallel multisplittings blockwise relaxation method).

Given the initial vector.

For $m = 0, 1, 2, \dots$ repeat (I) and (II), until convergence.

(I) For $k = 1, 2, \dots, \alpha$, (parallel) solving y_k:

$$M_k y_k = N_k x^m + b.$$

(II) Computing

$$x^{m+1} = \beta \sum_{k=1}^{\alpha} E_k y_k + (1-\beta)x^m.$$

Algorithm 2.2 associated with GMBGSAOR method can be written as

$$x^{m+1} = H_{\text{GMBGSAOR}} x^m + \beta G_{\text{GMBGSAOR}} b, \qquad m = 0, 1, \dots \tag{2.3}$$

where $H_{\text{GMBGSAOR}} = \beta H_{\text{GMBGSAOR}} + (1-\beta)I$.

3 Preliminaries

We shall use the following notations and Lemmas. A matrix $A = (a_{ij})$ is called a *Z-matrix* if for any $i \neq j$, $a_{ij} \leqslant 0$. A *Z-matrix* is a nonsingular *M-matrix* if A is nonsingular and $A^{-1} \geqslant 0$. Additionally, we denote the spectral radius of A by $\rho(A)$. It is well-known that if $A \geqslant 0$ and there exists a vector $x > 0$ such that $Ax < \alpha x$, then $\rho(A) < \alpha$ [3]. Let

$$\mathbb{L}_{n,I}(n_1, \dots, n_s) = \{A = (A_{ij}) \in \mathbb{R}^{n\times n} \mid A_{ii} \in \mathbb{R}^{n_i \times n_i} \text{ nonsingular}, \quad i = 1, \dots, s\}$$
$$\mathbb{L}^d_{n,I}(n_1, \dots, n_s) = \{A = \operatorname{diag}(A_{11}), (A_{22}), \dots, (A_{ss}) \mid A_{ii} \in \mathbb{R}^{n_i \times n_i} \text{ nonsingular}, \quad i = 1, \dots, s\}.$$

We will review the concepts of strictly block diagonally dominant matrix and block H-matrix.

Definition 3.1 ([1, 20]). Let $A \in \mathbb{L}_{n,I}$, (I) block comparison matrix $\mathcal{M}(A) = ((\mathcal{M}(A))_{ij}) \in \mathbb{R}^{s\times s}$ and (II) block comparison matrix $\mathcal{N}(A) = ((\mathcal{N}(A))_{ij}) \in \mathbb{R}^{s\times s}$ are defined respectively as follows:

$$(\mathcal{M}(A))_{ij} = \begin{cases} \|A_{ij}^{-1}\|^{-1}, & i = j, \\ -\|A_{ij}\|, & i \neq j, \end{cases} \qquad i, j = 1, \dots, s$$

$$(\mathcal{N}(A))_{ij} = \begin{cases} 1, & i = j, \\ -\|A_{ij}^{-1} A_{ij}\|, & i \neq j, \end{cases} \qquad i, j = 1, \dots, s$$

where $\|\cdot\|$ is a consistent matrix norm such that $\|I\| = 1$.

For block matrices $A \in \mathbb{L}_{n,I}$, we define $D(A) = \operatorname{diag}(A_{11}, A_{22}, \dots, A_{ss})$, $B(A) = D(A) - A$, $J(A) = D(A)^{-1}B(A)$, $\mu_1(A) = \rho(J_{\mathcal{M}(A)})$, $\mu_2(A) = \rho(I - \mathcal{N}(A))$. In [1], Liu et al. show that $\mathcal{M}(I - J(M)) = \mathcal{N}(I - J(M))$, $\mu_2(A) \leqslant \mu_1(A)$.

Definition 3.2 ([15, 16]). Let $A \in \mathbb{L}_{n,I}$. A matrix A is said to be a strictly (I) block diagonally dominant matrix, if

$$\|A_{ii}^{-1}\|^{-1} > \sum_{i\neq j} \|A_{ij}\|, \qquad j = 1, 2, \dots, s.$$

A matrix A is said to be a strictly (II) block diagonally dominant matrix, if

$$\sum_{i\neq j} \|A_{ii}^{-1} A_{ij}\| < 1, \qquad j = 1, 2, \dots, s.$$

Remark 3.1. From Definition 3.2, we know a strictly (I) block diagonally dominant matrix must be a strictly (II) block diagonally dominant matrix, but not conversely.

Definition 3.3 ([1, 15, 16]). Let $A \in \mathbb{L}_{n,I}$. A matrix A is said to be a (I) block H-matrix ($H_B^{(I)}(P,Q)$-matrix) with respect to nonsingular matrices P and Q if there are two matrices $P, Q \in \mathbb{L}_{n,I}^d$ such that $\mathcal{M}(PAQ)$ is an M-matrix; A matrix A is said to be a (II) block H-matrix ($H_B^{(II)}(P,Q)$-matrix) with respect to nonsingular matrices P and Q if $\mathcal{N}(PAQ)$ is an M-matrix.

Remark 3.2 ([1]). From Definition 3.3, we obtain a (I) block H-matrix ($H_B^{(I)}(P,Q)$-matrix) must be a (II) block H-matrix ($H_B^{(II)}(P,Q)$-matrix), but conversely, it is not true.

Combining Remarks 3.1 and 3.2, we have

Remark 3.3. A strictly (I) block diagonally dominant matrix must be $H_B^{(I)}(P,Q)$-matrix; A strictly (II) block diagonally dominant matrix must be a (II) block H-matrix $H_B^{(II)}(P,Q)$-matrix.

Definition 3.4 ([15]). If there exists a block diagonal matrix X such that AX is a strictly block diagonally dominant matrix, then A is said to be a block H-matrix ($H_B^{(I)}(P,Q)$-matrix and $H_B^{(II)}(P,Q)$-matrix).

Definition 3.5 ([1, 5, 6]). Let $A \in \mathbb{L}_n$. We call $[A] = (\|A_{ij}\|) \in \mathbb{R}^{N\times N}$ the block absolute value of the block matrix M. The block absolute value $[x] \in \mathbb{R}^N$ of a block vector $x \in V_n$ is defined in an analogous way.

These kinds of block absolute values have the following important properties.

Lemma 3.1 ([1, 5, 6]). *Let $L, M \in \mathbb{L}_n$, $x, y \in V_n$ and $r \in \mathbb{R}^1$. Then*
1) $|[L]-[M]| \leqslant [L+M] \leqslant [L]+[M]$ $(|[x]-[y]| \leqslant [x+y] \leqslant [x]+[y])$;
2) $[LM] \leqslant [L][M]$ $([xy] \leqslant [x][y])$;
3) $[rM] \leqslant |r|[M]$ $([rx] \leqslant |r|[x])$;
4) $\rho(M) \leqslant \rho(|M|) \leqslant \rho([M])$ *(here, $\|\cdot\|$ is either $\|\cdot\|_\infty$ or $\|\cdot\|_1$).*

Lemma 3.2 ([5, 6]). *Let $A \in \mathbb{L}_{n,I}$ be a strictly block diagonally dominant matrix, then*
1) *A is nonsingular;*
2) $[(A)^{-1}] \leqslant \mathcal{M}(A)^{-1}$;
3) $\rho(J(\mathcal{M}(A))) < 1$.

Let

$$\Omega_B^I(A) = \left\{F = (F_{ij}) \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s) \mid \|F_{ii}^{-1}\| = \|A_{ii}^{-1}\|, \quad \|F_{ij}\| = \|A_{ij}\|, \quad i \neq j, \quad i,j = 1,2,\ldots,s\right\}$$
$$\Omega_B^{II}(A) = \left\{F = (F_{ij}) \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s) \mid \|F_{ii}^{-1}F_{ij}\| = \|A_{ii}^{-1}A_{ij}\|, \quad i,j = 1,2,\ldots,s\right\}$$

denote respectively the set of (I) and (II) matrices such that the absolute values of whose elements are equal to absolute values of corresponding elements of matrix A.

4 Main results

For the present Algorithms 2.1 and 2.2, we give convergence theorems for block diagonally dominant matrices and block H-matrices.

Theorem 4.1. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$ be a strictly (I) block diagonally dominant matrix, $A \in \Omega_B^I(PMQ)$ and the collection of triples $(D-U_k, L_k, E_k)$ and $(D-L_k, U_k, E_k)$, $k = 1, 2, \ldots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \ldots, \alpha \tag{4.1}$$

if

$$0 < \omega_i < \frac{2}{1+\mu_1(PMQ)}, \qquad i = 1, 2, \ldots, n$$

then LMBGSAOR method converges for any initial vector $x^0 \in V_s$.

Proof. By Lemma 3.1, we know

$$\rho(H_{\mathrm{LMBGSAOR}}) \leqslant \rho(|H_{\mathrm{LMBGSAOR}}|) \leqslant \rho([H_{\mathrm{LMBGSAOR}}])$$

and then, the iteration (2.1) converges for any initial vector $x^0 \in V_N$ if and only if

$$\rho([H_{\mathrm{LMBGSAOR}}]) < 1.$$

Since $A \in \Omega_B^I(PMQ)$, then $\mathcal{M}(A) = \mathcal{M}(PMQ)$. Because

$$A \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$$

is a strictly (I) block diagonally dominant matrix, so $A \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$ is a strictly (I) block diagonally dominant matrix and $\mu_1 = \rho(J(\mathcal{M}(A))) = \rho(J(\mathcal{M}(PMQ))) = \mu_1(PMQ)$, and thus, we have $\mu_1 = \rho(J(\mathcal{M}(PMQ))) = \mu_1(PMQ) < 1$, it follows from Lemma 3.2. Let $B = L_k + U_k$, by (2.2), we know that $[B] = [L_k] + [U_k]$, $k = 1, 2, \ldots, \alpha$, clearly, $D - r\Omega L_k$, $k = 1, 2, \ldots, \alpha$, are strictly (I) block diagonally dominant matrix and $\mathcal{M}(D) - r[\Omega][B]$ are strictly diagonally dominant matrix for $0 < \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \ldots, n$, and $r \in [0, 1]$ which follow from A is a strictly (I) block diagonally dominant matrix. Since

$$\mathcal{M}(D) - r[\Omega][B] \leqslant \mathcal{M}(D) - r[\Omega][U_k] \leqslant \mathcal{M}(D)$$

for $0 < \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \ldots, n$, $r \in [0, 1]$, $k = 1, 2, \ldots, \alpha$, and $\mathcal{M}(A)$ is a strictly diagonally dominant matrix, we have $\mathcal{M}(D) - r[\Omega][B]$ and $\mathcal{M}(D)$ are strictly diagonally dominant M-matrices, for $0 < \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \ldots, n$, $r \in [0, 1]$. Therefore, $\mathcal{M}(D) - r[\Omega][U_k]$ are strictly diagonally dominant M-matrices, and then $D - r\Omega U_k$, $k = 1, 2, \ldots, \alpha$, are strictly (I) block diagonally dominant matrices, for $0 < \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \ldots, n$, and $r \in [0, 1]$.

Let $\overline{L}_k = D^{-1}L_k$ and $\overline{U}_k = D^{-1}U_k$, then $I - r\Omega\overline{L}_k$ and $I - r\Omega\overline{U}_k$ are also strictly (I) block diagonally dominant matrices, for $0 < \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \ldots, n$, $r \in [0, 1]$, $k = 1, 2, \ldots, \alpha$. Thus, by Lemma 3.1, we have

$$\leqslant (\mathcal{M}(I - r\Omega\overline{L}_k))^{-1} = (I - r[\Omega][\overline{L}_k])^{-1}$$
$$[(I - r\Omega\overline{U}_k)^{-1}] \leqslant (\mathcal{M}(I - r\Omega\overline{U}_k))^{-1} = (I - r[\Omega][\overline{U}_k])^{-1}.$$

From (4.1), we have

$$\begin{aligned}
[U^{(k)}(A)] &= [(D - r\Omega U_k)^{-1}\{(I - \Omega)D + (1 - r)\Omega U_k + \Omega L_k\}] \\
&= [(I - r\Omega\overline{U}_k)^{-1}\{I - \Omega + (1 - r)\Omega\overline{U}_k + \Omega\overline{L}_k\}] \\
&\leqslant [(I - r\Omega\overline{U}_k)^{-1}]\{[I - \Omega] + (1 - r)[\Omega\overline{U}_k] + [\Omega\overline{L}_k]\} \\
&\leqslant (I - r[\Omega][\overline{U}_k])^{-1}\{[I - \Omega] + (1 - r)[\Omega][\overline{U}_k] + [\Omega][\overline{L}_k]\} \\
&= I + (I - r[\Omega][\overline{U}_k])^{-1}\{[I - \Omega] - I + [\Omega][\overline{U}_k + \overline{L}_k]\}.
\end{aligned}$$

Since $\overline{L}_k = D^{-1}L_k$ and $\overline{U}_k = D^{-1}U_k$, we have $[\overline{L}_k] \leqslant \mathcal{M}(D)^{-1}[L_k]$ and $[\overline{U}_k] \leqslant \mathcal{M}(D)^{-1}[U_k]$ which follow from Lemma 3.1 and Lemma 3.2, and then

$$[\overline{U}_k] + [\overline{L}_k] \leqslant \mathcal{M}(D)^{-1}[U_k + L_k] = \mathcal{M}(D)e^{-1}[B] = J(\mathcal{M}(A)), \qquad k = 1, 2, \ldots, \alpha.$$

Therefore, we have

$$[U^{(k)}(A)] \leqslant I - (I - r[\Omega][\overline{U}_k])^{-1}(I - T([\Omega]))$$

where $T([\Omega]) = [I - \Omega] + [\Omega]J(\mathcal{M}(A))$. Note that $(I - r[\Omega][\overline{U}_k(k)])^{-1} \geqslant I$, $k = 1, 2, \ldots, \alpha$, and then

$$[U^{(k)}(A)] \leqslant I - I - T([\Omega])) = T([\Omega]).$$

Similar to the above proving process, we have

$$[L^{(k)}(A)] \leqslant I - (I - T([\Omega])) = T([\Omega]).$$

Let $\vartheta_1 = \max\{\omega_i\}$, $\vartheta_2 = \min\{\omega_i\}$ and $f(t) = |1-t|I + tJ(\mathcal{M}(A))$, where $t > 0$. Obviously, $f(t)$ is nonincreasing for $0 < t \leqslant 1$ and $f(t)$ is nondecreasing for $t \geqslant 1$. Therefore, we have

$$T([\Omega]) \leqslant (1-\vartheta_2)I + \vartheta_2 J(\langle A\rangle), \qquad 0 < \omega_i \leqslant 1, \quad i = 1, 2, \ldots, n$$
$$T([\Omega]) \leqslant (\vartheta_1 - 1)I + \vartheta_1 J(\mathcal{M}(A)), \qquad 1 < \omega_i < \frac{2}{1+\mu_1(PMQ)}, \quad i = 1, 2, \ldots, n.$$

Let e denotes the vector $e = (1, 1, \ldots, 1)^T \in V_s$ and $J_\varepsilon(\mathcal{M}(A)) = J(\mathcal{M}(A)) + \varepsilon ee^T$, since $J(\mathcal{M}(A))$ is nonnegative, the matrix $J_\varepsilon(\mathcal{M}(A))$ has only positive entries and irreducible for any $\varepsilon > 0$. By the Perron-Frobenius theorem [3] for any $\varepsilon > 0$, there exists a vector $x_\varepsilon > 0$ such that

$$J_\varepsilon(\mathcal{M}(A))x_\varepsilon = \rho(J_\varepsilon(\mathcal{M}(A)))x_\varepsilon = \rho_\varepsilon x_\varepsilon$$

where $\rho_\varepsilon = \rho(J_\varepsilon(\mathcal{M}(A)))$. Moreover, if $\varepsilon > 0$ is small enough, we have $\rho_\varepsilon < 1$ by continuity of the spectral radius. Thus, we can get $\vartheta_1 - 1 + \vartheta_1\rho_\varepsilon < 1$ and $1 - \vartheta_2 + \vartheta_2\rho_\varepsilon < 1$. And then

$$[H_{\mathrm{LMBGSAOR}}]x_\varepsilon \leqslant \sum_{k=1}^{\alpha}[E_k][U^{(k)}(A)][L^{(k)}(A)]x_\varepsilon \leqslant \sum_{k=1}^{\alpha}[E_k]T([\Omega])^2 x_\varepsilon.$$

Step 1. For $0 < \omega_i \leqslant 1$, $i = 1, 2, \ldots, n$,

$$[H_{\mathrm{LMBGSAOR}}]x_\varepsilon \leqslant (1-\vartheta_2+\vartheta_2\rho_\varepsilon)^2 x_\varepsilon < x_\varepsilon;$$

Step 2. For $1 < \omega_i < 2/(1+\mu_1(PMQ))$, $i = 1, 2, \ldots, n$,

$$[H_{\mathrm{LMBGSAOR}}]x_\varepsilon \leqslant (\vartheta_1 - 1 + \vartheta_1\rho_\varepsilon)^2 x_\varepsilon < x_\varepsilon.$$

Then, we have $[H_{\mathrm{LMBGSAOR}}]x_\varepsilon < x_\varepsilon$ and $\rho([H_{\mathrm{LMBGSAOR}}]) < 1$. □

Theorem 4.2. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$ be an $H_B^{(I)}(P, Q)$-matrix, $A \in \Omega_B^I(PMQ)$, and the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$, $k = 1, 2, \ldots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \ldots, \alpha.$$

If

$$0 < \omega_i < \frac{2}{1+\mu_1(PMQ)}, \qquad i = 1, 2, \ldots, n$$

then LMBGSAOR method converges for any initial vector $x^0 \in V_s$.

Proof. Since $A \in \Omega_B^I(PMQ)$, then $\mathcal{M}(A) = \mathcal{M}(PMQ)$. Because $M \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$ is an $H_B^{(I)}(P, Q)$-matrix, so $A \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$ is an block $H_B^{(I)}(I, I)$-matrix. From Definition 3.5, there exists a block diagonal matrix X such that AX is a strictly (I) block diagonally dominant matrix. Let $H_{\mathrm{LMBGSAOR}}(A)$ and $H_{\mathrm{LMBGSAOR}}(AX)$ denote the iterative matrices of LMBGSAOR methods for block matrix A and AX, respectively. By simple calculation, we have $H_{\mathrm{LMBGSAOR}}(A)$ and $H_{\mathrm{LMBGSAOR}}(AX)$ are similar. Since similar matrices have the same eigenvalues, it follows that $\rho(H_{\mathrm{LMBGSAOR}}(A)) = \rho(H_{\mathrm{LMBGSAOR}}(AX)) < 1$. □

Theorem 4.3. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$ be a strictly (II) block diagonally dominant matrix, $A \in \Omega_B^{II}(PMQ)$ and the collection of triples $(D - U_k, L_k, E_k)$ and $(D - L_k, U_k, E_k)$, $k = 1, 2, \ldots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \ldots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \ldots, \alpha.$$

If

$$0 < \omega_i < \frac{2}{1+\mu_2(PMQ)}, \qquad i = 1, 2, \ldots, n$$

then LMBGSAOR method converges for any initial vector $x^0 \in V_s$.

Proof. The proof goes along the same lines as in the Theorem 4.1 except that strictly (I) block diagonally dominant matrix and $\Omega_B^I(PMQ)$ play the roles of strictly (II) block diagonally dominant matrix and $\Omega_B^{II}(PMQ)$, respectively, which completes the proof. □

Theorem 4.4. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$ be an $H_B^{(II)}(P,Q)$-matrix, $A \in \Omega_B^{II}(PMQ)$, and the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$, $k = 1, 2, \dots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \dots, \alpha.$$

If

$$0 < \omega_i < \frac{2}{1 + \mu_2(PMQ)}, \qquad i = 1, 2, \dots, n$$

then LMBGSAOR method converges for any initial vector $x^0 \in V_s$.

Proof. The proof is similar to that given in Theorem 4.2, so omitted. □

Using GMBGSAOR method, we can also get the following convergence results.

Theorem 4.5. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$ be a strictly (I) block diagonally dominant matrix, $A \in \Omega_B^I(PMQ)$ and the collection of triples $(D - U_k, L_k, E_k)$ and $(D - L_k, U_k, E_k)$, $k = 1, 2, \dots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \dots, \alpha$$

(a) if $0 < \omega_i \leqslant 1$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_2^2)$, $h_2 = 1 - \vartheta_2 + \vartheta_2\mu_1(PMQ)$;
(b) if $1 \leqslant \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_1^2)$, $h_1 = \vartheta_1 - 1 + \vartheta_1\mu_1(PMQ)$.
Then GMBGSAOR method converges for any initial vector $x^0 \in V_s$, where $\vartheta_1 = \max\{\omega_i\}$, $\vartheta_2 = \min\{\omega_i\}$.

Proof. Since $\rho(H_{\mathrm{GMBGSAOR}}) \leqslant \rho(|H_{\mathrm{GMBGSAOR}}|) \leqslant \rho([H_{\mathrm{GMBGSAOR}}])$, the iteration (2.3) converges for any initial vector $x^0 \in V_s$ if and only if

$$\rho([H_{\mathrm{GMBGSAOR}}]) < 1.$$

Similar to the proof of Theorem 4.1, we have

$$\mu_1 = \rho(J(\mathcal{M}(A))) = \mu_1(PMQ) < 1$$

and there exists $\varepsilon > 0$ such that $J_\varepsilon(\mathcal{M}(A)) = J(\mathcal{M}(A)) + \varepsilon ee^T$ has only positive entries and irreducible for any $\varepsilon > 0$. By the Perron-Frobenius theorem for any $\varepsilon > 0$, there exists a vector $x_\varepsilon > 0$ such that

$$J_\varepsilon(\mathcal{M}(A))x_\varepsilon = \rho(J_\varepsilon(\mathcal{M}(A)))x_\varepsilon = \rho_\varepsilon x_\varepsilon$$

where $\rho_\varepsilon = \rho(J_\varepsilon(\mathcal{M}(A)))$. Moreover, if $\varepsilon > 0$ is small enough, we have $\rho_\varepsilon < 1$, by continuity of the spectral radius. Under the condition of Theorem 4.5, we not only get $h_1 = \vartheta_1 - 1 + \vartheta_1\rho_\varepsilon < 1$ and $h_2 = 1 - \vartheta_2 + \vartheta_2\rho_\varepsilon < 1$, but also $\beta h_1^2 + |1 - \beta| < 1$, and $\beta h_2^2 + |1 - \beta| < 1$, and then

$$\begin{aligned} x_\varepsilon &\leqslant \beta \sum_{k=1}^{\alpha} [E_k][U^{(k)}(A)][L^{(k)}(A)]x_\varepsilon + |1 - \beta|x_\varepsilon \\ &\leqslant \beta \sum_{k=1}^{\alpha} [E_k]T([\Omega])^2 x_\varepsilon + |1 - \beta|x_\varepsilon. \end{aligned}$$

Case (a). For $0 < \omega_i \leqslant 1$, $i = 1, 2, \dots, n$.

$$[H_{\mathrm{LMBGSAOR}}]x_\varepsilon \leqslant \beta(1 - \vartheta_2 + \vartheta_2\rho_\varepsilon)^2 x_\varepsilon + |1 - \beta|x_\varepsilon = (\beta h_2^2 + |1 - \beta|)x_\varepsilon < x_\varepsilon.$$

Case (b). For $1 < \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \dots, n$.

$$[H_{\mathrm{LMBGSAOR}}]x_\varepsilon \leqslant \beta(\vartheta_1 - 1 + \vartheta_1\rho_\varepsilon)^2 x_\varepsilon + |1 - \beta|x_\varepsilon = (\beta h_1^2 + |1 - \beta|)x_\varepsilon < x_\varepsilon.$$

Then $[H_{\text{GMBGSAOR}}]x_\varepsilon < x_\varepsilon$ and $\rho([H_{\text{GMBGSAOR}}]) < 1$. □

Theorem 4.6. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$ be an $H_B^{(I)}(P, Q)$-matrix, $A \in \Omega_B^I(PMQ)$, and the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$, $k = 1, 2, \dots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \dots, \alpha$$

(a) if $0 < \omega_i \leqslant 1$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_2^2)$, $h_2 = 1 - \vartheta_2 + \vartheta_2\mu_1(PMQ)$;
(b) if $1 \leqslant \omega_i < 2/(1 + \mu_1(PMQ))$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_1^2)$, $h_1 = \vartheta_1 - 1 + \vartheta_1\mu_1(PMQ)$.
Then GMBGSAOR method converges for any initial vector $x^0 \in V_s$, where $\vartheta_1 = \max\{\omega_i\}$, $\vartheta_2 = \min\{\omega_i\}$.

Theorem 4.7. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$ be a strictly (II) block diagonally dominant matrix, $A \in \Omega_B^{II}(PMQ)$ and the collection of triples $(D - U_k, L_k, E_k)$ and $(D - L_k, U_k, E_k)$, $k = 1, 2, \dots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \dots, \alpha$$

(a) if $0 < \omega_i \leqslant 1$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_2^2)$, $h_2 = 1 - \vartheta_2 + \vartheta_2\mu_2(PMQ)$;
(b) if $1 \leqslant \omega_i < 2/(1 + \mu_2(PMQ))$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_1^2)$, $h_1 = \vartheta_1 - 1 + \vartheta_1\mu_2(PMQ)$.
Then GMBGSAOR method converges for any initial vector $x^0 \in V_s$, where $\vartheta_1 = \max\{\omega_i\}$, $\vartheta_2 = \min\{\omega_i\}$.

Theorem 4.8. *Let $M \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$ be an $H_B^{(II)}(P, Q)$-matrix, $A \in \Omega_B^{II}(PMQ)$, and the collection of triples $(D - L_k, U_k, E_k)$ and $(D - U_k, L_k, E_k)$, $k = 1, 2, \dots, \alpha$, are BMM of the block matrix $A \in \mathbb{L}_{n,I}(n_1, n_2, \dots, n_s)$. Assume that*

$$\mathcal{M}(A) = \mathcal{M}(D) - [L_k] - [U_k] = \mathcal{M}(D) - [B], \qquad k = 1, 2, \dots, \alpha$$

(a) if $0 < \omega_i \leqslant 1$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_2^2)$, $h_2 = 1 - \vartheta_2 + \vartheta_2\mu_2(PMQ)$;
(b) if $1 \leqslant \omega_i < 2/(1 + \mu_2(PMQ))$, $i = 1, 2, \dots, n$, and $0 < \beta < 2/(1 + h_1^2)$, $h_1 = \vartheta_1 - 1 + \vartheta_1\mu_2(PMQ)$.
Then GMBGSAOR method converges for any initial vector $x^0 \in V_s$, where $\vartheta_1 = \max\{\omega_i\}$, $\vartheta_2 = \min\{\omega_i\}$.

Proof. Similar to the proof of Theorem 4.2, Theorem 4.3, and Theorem 4.4, we can prove Theorem 4.6, Theorem 4.7, and Theorem 4.8, respectively. So omitted. □

5 Conclusion

Liu *et al.* [1] consider the convergence of block parallel multisplittings GSAOR iterative methods for $1 \leqslant \omega_i < 2/(1 + \mu_1(PMQ))$ or $1 \leqslant \omega_i < 2/(1 + \mu_2(PMQ))$, $i = 1, 2, \dots, n$. In this paper, we extend interval of ω_i, $i = 1, 2, \dots, n$, to $(0, 2/(1 + \mu_1(PMQ)))$ or $(0, 2/(1 + \mu_2(PMQ)))$ for block parallel multisplittings GSAOR iterative methods.

Acknowledgment: We express our thanks to the anonymous referees who made much useful and detailed suggestions that helped us to correct some minor errors and improve the quality of the paper.

References

[1] Q. B. Liu, G. L. Chen, and J. Huang, On the parallel GSAOR method for block diagonally dominant matrices, *Appl. Math. Comput.*, **215** (2009), 707–715.
[2] K. R. James, Convergence of matrix iterations subject to diagonal dominance, *SIAM J. Numer. Anal.*, **10** (1986), 117–132.
[3] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, PA, 1994.
[4] D. P. O'Leary and R. E. White, Multi-splittings of matrices and parallel solution of linear systems, *SIAM J. Alg. Disc. Math.*, **6** (1985), 630–640.
[5] Z. Z. Bai, Parallel matrix multisplitting block relaxation iteration methods, *Math. Numer. Sinica*, **17** (1995), 238–252.

[6] Z. Z. Bai, A class of asynchronous parallel multisplitting blockwise relaxation methods, *Parallel Computing*, **25** (1999), 681–701.
[7] M. Neumann and R. J. Plemmons, Convergence of parallel multisplitting iterative methods for M-matrices, *Lin. Alg. Appl.*, **88-89** (1987), 559–573.
[8] L. T. Zhang, T. Z. Huang, T. X. Gu, and X. L. Guo, Convergence of relaxed multisplitting USAOR methods for H-matrices linear systems, *Appl. Math. Comput.*, **202** (2008), 121–132.
[9] L. Elsner, Comparisons of weak regular splittings and multisplitting methods, *Numer. Math.*, **56** (1989), 283–289.
[10] A. Frommer and G. Mayer, Convergence of relaxed parallel multisplitting methods, *Lin. Alg. Appl.*, **119** (1989), 141–152.
[11] R. E. White, Multisplitting of a symmetric positive deRnite matrix, *SIAM J. Matrix Anal. Appl.*, **11** (1990), 69–82.
[12] R. E. White, Multisplitting with different weighting schemes, *SIAM J. Matrix Anal. Appl.*, **10** (1989), 481–493.
[13] W. Li, Comparison results for parallel multisplitting methods with applications to AOR methods, *Lin. Alg. Appl.*, **206** (2008), 738–747.
[14] D. R. Wang, On the convergence of the parallel multisplitting AOR algorithm, *Lin. Alg. Appl.*, **154-156** (1991), 473–486.
[15] D. G. Feingold and R. S. Varga, Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem, *Pacific J. Math.*, **12** (1962), 1241–1249.
[16] B. Polman, Incomplete blockwise factorizations of (block) H-matrices, *Lin. Alg. Appl.*, **90** (1987), 119–132.
[17] R. Bru, V. Migallon, J. Penadés, and D. B. Szyld, Parallel synchronous and asynchronous two-stage multisplitting methods, *Elec. Tran. Num. Anal.*, **3** (1995), 24–38.
[18] D. B. Szyld, Synchronous and asynchronous two-stage multisplitting methods, in: *Proc. 5th SIAM Conference on Applied Linear Algebra*, (Ed. J. G. Lewis), SIAM, Philadelphia, 1994.
[19] J. Mos and D. B. Szyld, Nonstationary Parallel Relaxed Multisplitting Methods, *Lin. Alg. Appl.*, **241-243** (1996), 733–747.
[20] Y. Z. Song, On the convergence of GAOR methods, *Math. Numer. Sinica*, **4** (1989), 405–412 (in Chinese).