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Fractional vector calculus and fluid mechanics

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Abstract: Basic fluid mechanics equations are studied and revised under the prism of fractional continuum mechanics (FCM), a very promising research field that satisfies both experimental and theoretical demands. The geometry of the fractional differential has been clarified corrected and the geometry of the fractional tangent spaces of a manifold has been studied in Lazopoulos and Lazopoulos (Lazopoulos KA, Lazopoulos AK. *Progr. Fract. Differ. Appl.* 2016, 2, 85–104), providing the bases of the missing fractional differential geometry. Therefore, a lot can be contributed to fractional hydrodynamics: the basic fractional fluid equations (Navier Stokes, Euler and Bernoulli) are derived and fractional Darcy's flow in porous media is studied.

Keywords: fractional continuum mechanics; fractional Darcy flow; fractional derivative; fractional fluid mechanics; fractional geometry; fractional Navier Stokes equations; fractional Newtonian fluids.

1 Introduction

Fractional derivatives and integrals [1–5] have been applied in many fields, since they are considered as more advanced mathematical tools for formulating more realistic responses in various scientific problems in physics and engineering [6–8]. Especially in mechanics, researchers working in disordered (non-homogeneous) materials with microstructure, Vardoulakis et al. [9], Wyss et al. [10], have used fractional analysis for better description of mechanics of porous materials, colloidal aggregates, ceramics etc., since major factors in determining materials deformation are microcracks, voids, and material phases. Further viscoelasticity problems have been recently formulated applying fractional analysis.

Generally, those problems demand non-local theories. Just to satisfy that requirement, gradient strain theories appeared, Toupin [11], Mindlin [12], Aifantis [13], Eringen [14]. In these theories, the authors introduced intrinsic material lengths that accompany the higher order derivatives of the strain. Many problems have been solved employing those theories concerning size effects, lifting of various singularities, porous materials, Aifantis [13, 15, 16], Askes and Aifantis [17], Lazopoulos [18, 19]. Another non-local approach was introduced by Kunin [20, 21].

There are many studies considering fractional elasticity theory, introducing fractional strain, Drapaca and Sivaloganathan [22], Carpinteri et al. [23, 24], Di Paola et al. [25], Atanackovic and Stankovic [26], Agrawal [27], Sumelka [28]. Baleanu and his co-workers [29–31] has presented a long list of publications concerning various applications of fractional calculus in physics, in control theory in solving differential equations and numerical solutions. In addition Tarasov [32, 33] has presented a fractional vector fields theory with applications.

Lazopoulos [34] introduced fractional derivatives of the strain in the strain energy density function in an attempt to introduce non-locality in the elastic response of materials. Fractional calculus was used by many researchers, not only in the field of mechanics but mainly in physics and especially in quantum mechanics, to develop the idea of introducing non-locality through. In fact, the history of fractional calculus is dated since 17th century. Particle physics, electromagnetics, mechanics of materials, hydrodynamics, fluid flow, rheology, viscoelasticity, optics, electrochemistry and corrosion, chemical physics are some fields where fractional calculus has been introduced.

Nevertheless, the formulation of the various physical problems, into the context of fractional mathematical analysis, follows a procedure that might be questionable. In fact, although the various laws in physics have been derived through differentials, this is not the case for the well known fractional derivatives which are not related to differentials. Simple substitution of the conventional differentials to fractional ones is not able to express realistic behavior of the various physical problems.

Lazopoulos [35] introduced Leibnitz (L-fractional) derivative corresponding to fractional differential. Furthermore, the L-fractional derivative was improved in Lazopoulos and Lazopoulos [36], applying it to continuum mechanics. Lazopoulos and Lazopoulos [36, 37] have presented the application of the L-fractional derivative into the fractional differential geometry, fractional vector field theory and fractional continuum mechanics. Recently, Lazopoulos et al. [38] have presented application of the L-fractional derivative into the viscoelastic problems.

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The present work deals with the study of fractional fluid flows, introducing the L-fractional derivative into the various flows concerning the fractional Navier-Stokes equations and the fractional Euler and Bernoulli ones. Further the fractional flows are introduced to Darcy's flows, just to study the flows through porous media. There exists a long list of works concerning fractional fluid dynamics [39–41]. However, the use of fractional derivatives, into conventional fluid flow laws, exhibit the handicap of having no-differentials. After a brief introduction of the L-fractional derivative and fractional vector field theory, the fractional fluid flows are studied using the L-fractional derivative, trying to introduce non-local fluid mechanics both in time and space considering in homogeneities, porous, cracks, etc.

2 Basic properties of fractional calculus

Fractional calculus has recently become a branch of pure mathematics, with many applications in physics and engineering, Tarasov [32, 33]. Many definitions of fractional derivatives exist. In fact, fractional calculus originated by Leibniz, looking for the possibility of defining the derivative $\frac{d^n g}{dx^n}$ when $n=\frac{1}{2}$. The various types of the fractional derivatives exhibit some advantages over the others. Nevertheless they are almost all non local, contrary to the conventional ones.

The detailed properties of fractional derivatives may be found in Kilbas et al. [42], Podlubny [43] and Samko et al. [2]. Starting from Cauchy formula for the n-fold integral of a primitive function $f(x)$

$$I^n f(x)=\int_{\mathrm{a}}^{x} f(s)(ds)^n=\int_{\mathrm{a}}^{x} dx_n \int_{\mathrm{a}}^{x_n} dx_{n-1} \int_{\mathrm{a}}^{x_{n-1}} dx_{n-2} \ldots . . \int_{\mathrm{a}}^{x_2} f(x_1) dx_1 \quad (1)$$

expressed by

$$ {}_{\mathrm{a}}I_x^n f(x)=\frac{1}{(n-1)!}\int_{\mathrm{a}}^{x}(x-s)^{n-1} f(s)ds, \quad x>0,\ n\in N \quad (2)$$

and

$$ {}_{x}I_b^n f(x)=\frac{1}{(n-1)!}\int_{x}^{b}(s-x)^{n-1} f(s)ds, \quad x>0,\ n\in N \quad (3)$$

the left and right fractional integral of f are defined as:

$$ {}_{\mathrm{a}}I_x^a f(x)=\frac{1}{\Gamma(\alpha)}\int_{\mathrm{a}}^{x}\frac{f(s)}{(x-s)^{1-a}}ds \quad (4)$$

$$ {}_{x}I_b^a f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{f(s)}{(s-x)^{1-a}}ds \quad (5)$$

In Eqs. (4, 5) we assume that α is the order of fractional integrals with $0<\alpha\leq 1$, considering $\Gamma(x)=(x-1)!$ with $\Gamma(\alpha)$ Euler's Gamma function.

Thus the left and right Riemann-Liouville (R-L) derivatives are defined by:

$$ {}_{\mathrm{a}}D_x^a f(x)=\frac{d}{dx}\left({}_{\mathrm{a}}I_x^{1-a} f(x)\right) \quad (6)$$

and

$$ {}_{x}D_b^a f(x)=-\frac{d}{dx}\left({}_{b}I_x^{1-a} f(x)\right) \quad (7)$$

Pointing out that the R-L derivatives of a constant c are non zero, Caputo's derivative has been introduced, yielding zero for any constant. Thus, it is considered as more suitable in the description of physical systems.

In fact Caputo's derivative is defined by:

$$ {}_{\mathrm{a}}^{c}D_x^a f(x)=\frac{1}{\Gamma(1-\alpha)}\int_{\mathrm{a}}^{x}\frac{f'(s)}{(x-s)^a}ds \quad (8)$$

and

$$ {}_{x}^{c}D_b^a f(x)=-\frac{1}{\Gamma(1-\alpha)}\int_{x}^{b}\frac{f'(s)}{(s-x)^a}ds \quad (9)$$

Evaluating Caputo's derivatives for functions of the type:

$f(x)=(x-\mathrm{a})^n$ or $f(x)=(b-x)^n$ we get:

$$ {}_{\mathrm{a}}^{c}D_x^a (x-\mathrm{a})^\nu=\frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}(x-\mathrm{a})^{\nu-\alpha}, \quad (10)$$

and for the corresponding right Caputo's derivative:

$$ {}_{x}D_b^a (b-x)^\nu=\frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)}(b-x)^{\nu-\alpha}$$

Likewise, Caputo's derivatives are zero for constant functions:

$$f(x)=c. \quad (11)$$

Finally, Jumarie's derivatives are defined by,

$$ {}_{\mathrm{a}}^{J}D_x^a f(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{\mathrm{a}}^{x}\frac{f(s)-f(a)}{(x-s)^a}ds$$

and

$$ {}_x^J D_b^a f(x) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(s)-f(b)}{(s-x)^a}ds $$

Although those derivatives are accompanied by some derivation rules that are not valid, the derivatives themselves are valid and according to our opinion are better than Caputo's, since they accept functions less smooth than the ones for Caputo. Also Jumarie's derivative is zero for constant functions, basic property, advantage, of Caputo derivative. Nevertheless, Caputo's derivative will be employed in the present work, having in mind that Jumarie's derivative may serve better our purpose.

3 The geometry of fractional differential

It is reminded, the n-fold integral of the primitive function $f(x)$, Eq. (1), is

$$ I^n f(x) = \int_a^x f(s)(ds)^n \tag{12} $$

which is real for any positive or negative increment ds. Passing to the fractional integral

$$ I^\alpha(f(x)) = \int_a^x f(s)(ds)^\alpha \tag{13} $$

the integer n is simply substituted by the fractional number α. Nevertheless, that substitution is not at all straightforward. The major difference between passing from Eq. (11) to Eq. (12) is that although $(ds)^n$ is real for negative values of ds, $(ds)^\alpha$ is complex. Therefore, the fractional integral, Eq. (13), is not compact for any increment ds. Hence the integral of Eq. (13) is misleading. In other words, the differential, necessary for the existence of the fractional integral, Eq. (13), is wrong. Hence, a new fractional differential, real and valid for positive and negative values of the increment ds, should be established.

It is reminded that the a-fractional differential of a function $f(x)$ is defined by Adda [44]:

$$ d^a f(x) = {}_a^c D_x^a f(x)(dx)^a \tag{14} $$

It is evident that the fractional differential, defined by Eq. (14), is valid for positive incremental dx, whereas for negative ones, that differential might be complex. Hence considering for the moment that the increment dx is positive, and recalling that ${}_a^c D_x^a x \neq 1$, the α-fractional differential of the variable x is:

$$ d^a x = {}_a^c D_x^a x(dx)^a \tag{15} $$

hence

$$ d^a f(x) = \frac{{}_a^c D_x^a f(x)}{{}_a^c D_x^a x} d^a x. \tag{16} $$

It is evident that $d^a f(x)$ is a non-linear function of dx, although it is a linear function of $d^a x$. That fact suggests the consideration of the fractional tangent space that we propose. Now the definition of fractional differential, Eq. (16), is imposed either for positive or negative variable differentials $d^\alpha x$. In addition the proposed L-fractional (in honour of Leibniz) derivative ${}_a^L D_x^a f(x)$ is defined by,

$$ d^a f(x) = {}_a^L D_x^a f(x) d^a x \tag{17} $$

with the Leibniz L-fractional derivative,

$$ {}_a^L D_x^a f(x) = \frac{{}_a^c D_x^a f(x)}{{}_a^c D_x^a x}. \tag{18} $$

Hence only Leibniz's derivative has any geometrical of physical meaning.

In addition, Eq. 3, is deceiving and the correct form of Eq. (3) should be substituted by,

$$ \begin{aligned} f(x) - f(\mathrm{a}) &= {}_a^L I_x^a({}_a^L D_x^\alpha f(x)) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(2-a)}\int_a^x \frac{(s-\mathrm{a})^{1-\alpha}}{(x-s)^{1-a}} {}_a^L D_x^\alpha f(s)ds \end{aligned} \tag{19} $$

It should be pointed out that the correct forms are defined for the fractional differential by Eq. (17), the Leibniz derivative, Eq. (18) and the fractional integral by Eq. (19). All the other forms are misleading.

Configurating the fractional differential, along with the first fractional differential space (fractional tangent space), the function $y = f(x)$ has been drawn in Figure 1,

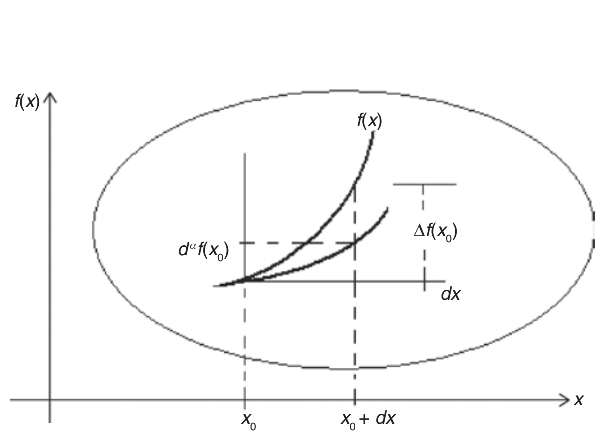

Figure 1: The non-linear differential of $f(x)$.

with the corresponding first differential space at a point x, according to Adda's definition, Eq. (14).

The tangent space, according to [44] definition, Eq. (14), is configurated by the nonlinear curve $d^\alpha f(x)$ versus dx. Nevertheless, there are some questions concerning the correct picture of the configuration, Figure 1, concerning the fractional differential presented by Adda [44]. Indeed,

a. The tangent space should be linear. There is not conceivable reason for the nonlinear tangent spaces.
b. The differential should be configured for positive and negative increments dx. However, the tangent spaces, in the present case, do not exist for negative increments dx.
c. The axis $d^\alpha f(x)$, in Figure 1, presents the fractional differential of the function $f(x)$, however dx denotes the conventional differential of the variable x. It is evident that both axes along x and $f(x)$ should correspond to differentials of the same order.

Therefore, the tangent space (first differential space), should be configured in the coordinate system with axes $(d^\alpha x, d^\alpha f(x))$. Hence, the fractional differential, defined by Eq. (17), is configured in the plane $(d^\alpha x, d^\alpha f(x))$ by a line, as it is shown in Figure 2.

It is evident that the differential space is not tangent (in the conventional sense) to the function at x_0, but intersects the figure $y=f(x)$ at least at one point x_0. This space, we introduce, is the tangent space. Likewise, the normal is perpendicular to the line of the fractional tangent. Hence we are able to establish fractional differential geometry of curves and surfaces with the fractional field theory. Consequently when $\alpha=1$, the tangent spaces, we propose, coincide with the conventional tangent spaces. As a last comment concerning the proposed L-fractional derivative, the physical dimensions of the various quantities remain unaltered from the conventional to any order fractional calculus.

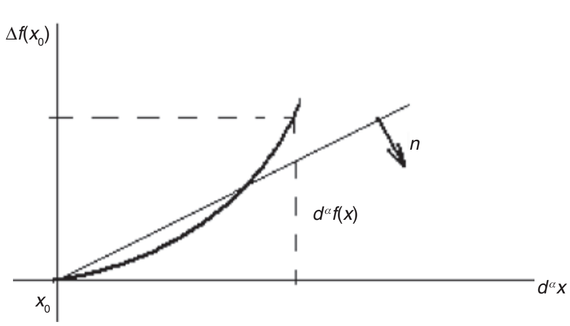

Figure 2: The virtual tangent space of the $f(x)$ at the point $x=x_0$.

4 Fractional vector calculus

For Cartesian coordinates, fractional generalizations of the gradient, divergence and rotation operators are defined by (see Lazopoulos and Lazopoulos [36]):

(A) Grad$^{(\alpha)}$ F(x)

$$\nabla^{(\alpha)} f(\mathbf{x}) = grad^{(\alpha)} f(\mathbf{x}) = \nabla_i^{(\alpha)} f(x)\mathbf{e}_i = \frac{{}_{a}^{c}D_i^{a} f(\mathbf{x})}{{}_{a}^{c}D_{\underline{i}}^{a} x_{\underline{i}}}\mathbf{e}_i = {}_{a}^{L}D_i^{a} f(\mathbf{x})\mathbf{e}_i \quad (20)$$

where ${}_{a}^{c}D_i^{a}$ are Caputo's fractional derivatives of order α and the sub line meaning no contraction.

(B) Div$^{(\alpha)}$ $\mathbf{F}(\mathbf{x})$

$$\nabla^{(a)} \cdot \mathbf{F}(x) = div^{(a)}\mathbf{F}(x) = \frac{{}_{a}^{c}D_k^{a} F_k(\mathbf{x})}{{}_{a}^{c}D_{\underline{k}}^{a} x_{\underline{k}}} = {}_{a}^{L}D_k^{a} F_k(\mathbf{x}) \quad (21)$$

where the sub-line denotes no contraction.

(C) Curl$^{(\alpha)}$ $\mathbf{F}(\mathbf{x})$

$$curl^{(a)}\mathbf{F} = \mathbf{e}_l \varepsilon_{lmn} \frac{{}_{a}^{c}D_m^{a} F_n}{{}_{a}^{c}D_{\underline{m}}^{a} x_{\underline{m}}} = \mathbf{e}_l \varepsilon_{lmn}\, {}_{a}^{L}D_m^{a} F_n \quad (22)$$

Moreover the circulation and the fractional Gauss' formula for a field are given by:

(D) Circulation

For a vector field

$$\mathbf{F}(x_1, x_2, x_3) = \mathbf{e}_1 F_1(x_1, x_2, x_3) + \mathbf{e}_2 F_2(x_1, x_2, x_3) + \mathbf{e}_3 F_3(x_1, x_2, x_3) \quad (23)$$

where $F_i(x_1, x_2, x_3)$ are absolutely integrable, the circulation is defined by:

$$C_L^{(\alpha)}(\mathbf{F}) = ({}_{a}I_L^{(a)}, \mathbf{F}) = \int_L (dL, \mathbf{F}) = {}_{a}I^{(a)}{}_L(F_1 d^\alpha x_1) + {}_{a}I^{(a)}{}_L(F_2 d^\alpha x_2) + {}_{a}I^{(a)}{}_L(F_3 d^\alpha x_3) \quad (24)$$

(E) Fractional Gauss' formula

For the conventional fields theory, let $\mathbf{F} = \mathbf{e_1}F_1 + \mathbf{e_2}F_2 + \mathbf{e_3}F_3$ be a continuously differentiable real-valued function in a domain W with boundary ∂W. Then the conventional divergence Gauss' theorem is expressed by,

$$\iint_{\partial W} \mathbf{F} \cdot \mathbf{dS} = \iiint_{W} div\mathbf{F} dV. \quad (25)$$

Since

$$\mathbf{d}^{(\alpha)}\mathbf{S} = \mathbf{e}_1 d^\alpha x_2 d^\alpha x_3 + \mathbf{e}_2 d^\alpha x_3 d^\alpha x_1 + \mathbf{e}_3 d^\alpha x_1 d^\alpha x_2 \quad (26)$$

where $d^a x_i\; i=1, 2, 3$ is expressed by:

$$d^{(\alpha)}V = d^\alpha x_1 d^\alpha x_2 d^\alpha x_3 \quad (27)$$

Furthermore

$$div^{(\alpha)}\mathbf{F}(x)=\frac{{}_{a}^{c}D_{k}^{a}F_{k}(\mathbf{x})}{{}_{a}^{c}D_{k}^{a}x_{m}}\delta_{km}={}_{a}^{L}D_{k}^{a}F_{k}(\mathbf{x}) \quad (28)$$

The fractional Gauss divergence theorem becomes:

$${}_{a}^{(\alpha)}\iint_{\partial W}\mathbf{F}\cdot d^{(\alpha)}\mathbf{S}={}_{a}^{(\alpha)}\iiint_{w}div^{(\alpha)}\mathbf{F}d^{(\alpha)}V \quad (29)$$

Remember that the differential $d^{\alpha}\mathbf{S}=\mathbf{n}^{\alpha}d^{\alpha}S$, where $\mathbf{n}^{\alpha}$ is the unit normal of the fractional tangent space as it has been defined in Section 3.

5 Fractional stresses

In a fractional tangent space the fractional normal vector $\mathbf{n}^{\alpha}$ does not coincide with the conventional normal vector $\mathbf{n}$ (Lazopoulos and Lazopoulos [36]). Hence we should expect the stresses and consequently the stress tensor to differ from the conventional ones not only in the values but in the directions as well.

If $d^{\alpha}\mathbf{P}$ is a contact force acting on the deformed area $d^{\alpha}\mathbf{a}=\mathbf{n}^{\alpha}d^{\alpha}\mathrm{a}$, lying on the fractional tangent plane where $\mathbf{n}^{\alpha}$ is the unit outer normal to the element of area $d^{\alpha}\mathbf{a}$, then the α-fractional stress vector is defined by:

$$\mathbf{t}^{\alpha}=\lim_{d^{\alpha}\mathrm{a}\to 0}\frac{d^{\alpha}\mathbf{P}}{d^{\alpha}\mathrm{a}} \quad (30)$$

However, the α-fractional stress vector does not have any connection with the conventional one:

$$\mathbf{t}=\lim_{d\mathrm{a}\to 0}\frac{d\mathbf{P}}{d\mathrm{a}} \quad (31)$$

since the conventional tangent plane has different orientation from the α-fractional tangent plane and the corresponding normal vectors too.

Following similar procedures as the conventional ones we may establish Cauchy's fundamental theorem, see [45].

If $\mathbf{t}^{\alpha}(\mathbf{n}^{\alpha})$ is a continuous function of the transplacement vector $\mathbf{y}$, there is an α-fractional Cauchy stress tensor field.

$$T^{a}=[\tau_{ij}^{a}] \quad (32)$$

For a Newtonian fluid the following relations apply:

$$\tau_{xx}^{a}=\lambda\nabla^{(\alpha)}\mathbf{V}+2\mu\,{}_{a}^{L}D_{x}^{\alpha}u$$
$$\tau_{yy}^{a}=\lambda\nabla^{(\alpha)}\mathbf{V}+2\mu\,{}_{a}^{L}D_{y}^{\alpha}v$$
$$\tau_{zz}^{a}=\lambda\nabla^{(\alpha)}\mathbf{V}+2\mu\,{}_{a}^{L}D_{z}^{\alpha}w$$

$$\tau_{xy}^{a}=\tau_{yx}^{a}=\mu[{}_{a}^{L}D_{x}^{\alpha}v+{}_{a}^{L}D_{y}^{\alpha}u]$$
$$\tau_{xz}^{a}=\tau_{zx}^{a}=\mu[{}_{a}^{L}D_{z}^{\alpha}u+{}_{a}^{L}D_{x}^{\alpha}w]$$
$$\tau_{yz}^{a}=\tau_{zy}^{a}=\mu[{}_{a}^{L}D_{y}^{\alpha}w+{}_{a}^{L}D_{z}^{\alpha}v] \quad (33)$$

where λ, μ the Lamè constants and $\mathbf{V}$ the velocity array $(ui+vj+wk)$.

6 The balance principles

Almost all balance principles are based upon Reynold's transport theorem. Hence the modification of that theorem, just to conform to fractional analysis is presented. The conventional Reynold's transport theorem is expressed by:

$$\frac{d}{dt}\int_{W}AdV=\int_{W}\frac{dA}{dt}dV+\int_{\partial W}A\mathbf{v}_{\mathbf{n}}dS. \quad (34)$$

For a vector field A applied upon region W with boundary ∂W and $\mathbf{v}_{\mathbf{n}}$ is the normal velocity of the boundary ∂W.

a. The balance of mass

The conventional balance of mass, expressing the mass preservation is expressed by:

$$\frac{d}{dt}\int_{W}\rho dV=0 \quad (35)$$

In the fractional form it is given by:

$$\frac{d}{dt}\int_{\omega\,W}^{(\alpha)}\rho d^{\alpha}V=0 \quad (36)$$

where $\frac{d}{dt}$ is the total time derivative.

Recalling the fractional Reynold's transport theorem, we get:

$$\frac{d}{dt}\int_{a\,V}^{(\alpha)}\rho d^{\alpha}V=\int_{a\,V}^{(\alpha)}(\partial_{t}\rho(x,t)+div^{\alpha}[\mathbf{v}\rho])d^{\alpha}V \quad (37)$$

Since Eq. (37) is valid for any volume V, the continuity equation is:

$$\partial_{t}\rho+div^{a}[\mathbf{v}\rho]=0. \quad (38)$$

where, div^{α} has already been defined by Eq. (28). That is the continuity equation expressed in fractional form.

b. Balance of linear momentum principle

It is reminded that the conventional balance of linear momentum is expressed in continuum mechanics by:

$$\frac{d}{dt}\int_{\Omega}\rho\mathbf{v}\delta V=\int_{\partial\Omega}\mathbf{t}^{(n)}dS+\int_{\Omega}\rho\mathbf{b}dV \quad (39)$$

where $\mathbf{v}$ is the velocity, $\mathbf{t}^{(n)}$ is the traction on the boundary and $\mathbf{b}$ is the body force per unit mass. Likewise that principle in fractional form is expressed by:

$$\frac{d}{dt}\int_{a\ \Omega}^{(\alpha)} \rho \mathbf{v} d^{(\alpha)}V = \int_{a\ \Omega}^{(\alpha)} [\rho \mathbf{b} + div^{a}(\mathbf{T}^{\alpha})] d^{(\alpha)}V \tag{40}$$

Hence the equation of linear motion, expressing the balance of linear momentum is defined by,

$$div^{a}[\mathbf{T}^{a}] + \rho \mathbf{b} - \rho \frac{d\mathbf{v}}{dt} = 0 \tag{41}$$

In Eq. (40) we must introduce pressure in the term $\mathbf{T}^{\alpha}$. This is done by adding the term $-p\mathbf{I}$. Therefore Eq. (40) becomes:

$$div^{a}[\mathbf{T}^{a} - p\mathbf{I}] + \rho \mathbf{b} - \rho \frac{d\mathbf{v}}{dt} = 0 \tag{42}$$

It should be pointed out that div^{α} has already been defined by Eq. (28) and is different from the conventional definition of the divergence.

Following similar steps as in the conventional case, the balance of rotational momentum yields the symmetry of Cauchy stress tensor.

7 First law of thermodynamics

This law occurs from Eq. (42). Firstly we replace $\mathbf{T}^{\alpha} - p\mathbf{I}$ with $\mathbf{\Sigma}^{\alpha}$.

$$\mathbf{\Sigma}^{a} = \mathbf{T}^{a} - p\mathbf{I} \tag{43}$$

$$\mathbf{\Sigma}^{a} = [\sigma_{ij}] \tag{44}$$

We then derive the equation of conservation of energy from Eq. (42). This is accomplished by multiplying the equation Eq. (42) with the velocity υ_i and the result is integrated over the volume V:

$$\int_V \rho \upsilon_i \frac{d\upsilon_i}{dt} d^{(a)}V = \int_V \upsilon_i\, {}_{a}^{L}D_{x_j}^{\beta} \sigma_{ij} d^{(a)}V + \int_V \rho \upsilon_i b_i d^{(a)}V \tag{45}$$

But the left side of Eq. 45 becomes:

$$\int_V \rho \upsilon_i \frac{d\upsilon_i}{dt} d^{(a)}V = \frac{d}{dt}\int_V \frac{\rho \upsilon^2}{2} d^{(a)}V \tag{46}$$

which represents the time rate of change of the kinetic energy K in the continuum.

In Eq. (45) we can write (related to the right side):

$$\begin{aligned} \upsilon_i\, {}_{a}^{L}D_{x_i}^{\beta} \sigma_{ij} &= {}_{a}^{L}D_{x_i}^{\beta}(\upsilon_i \sigma_{ji}) - \sigma_{ji}\, {}_{a}^{L}D_{x_j}^{\beta} \upsilon_i = {}_{a}^{L}D_{x_i}^{\beta}(\upsilon_i \sigma_{ji}) \\ &- (D_{ij} + V_{ij})\sigma_{ji} = {}_{a}^{L}D_{x_i}^{\beta}(\upsilon_i \sigma_{ji}) - D_{ij}\sigma_{ji} \end{aligned} \tag{47}$$

In Eq. (47) we have:

$${}_{a}^{L}D_{x_i}^{\beta} \upsilon_i = D_{ij} + V_{ij}$$

where:

$$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{v}\overset{(a)}{\nabla}_{\boldsymbol{x}} + \overset{(a)}{\nabla}_{\boldsymbol{x}}\boldsymbol{v}\right) \quad \boldsymbol{V} = \frac{1}{2}\left(\boldsymbol{v}\overset{(a)}{\nabla}_{\boldsymbol{x}} - \overset{(a)}{\nabla}_{\boldsymbol{x}}\boldsymbol{v}\right) \tag{48}$$

We consider that: $V_{ij}\sigma_{ij} = 0$, since it is a product of a symmetric times an antisymmetric tensor. Therefore Eq. (45) with the help of Eq. (46) and Eq. (47) becomes:

$$\begin{aligned} \frac{d}{dt}\int_V \frac{\rho \upsilon^2}{2} d^{(a)}V + \int_V D_{ij}\sigma_{ji} d^{(a)}V &= \int_V {}_{a}^{L}D_{x_j}^{a}(\upsilon_i \sigma_{ji}) d^{(a)}V \\ &+ \int \rho b_i \upsilon_i d^{(a)}V \end{aligned} \tag{49}$$

The term $\int_V {}_{a}^{L}D_{x_j}^{a}(\upsilon_i \sigma_{ji}) d^{(a)}V$ in Eq. (49), due to the Gauss theorem, takes the form:

$$\int_V {}_{a}^{L}D_{x_j}^{a}(\upsilon_i \sigma_{ji}) d^{(a)}V = \int_S \upsilon_i \sigma_{ij} n_j d^{(a)}S \tag{50}$$

With the help of Eq. (50), Eq. (49) becomes:

$$\frac{d}{dt}\int_V \frac{\rho \upsilon^2}{2} d^{(a)}V + \int_V D_{ij}\sigma_{ji} d^{(a)}V = \int_S \upsilon_i \sigma_{ij} n_j d^{(a)}S + \int_V \rho b_i \upsilon_i d^{(a)}V \tag{51}$$

In Eq. (51) $\frac{d}{dt}\int_V \frac{\rho \upsilon^2}{2} d^{(a)}V$ is the rate of change of the kinetic energy of the system $\left(\frac{d}{dt}K\right)$, $\int_V D_{ij}\sigma_{ji} dV$ is the rate of change of the internal energy $\left(\frac{d}{dt}U\right)$, and $\int_S \upsilon_i \sigma_{ij} n_j d^{(a)}S + \int_V \rho b_i \upsilon_i d^{(a)}V$ is the rate of work of the external forces in the system $\left(\frac{d}{dt}W\right)$.

For a thermomechanical continuum it is customary to express the time rate of change of internal energy by the integral expression:

$$\frac{d}{dt}U = \frac{d}{dt}\int_V \rho e d^{(a)}V = \int_V \rho \frac{d}{dt} e d^{(a)}V \tag{52}$$

where e is called the specific internal energy. Moreover, if the vector c_i is defined as the heat flux per unit area per unit time by conduction, and z is taken as the radiant heat constant per unit mass per unit time, the rate of increase of total heat into the continuum is given by:

$$\frac{d}{dt}Q = -\int_S c_i n_i d^{(a)}S + \int_V \rho z d^{(a)}V \tag{53}$$

Therefore the energy principle for a thermomechanical continuum is given by:

$$\frac{d}{dt}\mathrm{K} + \frac{d}{dt}U = \frac{d}{dt}W + \frac{d}{dt}Q \tag{54}$$

or, in terms of the energy integrals as:

$$\frac{d}{dt}\int_V \frac{\rho v^2}{2} d^{(a)}V + \int_V \frac{d}{dt} e d^{(a)}V = \int_S v_i \sigma_{ij} n_j d^{(a)}S + \int_V \rho b_i v_i d^{(a)}V + \int_V \rho z d^{(a)}V - \int_S c_i n_i d^{(a)}S \tag{55}$$

Converting the surface integrals in Eq. (55) to volume integrals by the divergence theorem of Gauss, and again the fact that V is arbitrary, leads to the local form of the energy equation:

$$\frac{d}{dt}\left(\frac{v^2}{2} + e\right) = \frac{1}{\rho}\,{}_{\mathrm{a}}^{L}D_{x_j}^{\beta}(\sigma_{ij} v_i) + b_i v_i - \frac{1}{\rho}\,{}_{\mathrm{a}}^{L}D_{xi}^{\beta} c_i + z \tag{56}$$

8 Navier-Stokes equations

The Navier-Stokes equations are consisted by the following equations:

a) Balance of mass

$$\frac{d}{dt}\rho + div^a[\mathbf{v}\rho] = 0 \tag{57}$$

b) Balance of linear momentum

$$div^a[\mathbf{T}^a - p\mathbf{I}] + \rho\mathbf{b} - \rho\frac{d}{dt}\mathbf{v} = 0 \tag{58}$$

c) First law of thermodynamics

$$\frac{d}{dt}\left(\frac{v^2}{2} + e\right) = \frac{1}{\rho}\,{}_{\mathrm{a}}^{L}D_{x_j}^{\beta}(\sigma_{ij} v_i) + b_i v_i - \frac{1}{\rho}\,{}_{\mathrm{a}}^{L}D_{xi}^{\beta} c_i + z \tag{59}$$

d) Constitutive equations

$$\begin{aligned}
\tau_{xx}^a &= \lambda\nabla^{(\alpha)}\mathbf{V} + 2\mu\,{}_{\mathrm{a}}^{L}D_x^{\alpha}u \\
\tau_{yy}^a &= \lambda\nabla^{(\alpha)}\mathbf{V} + 2\mu\,{}_{\mathrm{a}}^{L}D_y^{\alpha}v \\
\tau_{zz}^a &= \lambda\nabla^{(\alpha)}\mathbf{V} + 2\mu\,{}_{\mathrm{a}}^{L}D_z^{\alpha}w \\
\tau_{xy}^a &= \tau_{yx}^a = \mu[{}_{\mathrm{a}}^{L}D_x^{\alpha}v + {}_{\mathrm{a}}^{L}D_y^{\alpha}u] \\
\tau_{xz}^a &= \tau_{zx}^a = \mu[{}_{\mathrm{a}}^{L}D_z^{\alpha}u + {}_{\mathrm{a}}^{L}D_x^{\alpha}w] \\
\tau_{yz}^a &= \tau_{zy}^a = \mu[{}_{\mathrm{a}}^{L}D_y^{\alpha}w + {}_{\mathrm{a}}^{L}D_z^{\alpha}v]
\end{aligned} \tag{60}$$

e) The kinetic equation of state:

$$p = p(\rho, \mathrm{T}) \tag{61}$$

f) The Fourier law of heat conduction:

$$\mathbf{c} = -k\nabla^{(\alpha)}T \tag{62}$$

g) The caloric equation of state:

$$e = e(\rho, T) \tag{63}$$

The system of these 16 equations contains 16 unknowns therefore it is determinate. Usually the studies concerning the Newtonian fluid, are restricted to the equations of conservation of mass (57), linear momentum (58) and constitutive equations (60).

9 The Euler and Bernoulli equations

The Euler equations occur from Eq. (27) for non-viscous fluid, for which $\mathbf{T}^{\alpha} = \mathbf{O}$. This holds because the Lamé constants λ, μ are considered zero in this case. Therefore, Eq. (49) takes the form:

$$\rho\frac{d}{dt}\mathbf{v} = \rho\mathbf{b} - div^a[p\mathbf{I}] \tag{64}$$

Lets assume a barotropic condition $\rho = \rho(p)$. A pressure function may be defined:

$$P(\rho) = \int_{p_0}^{p} \frac{d^a p}{\rho} \tag{65}$$

Moreover, the body force may be described by a potential function:

$$\boldsymbol{b} = -\nabla^{(\alpha)}\Omega \tag{66}$$

With these two conditions Eq. (42) becomes:

$${}_{a}^{L}D_t^{\beta}\mathbf{v} = -\nabla^{(\alpha)}(\Omega + P) \tag{67}$$

If Eq. (67) is integrated along *a* streamline, the result is the Bernoulli equation in the fractional form:

$$\Omega + \mathrm{P} + \frac{v^2}{2} + \int \frac{d}{dt} v_i d^a x_i = C(t) \tag{68}$$

10 Darcy's law

Consider a viscous fluid, flowing in a straight pipe of a constant circular cross section of inner radius R and cross section of $A=\pi R^2$ with perimeter $\Gamma=2\pi R$. If L is the length of the considered segment of the pipe defined by positions $x=\alpha$ and $x=b$, then $\mathrm{L}=b-\alpha$.

Denoting $Q(x, t)$ the discharge of the fluid, continuity of flow demands that $\mathbf{Q}(\mathbf{x}, \mathbf{t})$ be the same at any cross section of the pipe segment, depending only upon time

$$\mathbf{Q}(\mathbf{x}, \mathbf{t})=Q(t) \tag{69}$$

The cross sectional velocity of the flow may be computed by:

$$V=\frac{Q}{A}={}_{0}^{L}D_t^{\gamma}u(t) \tag{70}$$

Due to the fluid's internal friction, shear stresses τ are developed, between the fluid and the inner pipe-wall. Similarly to fractional viscoelasticity, the interface friction shear stress in circular pipes is equal to:

$$\tau=\frac{4\mu_f}{R}V=\frac{4\mu_f}{R}{}_{0}^{L}D_t^{\gamma}u(t) \tag{71}$$

where μ_f is fluid viscosity.

The friction stresses may be substituted by a fictitious body force per unit length of the pipe,

$$f_b=\tau\cdot V=8\pi\mu_f\,{}_{0}^{L}D_t^{\gamma}u(t) \tag{72}$$

The conservation of linear momentum equation yields:

$$-\frac{\partial p}{\partial x}Adx-f_b dx=\rho_f A V=\rho_f A\,{}_{0}^{L}D_t^{\gamma^{\cdot}}\left({}_{0}^{L}D_t^{\gamma}u(t)\right) \tag{73}$$

or

$$\frac{\partial p}{\partial x}+f=-\rho_f A\,{}_{0}^{L}D_t^{\gamma^{\cdot}}\left({}_{0}^{L}D_t^{\gamma}u(t)\right) \tag{74}$$

Since, according to Poiseille's law, f is proportional to the mean flow-velocity

$$f=cV=c\,{}_{0}^{L}D_t^{\gamma^{\cdot}}u(t) \tag{75}$$

Furthermore the coefficient c of viscous friction is proportional to the fluid viscosity and inversely proportional to the square of the pipe radius:

$$c=\pi\cdot\frac{\mu_f}{R^2} \tag{76}$$

Thus we obtain the differential equation:

$$-\frac{\partial p}{\partial x}=\rho_f\,{}_{0}^{L}D_t^{\gamma^{\cdot}}\left({}_{0}^{L}D_t^{\gamma}u(t)\right)+c\,{}_{0}^{L}D_t^{\gamma^{\cdot}}u(t) \tag{77}$$

Considering:

$$J(t)=\rho_f\,{}_{0}^{L}D_t^{\gamma^{\cdot}}\left({}_{0}^{L}D_t^{\gamma}u(t)\right)+c\,{}_{0}^{L}D_t^{\gamma^{\cdot}}u(t) \tag{78}$$

The governing Eq. (77) becomes:

$$-\frac{\partial p}{\partial x}=J(t) \tag{79}$$

For the case that $J(t)=J$ (constant) between the segments $x=\alpha$ and $x=b$, the fluid velocity is defined by:

$$\rho_f\,{}_{0}^{L}D_t^{\gamma}\left({}_{0}^{\dot{L}}D_t^{\gamma}u(t)\right)+c\,{}_{0}^{L}D_t^{\gamma}u(t)=\dot{J}=\text{constant} \tag{80}$$

Solution of fractional D.E:

In case Δp is the pressure drop from the one end to the other of the pipe segment with:

$$\Delta p=p(b)-p(\alpha)<0 \tag{81a}$$

Then the total discharge is defined by:

$$Q=AV=-\left(\frac{A}{c}\right)\frac{\Delta p}{L} \tag{81b}$$

Eq. (81b) is the famous Darcy's law.

11 Fractional flow in porous media

Considering the difference Δp to the hydrostatic we get:

$$\Delta p=-\rho_f g\Delta h \tag{82}$$

Then we may assume that for the wetted area A_v (The area of the voids), Darcy's law may be applied with:

$$Q=-\left(\frac{A_v}{c}\right)\frac{\Delta p}{L} \tag{83}$$

Assuming that the surface porosity Φ_A is equal to volume porosity Φ we get:

$$\Phi_A=\frac{A_v}{A}\approx\frac{V_v}{V}=\Phi \tag{84}$$

And thus:

$$A_v\approx\Phi\cdot A \tag{85}$$

For the specific fluid discharge:

$$q=\frac{Q}{A} \tag{86}$$

we get:

$$V=\frac{Q}{A_v}\approx\frac{Q}{A}\frac{A}{A_v}=\frac{q}{\Phi}\rightarrow q=\Phi V \tag{87}$$

Hence Darcy's law becomes:

$$q=\frac{Q}{A}=-K\frac{\Delta p}{L} \tag{88}$$

with $K=\frac{\Phi}{c}$

Finally, Darcy's law may be generalized by:

$$q=-\frac{1}{f}\frac{\partial p}{\partial x},\ \text{where}\ f=\frac{\mu_f}{K} \tag{89}$$

with p: The pore fluid pressure.

q: The specific fluid discharge.

K: The permeability of the porous medium.

μ_f: The viscosity of the fluid.

12 Fractional flow in elastic tubes

Consider a thin elastic tube, at its (unstressed) reference placement, with its inner radius R and its thickness $\delta\ll R$. The elastic tube is considered into the context of linear elasticity with modulus E. Loading the tube with pressure p, its radius increases by ΔR. Hence its cross-sectional area becomes:

$$A=\pi\cdot(R+\Delta R)^2 \tag{90}$$

and its current radius $r=R+\Delta R$ is given by:

$$r=R+\Delta R=R+\frac{R^2}{E\cdot\delta}P \tag{91}$$

Restricting to the linear elasticity where the changes of the radius are infinitesimal, we get:

$$A=A_0+\Delta A=A_0\left(1+2\frac{R_0\cdot p}{E\cdot\delta}\right) \tag{92}$$

Therefore the relation between the variable cross-section area and the fluid pressure and cross-section area:

$$A=A_0\cdot\left(1+\frac{p}{\kappa}\right),\ \kappa=\frac{2\cdot R}{E\cdot\delta} \tag{93}$$

Since mass balance is expressed by:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0 \tag{94}$$

Recalling Darcy's law with:

$$Q=-\left(\frac{A}{C}\right)\frac{\partial p}{\partial x},\ c=\pi^2\frac{\mu_f}{A} \tag{95}$$

or

$$Q=-\frac{A^2}{\pi^2\mu_f}\frac{\partial p}{\partial x} \tag{96}$$

Eliminating the discharge Q, we end up to:

$$\frac{\partial A}{\partial t}=\frac{1}{\pi^2\mu_f}\frac{\partial\left(A^2\frac{\partial p}{\partial x}\right)}{\partial x} \tag{97}$$

Since:

$$\frac{\partial A}{\partial t}=\frac{\partial A}{\partial p}\frac{\partial p}{\partial t}=\frac{A_0}{\kappa}\frac{\partial p}{\partial x} \tag{98}$$

Consequently, Eq. (97) yields:

$$\frac{\partial p}{\partial t}=\frac{1}{\pi^2\mu_f}\left(2\cdot k\cdot\left(\frac{\partial p}{\partial x}\right)^2+A\cdot\kappa\cdot\frac{\partial^2 p}{\partial x^2}\right) \tag{99}$$

However, for small changes in pressure, the linear problem is considered, resulting in:

$$\frac{\partial p}{\partial t}=c_p\frac{\partial^2 p}{\partial x^2} \tag{100}$$

where:

$$c_p=\frac{A_0\cdot\kappa}{\pi^2\mu_f}=\frac{2\cdot R^3}{\pi\cdot\mu_f\cdot E\cdot\delta} \tag{101}$$

Solution of parabolic equation.

Following just the same procedure, but for the fractional time derivative fields, we get the fractional parabolic equation,

$${}_0^LD_t^\alpha p=c_p\frac{\partial^2 p}{\partial x^2} \tag{102}$$

Furthermore, the initial condition expresses the constant pressure value,

$$t=0:\quad p=p_0,\ \forall x\varepsilon[0,L]. \tag{103}$$

Further, at the time $=0^+$, the pressure at the entry point of the tube is increased by Δp, keeping constant the pressure value at the exit point of the tube. Consequently the boundary conditions for $t>0$ become,

$$x=0,\ p=p_1=p_0+\Delta p \tag{104}$$

$$x=L,\ p=p_0 \tag{105}$$

Nondimensionaling the variables we get:

$$p^*=\frac{p}{p_o},\ x^*=\frac{x}{L}\ (0\le x^*\le 1),\ \text{and}\ t^*=\frac{t}{t_c}\ \text{where}\ t_c=\frac{L^2}{c_p}. \tag{106}$$

Omitting the upper-stars the non dimensional governing equation of the fractional flow become,

$${}_0^L D_t^\alpha p=\frac{\partial^2 p}{\partial x^2} \tag{107}$$

with the initial condition,

$$x=0;\ p=1,\ \forall x\ \varepsilon[0,\ 1] \tag{108}$$

and the boundary conditions,

$$x=0,\ p=p_1=1+\lambda,\ \lambda=\frac{\Delta p}{p_0} \tag{109}$$

$$x=1,\ p=p_2=1.$$

$$\frac{\partial p}{\partial t}=c_p\frac{\partial^2 p}{\partial x^2}$$

The initial condition for the pressure

$$t=0:p=p_0,\ \forall x\ \varepsilon[0,\ L] \tag{110}$$

At $t=0^+$, the pressure at the entry point is increased by Δp with constant pressure at the exit point. Hence the boundary conditions for $t>0$ become:

$$x=0,\ p=p_1\pm p_0+\Delta p \tag{111}$$

$$x=L,\ p=p_0$$

Non-dimensional variables become:

$$p^*=\frac{p}{p_0},\ x^*=\frac{x}{L}\ (0\le x^*\le 1),\ \text{and}\ t^*=\frac{t}{t_c}\ \text{where}\ t_c=\frac{L^2}{c_p}. \tag{112}$$

The non-dimensional equation becomes:

$$\frac{\partial p}{\partial t}=\frac{\partial^2 p}{\partial x^2} \tag{113}$$

With the initial conditions.

$$t=0:\qquad p=1,\ \forall x\varepsilon[0,\ L] \tag{114}$$

And boundary conditions:

$$x=0,\ p=p_1\pm 1+\lambda,\ \lambda=\frac{\Delta p}{p_0} \tag{115}$$

$$x=1,\ p=p_2=1$$

Considering first the steady solution:

$${}_0^L D_t^\alpha p=0\qquad \frac{\partial^2 p}{\partial x^2}=0\qquad p=\bar{p}=p_1+(p_1-p_2)x \tag{116}$$

And introducing a renormalized pressure:

$$\hat{p}=\frac{p-\bar{p}}{p_1-p_2} \tag{117}$$

From the above formulas occurs:

$$\frac{\partial\hat{p}}{\partial t}=\frac{\partial^2\hat{p}}{\partial x^2}\qquad 0\le x\le 1 \tag{118}$$

The initial condition is:

$$\hat{p}=h(x)=x-1,\ t=0,\ 0\le \tag{119}$$

And the boundary conditions are:

$$\hat{p}=h(x)=x-1,\ t=0,\ 0\le x\le 1 \tag{120}$$

And the boundary conditions:

$$\hat{p}=0,\ t>0,\ x=0\ \text{and}\ x=1 \tag{121}$$

Applying the separation of variables technique for the fractional diffusion equation we get:

$$\hat{p}(x,\ t)=X(x)T(t) \tag{122}$$

Following the conventional procedure for the diffusion equation we get:

$$\frac{{}_0^L D_t^\alpha T(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda^2=\text{constant} \tag{123}$$

Due to appendix, the solution to the equation (101) is given by:

$$\hat{p}(x,\ t)=\sum_{n=1}^\infty b_n\sin(n\pi x)\Phi^\alpha(-(n\pi)^2,\ t). \tag{124}$$

where:

$$\Phi^\alpha(-(n\pi)^2,\ t)=\sum_{k=1}^\infty(-(n\pi)^2)^\kappa\frac{\Gamma\left(\kappa+1-\alpha\right)\ldots\ldots\ldots t^k}{\Gamma(\kappa+1)\ldots\Gamma(3)\Gamma(2-\alpha)^{\kappa-1}} \tag{125}$$

For $h=x-1$, (see Appendix)

$$\hat{p}\approx-\frac{2}{\pi}\sum_{n=1}^\infty\frac{1}{n}\sin(n\pi x)\Phi^\alpha((n\pi)^2,\ t). \tag{126}$$

13 Conclusion

Since the preliminary elements have been defined (Leibniz fractional derivative, fractional gradient, fractional rotation, fractional divergence, fractional circulation and fractional Gauss' theorem), the basic equations of fluid mechanics are reinstated and analyzed. Further, Fractional Darcy's flow was studied as an application of the fractional flows into porous media. The main issue in our case is experimental validation of the occurring equations.

Appendix

The fractional diffusion equation

The conventional diffusion equation is:

$$\frac{\partial \hat{p}}{\partial t}=\frac{\partial^2 \hat{p}}{\partial x^2} \tag{A.1}$$

with the initial conditions

$$\hat{p}=h(x)=x-1,\ t=0,\ 0\le x\le 1 \tag{A.2}$$

And the boundary conditions:

$$x=0, \qquad p=p_1+\lambda, \qquad \lambda=\frac{\Delta p}{p_0} \tag{A.3}$$

$$x=1, \qquad p=p_2=1$$

Application of variables technique gives:

$$p(x,\ t)=f(x)g(t) \tag{A.4}$$

Therefore, the diffusion equation yields:

$$\begin{gathered} f''+\eta^2 f=0 \\ {}_0^L D_t^a g(t)+\eta^2 g=0 \end{gathered} \tag{A.5}$$

Taking under consideration the boundary conditions the solution is expressed by:

$$\hat{p}(x,\ t)=\sum_{n=1}^{\infty} b_n \sin(n\pi x)\exp(-n^2\pi^2 t^2) \tag{A.6}$$

The fractional diffusion equation is:

$${}_0^L D_t^\alpha p=\frac{\partial^2 p(x,\ t)}{\partial x^2} \tag{A.7}$$

With its initial and boundary conditions the same as the conventional ones stated above.

The solution:

$\hat{p}(x,\ t)=X(x)T(t)$ is defined by the solutions the system:

$${}_0^L D_t^a \varphi(t)+\lambda^2\varphi(t)=0 \tag{A.8}$$

Looking for the solution of $\varphi(t)$ into the power series of expansions of t we get:

After substituting into the Eq. (A.8) the relation:

$$\sum_{n=1}^{\infty} y_\kappa \frac{\Gamma(\kappa+1)\Gamma(2-\alpha)}{\Gamma(\kappa+1-\alpha)} t^{k+1}+A^2\sum_{\kappa=0}^{\infty} y_\kappa t^\kappa=0 \tag{A.9}$$

Equating the coefficients of the same power of t to zero, we have:

$$\begin{aligned} y_\kappa &= -A^2\frac{\Gamma(\kappa+1-\alpha)}{\Gamma(\kappa+1)\Gamma(2-\alpha)} y_{\kappa-1} \\ y_\kappa &= (-A^2)^{\mu+1}\frac{\Gamma(\kappa+1-\alpha)\ldots\Gamma(\kappa-\mu+1-\alpha)}{\Gamma(\kappa+1)\ldots\Gamma(\kappa-\mu+1)\Gamma(2-\alpha)^{\mu+1}} y_{\kappa-\mu-1} \\ y_\kappa &= (-A^2)^{\mu+1}\frac{\Gamma(\kappa+1-\alpha)\ldots}{\Gamma(\kappa+1)\ldots\Gamma(3)\Gamma(2-\alpha)^{\kappa-1}} y_0 \end{aligned} \tag{A.10}$$

Therefore, it occurs:

$$\begin{aligned} y_1 &= -A^2 y_0 \\ y_2 &= (-A^2)^2\frac{2-a}{2} y_0 \\ y_3 &= (-A^2)^3\frac{(3-a)(2-a)}{12} y_0 \\ y_4 &= (-A^2)^4\frac{(4-\alpha)(3-a)(2-a)}{288} y_0 \end{aligned} \tag{A.11}$$

and

$$y_\lambda = y_{0,\lambda}\sum_{\kappa=1}^{\infty}(-\lambda^2)^\kappa \frac{\Gamma(\kappa+1-\alpha)\ldots}{\Gamma(\kappa+1)\ldots\Gamma(2-\alpha)^{\kappa-1}} \tag{A.12}$$

and

$$\begin{aligned} &\hat{p}(x,\ t)=\sum_{n=1}^{\infty} b_n \sin(n\pi x) \\ &\sum_{\kappa=1}^{\infty}(-(n\pi)^2)^\kappa \frac{\Gamma(\kappa+1-\alpha)\ldots}{\Gamma(\kappa+1)\ldots\Gamma(2-\alpha)^{\kappa-1}} t^\kappa \end{aligned} \tag{A.13}$$

If we define:

$$\Phi^\alpha((n\pi)^2,\ t)=\sum_{\kappa=1}^{\infty}(-(n\pi)^2)^\kappa \frac{\Gamma(\kappa+1-\alpha)\ldots\Gamma(3-\alpha)}{\Gamma(\kappa+1)\ldots\Gamma(3)\Gamma(2-\alpha)^{\kappa-1}} t^\kappa \tag{A.14}$$

Then we have:

$$\hat{p}=\sum_{n=1}^{\infty} b_n \sin(n\pi x)\Phi^\alpha((n\pi)^2,\ t) \tag{A.15}$$

$$\text{For } h(x)=\sum_{n=1}^{\infty} b_n \sin(n\pi x),\ 0\le x\le 1 \tag{A.16}$$

$$b_n=2\int_0^1 h(x)\sin(n\pi x)dx \tag{A.17}$$

With $h=x-1$ we get:

$$\int_0^1 (x-1)\sin(n\pi x)dx = -\frac{1}{n\pi} \tag{A.18}$$

hence

$$h(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n} \tag{A.19}$$

and

$$\hat{p} \approx -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)\Phi^{\alpha}((n\pi)^2,\ t) \tag{A.20}$$

where

$$\Phi^{\alpha}((n\pi)^2,\ t) = \sum_{\kappa=1}^{\infty}(-(n\pi)^2)^{\kappa}\frac{\Gamma(\kappa+1-\alpha)\ldots\Gamma(3-\alpha)}{\Gamma(\kappa+1)\ldots\Gamma(3)\Gamma(2-\alpha)^{\kappa-1}}t^{\kappa} \tag{A.21}$$

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