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A note on dislocation-based mode III gradient elastic fracture mechanics

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A dislocation-based fracture mechanics approach is applied to a mode III crack within the first strain gradient elasticity (GradEla) framework [1–4]. The mode III crack is modeled by the convolution of distributed screw dislocations, for which the stress and dislocation density have been discussed in [4–7]. The unknown dislocation density is determined by using the nonstandard boundary conditions resulting from a variational formulation of GradEla [3, 8, 9]. In particular, the GradEla counterpart of the classical traction vector and the double stress traction vector are taken into account in modeling the crack faces. Then, the dislocation density is determined, resulting in nonsingular elastic stresses. To investigate the effect of the nonstandard boundary condition, the dislocation density and stress components are compared for the two cases involving nonstandard and classical boundary conditions.

Within an incompatible GradEla framework, we have the following basic equations

$$u_{i,j}=\beta^T_{ij}=\beta_{ij}+\beta^P_{ij};\ \varepsilon_{ij}=\frac{1}{2}(\beta_{ij}+\beta_{ji}) \tag{1}$$

$$\alpha_{ij}=e_{jkl}\beta_{ik,l};\ \alpha_{ij}=-e_{jkl}\beta^P_{ik,l};\ \alpha_{ij,j}=0, \tag{2}$$

$$\tau_{ij}=\lambda\delta_{ij}\varepsilon_{mm}+2\mu\varepsilon_{ij},\ \tau_{ijk}=\ell^2\tau_{ij,k};\ \sigma_{ij}=\tau_{ij}-\ell^2\tau_{ijk,k} \tag{3}$$

$$\sigma_{ij,j}=\tau_{ij,j}-\tau_{ijk,kj}=0 \tag{4}$$

$$\left.\begin{aligned} t_i&=(\tau_{ij}-\partial_k\tau_{ijk})n_j-\partial_j(\tau_{ijk}n_k)+n_j\partial_l(\tau_{ijk}n_kn_l),\\ q_i&=\tau_{ijk}n_jn_k.\end{aligned}\right\} \tag{5}$$

In the above-mentioned relations, (u_i, ε_{ij}, α_{ij}) are kinematic quantities denoting the displacement vector, the elastic strain tensor, and the dislocation density tensor [10]. The total distortion tensor β^T_{ij} is curl-free, while the elastic β_{ij} and plastic β^P_{ij} distortions are not curl-free within an incompatible framework. The constitutive quantities τ_{ij} and τ_{ijk} denote the elastic stress and the double (third-order) stress tensor, while σ_{ij} denotes the total (Cauchy-like) stress tensor satisfying the usual equilibrium equation. The symbols δ_{ij} and e_{ijk} denote the Kronecker delta and the Levi-Chivita tensors, respectively, while ℓ is the internal length and (λ, μ) the usual Lamé constants. The "natural" boundary conditions for the nonclassical traction vector t_i and the double traction vector q_i listed in Eq. (5) are to be used in the sequel result from a variational formulation of GradEla [2], with n_i denoting the unit normal to a smooth boundary [8, 9].

For infinite domains, the "boundary conditions" do not imply any constraint on the solution of the governing equations, other than the requirement of having finite values at infinity. In fact, the nonsingular solution for the stress field τ_{ij} of a screw dislocation was first derived by Gutkin and Aifantis [5, 6] and Lazar and Maugin [4] as

$$\begin{aligned}\tau_{xz}(x,y)&=-\frac{\mu b_z}{2\pi}\frac{y}{r^2}\left(1\ \frac{r}{\ell}K_1(r/\ell)\right);\\ \tau_{yz}(x,y)&=\frac{\mu b_z}{2\pi}\frac{x}{r^2}\left(1-\frac{r}{\ell}K_1(r/\ell)\right),\end{aligned} \tag{6}$$

where b_z denotes the Burgers vector and K_1 is the modified Bessel function of order 1. The singularity of the classical stress is regularized in GradEla by the term inside the parenthesis, which includes the Bessel function. Analogous expressions hold for the elastic strains, whereas the only non-vanishing component of the dislocation density tensor α_{zz} reads [11]

$$\alpha_{zz}=\frac{b_z}{2\pi\ell^2}K_0(r/\ell), \tag{7}$$

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where K_0 denotes the Bessel function of order 0. Furthermore, the non-vanishing components of the double stress tensor of the screw dislocation are [4, 11]

$$\tau_{zyx}(x,y)=\frac{\mu b_z}{2\pi}\left[\frac{l^2}{r^2}-2\frac{l^2x^2}{r^4}+\frac{x^2}{r^2}K_0\left(\frac{r}{l}\right)-\frac{l}{r}K_1\left(\frac{r}{l}\right)+2\frac{lx^2}{r^3}K_1\left(\frac{r}{l}\right)\right],$$
$$\tau_{zxy}(x,y)=-\frac{\mu b_z}{2\pi}\left[\frac{l^2}{r^2}-2\frac{l^2y^2}{r^4}+\frac{y^2}{r^2}K_0\left(\frac{r}{l}\right)-\frac{l}{r}K_1\left(\frac{r}{l}\right)+2\frac{ly^2}{r^3}K_1\left(\frac{r}{l}\right)\right],$$
$$\tau_{zyy}(x,y)=-\tau_{zxx}=\frac{\mu b_z}{2\pi}\frac{xy}{r^2}\left\{-2\frac{l^2}{r^2}+K_0\left(\frac{r}{l}\right)+2\frac{l}{r}K_1\left(\frac{r}{l}\right)\right\}. \tag{8}$$

It turns out that the stresses τ_{ij} and τ_{ijk} are divergent-free outside the dislocation core region. It should also be noted that the double stress tensor in first strain GradEla is still singular. Non-singular expressions for τ_{ijk} can be derived within a second strain GradEla theory [11].

Dislocations are elementary defects in solids, which can be used to represent more complex (composite) defects [12]. For example, using the distributed dislocation technique (DDT), the arbitrary configuration of cracks can be modeled [13]. In this technique, the dislocations are distributed in the locations of cracks, and then, the stress fields for the cracked medium can be derived by using the stress field of dislocations. In other words, the basic idea of DDT is that the field tensor for cracks can be determined by the convolution of the field tensor for dislocations with a dislocation distribution function.

In general, DDT is capable of analysis of multiple curved cracks. Here, for simplicity, we consider one straight crack, i.e. a plane weakened by one straight crack of length $2a$ along the x-axis (Figure 1). The parametric form of the crack is

$$x_1=\alpha(s)=as;\; y_1=\beta(s)=0,\; -1<s<1. \tag{9}$$

The antiplane Cauchy traction vector (t_z) and the double stress traction vector (q_z) on the surface of the crack (for which $n_x=0$, $n_y=1$, Figure 1) in terms of shear stress components in the Cartesian coordinates (x, y) can be determined by using Eq. (5). Considering the non-vanishing stress components, the antiplane Cauchy traction along the crack face is

$$t_z=\tau_{yz}-(\tau_{zyx,x}+\tau_{zyy,y}+\tau_{zxy,x}), \tag{10}$$

and the double stress traction is

$$q_z=\tau_{zyy}. \tag{11}$$

A crack is constructed by a continuous distribution of dislocations. Consequently, using the principal of superposition, the antiplane Cauchy and double stress tractions on the surface of the crack due to the presence of the above-mentioned distribution of dislocations on the crack are

$$\begin{cases} t_z(\alpha_k(s))=a\int_{-1}^{1}K^t(s,t)B_z dt \\ q_z(\alpha_k(s))=a\int_{-1}^{1}K^q(s,t)B_z dt \end{cases} \tag{12}$$

while $B_z(t)$ is the unknown effective dislocation density of a crack for which (Hills et al. [13])

$$B_z(t)=\frac{db_z(t)}{dt}. \tag{13}$$

The kernels $K^t(\alpha(s),\beta(t))=K^t(s,t)$ and $K^q(\alpha(s),\beta(t))=K^q(s,t)$ are

$$K^t(s,t)=\frac{1}{b_z}[\tau_{yz}(X,Y)-\tau_{zyx,x}(X,Y)-\tau_{zyy,y}(X,Y)-\tau_{zxy,x}(X,Y)];\; K^q(s,t)=\frac{1}{b_z}\tau_{zyy}(X,Y), \tag{14}$$

in which $X=\alpha(s)-\alpha(t)$, $Y=\beta(s)-\beta(t)$.

Using Eqs. (6) and (8), the kernels in Eq. (14) are simplified to

$$K^t(s,t)=\frac{\mu}{2\pi}\frac{X}{R^2}\left\{1-\frac{R}{l}K_1(R/l)+6\frac{l^2}{R^2}-3K_0(R/l)-6\frac{l}{R}K_1(R/l)\right\}$$
$$-\frac{\mu}{\pi}\frac{X^3}{R^4}\left\{4\frac{l^2}{R^2}-2K_0(R/l)-\left(\frac{1}{2}\frac{R}{l}+4\frac{l}{R}\right)K_1(R/l)\right\},$$
$$K^q(s,t)=\frac{\mu}{2\pi}\frac{XY}{R^2}\left\{-2\frac{l^2}{R^2}+K_0(R/l)+2\frac{l}{R}K_1(R/l)\right\}, \tag{15}$$

where $R=\sqrt{X^2+Y^2}$.

For a single horizontal crack, in view of Eq. (9), (X, Y) are simplified to $X=\alpha(s)-\alpha(t)=a(s-t)$, $Y=\beta(s)-\beta(t)=0$. Consequently, the kernel corresponding to double stress traction vanishes, i.e. $K^q(s,t)=0$, and Eq. (12) reduces to a single equation, i.e. Eq. $(12)_1$, where the unknown effective dislocation density $B_z(t)$ needs to be determined by using the traction-free boundary conditions, i.e.

$$t_i=q_i=0. \tag{16}$$

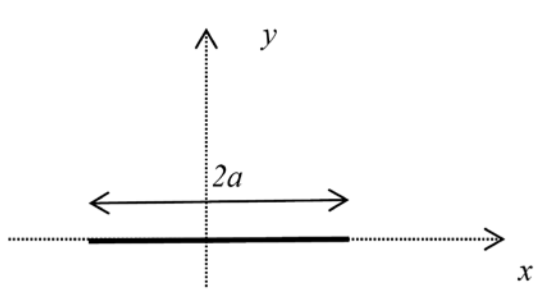

Figure 1: Plane weakened by one crack.

For the mode III crack problem considered herein, we assume the following conditions at infinity

$$\tau^{\infty}_{xz}=0;\ \tau^{\infty}_{yz}=\tau_{yz0}. \tag{17}$$

The defectless plane under the above-mentioned loading experiences the following stress components:

$$\tau_{xz}=0;\ \tau_{yz}=\tau_{yz0}, \tag{18}$$

and thus, the tractions at the location (x_k, y_k) of the crack in the defectless plane read

$$t_z(x_k, y_k)=\tau_{yz0};\ q_z(x_k, y_k)=0. \tag{19}$$

By virtue of the Bueckner superposition principle [14] to satisfy the boundary conditions at the crack face, the left-hand side of Eq. $(12)_1$ is identical to the traction with opposite sign in Eq. (19), i.e.

$$a\int_{-1}^{1}K^{t}B_z dt=-\tau_{yz0}. \tag{20}$$

The closure requirement should be satisfied to ensure single-valued field around the crack surface, i.e.

$$\int_{-1}^{1}B_z(t)dt=0, \tag{21}$$

which constitutes N closure equations. The unknown dislocation densities (B_z) can be determined by solving the above-mentioned system of integral equations, i.e. Eqs. (20) and (21).

Having calculated the dislocation density functions, by using the superposition principle, the stress field inside the medium at an arbitrary point with coordinates (x, y) is given by

$$\begin{aligned}\tau^{crack}_{xz}(x,y)&=-\frac{a\mu}{2\pi}\int_{-1}^{1}\frac{Y}{R^2}\left[1-\frac{R}{\ell}K_1\left(\frac{R}{\ell}\right)\right]B_z dt;\\ \tau^{crack}_{yz}(x,y)&=\tau_{yz0}+\frac{a\mu}{2\pi}\int_{-1}^{1}\frac{X}{R^2}\left[1-\frac{R}{\ell}K_1\left(\frac{R}{\ell}\right)\right]B_z dt,\end{aligned} \tag{22}$$

with the double stress tensor components given by

$$\begin{aligned}\tau^{\text{crack}}_{zyx}(x,y)&=\frac{a\mu}{2\pi}\int_{-1}^{1}\left(\frac{X^2}{R^2}\left\{-2\frac{\ell^2}{R^2}+K_0\left(\frac{R}{\ell}\right)+2\frac{\ell}{R}K_1\left(\frac{R}{\ell}\right)\right\}\right)B_z dt,\\ \tau^{\text{crack}}_{zxy}(x,y)&=-\frac{a\mu}{2\pi}\int_{-1}^{1}\left(\frac{Y^2}{R^2}\left\{-2\frac{\ell^2}{R^2}+K_0\left(\frac{R}{\ell}\right)+2\frac{\ell}{R}K_1\left(\frac{R}{\ell}\right)\right\}\right)B_z dt,\\ \tau^{\text{crack}}_{zyy}(x,y)&=-\tau^{c}_{zxx}(x,y)=\frac{a\mu}{2\pi}\int_{-1}^{1}\left(\frac{XY}{R^2}\left\{-2\frac{\ell^2}{R^2}+K_0\left(\frac{R}{\ell}\right)\right.\right.\\ &\left.\left.+2\frac{\ell}{R}K_1\left(\frac{R}{\ell}\right)\right\}\right)B_z dt,\end{aligned} \tag{23}$$

where $X=x-\alpha(t)$, $Y=y-\beta(t)$.

We proceed by considering the horizontal crack (Figure 1) described by the parametric form given by Eq. (9) loaded as shown by Eq. (17), with $\tau_{yz0}=\mu$. The dislocation density $B_{zj}(t)$ is determined by solving the system of Eqs. (20) and (21) and is depicted in Figure 2. To shed light on the effect of the gradient parameter, the results for $\ell=0.05a$, $0.1a$, and $0.5a$ are compared with the classical dislocation density for this crack. The stress components corresponding to the cracked plate are denoted by the superscript "crack".

It is observed that by reducing the gradient parameter, the GradEla results approach those in classical elasticity everywhere (even in the vicinity of the crack tip). It is worth mentioning that, in nonlocal elasticity, it turned out that by reducing the nonlocal parameter, the dislocation density of crack approaches to the one in classical elasticity everywhere, except at the crack tip [15].

By using the stress field given by Eq. (22) for a single horizontal crack, the stress along the x-axis reads

$$\tau^{\text{crack}}_{yz}(x)=\tau_{yz0}+\frac{a\mu}{2\pi}\int_{-1}^{1}\frac{1}{X}\left[1-\frac{R}{\ell}K_1\left(\frac{|X|}{\ell}\right)\right]B_z dt, \tag{24}$$

while $\tau^{\text{crack}}_{xz}(x)=0$. This result is derived using a dislocation-based approach within an incompatible GradEla framework. It is interesting to compare this result with the analytical expression of Aifantis [3, 16] and Isaksson and Hägglund [17], as well as the incompatible nonlocal elasticity dislocation-based approach [15]. The analytical asymptotic expression for the "microstress" field in these works [3, 16] reads

$$\tau^{\text{crack}}_{yz}(x)=\frac{K_{III}}{\sqrt{2\pi r}}(1-e^{-r/\ell}), \tag{25}$$

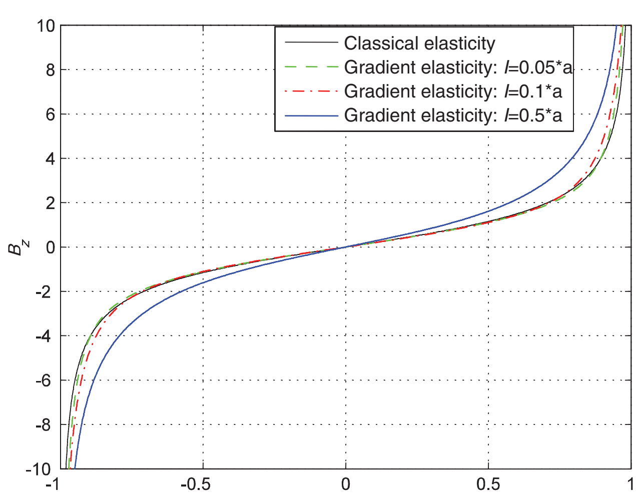

Figure 2: Dislocation density of a crack.

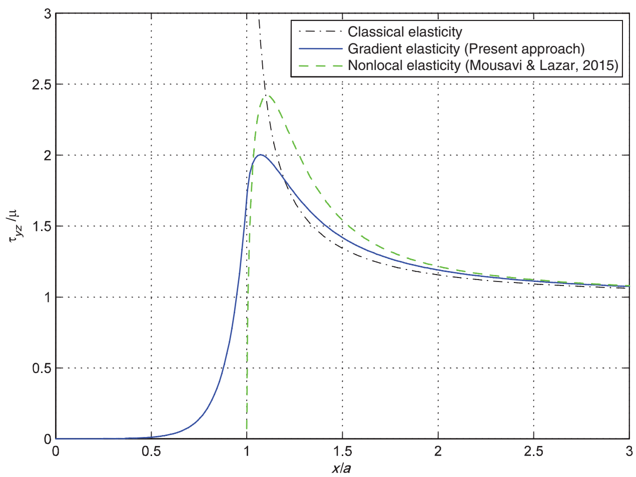

Figure 3: Normalized stress component τ_{yz} for $\ell=0.1a$, in comparison with classical and nonlocal elasticity.

where K_{III} is the mode III stress intensity factor, and $r=|x-a|$ is the polar coordinate centered at the crack tip.

Figure 3 depicts the normalized stress component τ_{yz} in the above-mentioned theories, in comparison with classical elasticity. It should also be noted that since the non-classical traction-free boundary condition is satisfied at the crack faces, consequently it is the traction given by Eq. (10) that vanishes at the crack faces, while the stress component (τ_{yz}) is nonzero at the crack faces.

Moreover, for completeness, we depict in Figure 4 the asymptotic stress field expressions of classical elasticity and GradEla solutions [16] derived without the use of the present continuously distributed dislocation approach. The gradient parameter is assumed to be $\ell=0.2a$. As expected, the gradient theory predicts nonsingular components as opposed to the singular classical field.

The only nonzero double stress component along the crack line, i.e. τ_{zyx} is

$$\tau_{zyx}^{\text{crack}}(x,y)=\frac{a\mu}{2\pi}\int_{-1}^{1}\left(-2\frac{\ell^2}{X^2}+K_0\left(\frac{|X|}{\ell}\right)+2\frac{\ell}{R}K_1\left(\frac{|X|}{\ell}\right)\right)B_z dt, \tag{26}$$

where $X=x-at$. The other double stress components vanish on the x-axis. As expected, the double stress component is singular at the crack tip. The total stress tensor σ_{zy} along the x-axis ($y=0$) reads

$$\sigma_{zy}(x,0)=\tau_{yz0}+\frac{a\mu}{2\pi}\int_{-1}^{1}\frac{1}{X}\left[1-4\frac{\ell^2}{X^2}+2K_0\left(\frac{|X|}{\ell}\right)+4\frac{\ell}{|X|}K_1\left(\frac{|X|}{\ell}\right)\right]B_z dt, \tag{27}$$

and it is singular at the crack tip.

To shed further light on the effect of the non-classical terms in the expression for the traction given by Eq. (10), the classical traction-free condition $t_z=\tau_{yz}=0$ is examined. This condition was also used by Mousavi et al. [18]. The appropriate kernel in this case reads

$$K^t(s,t)=\frac{\tau_{yz}(X,Y)}{b_z}=\frac{\mu}{2\pi}\frac{X}{R^2}\left[1-\frac{R}{\ell}K_1(R/\ell)\right], \tag{28}$$

and the corresponding dislocation density is determined and is compared with the one derived from the nonclassical traction condition in Figure 5. It is observed that at the vicinity of the right (left) crack tips, the dislocation density

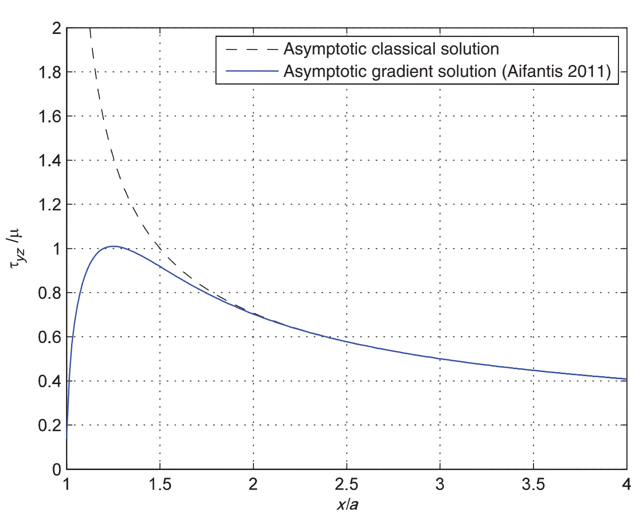

Figure 4: Asymptotic stress fields around the crack tip, $\ell=0.2a$ (Aifantis, [16]).

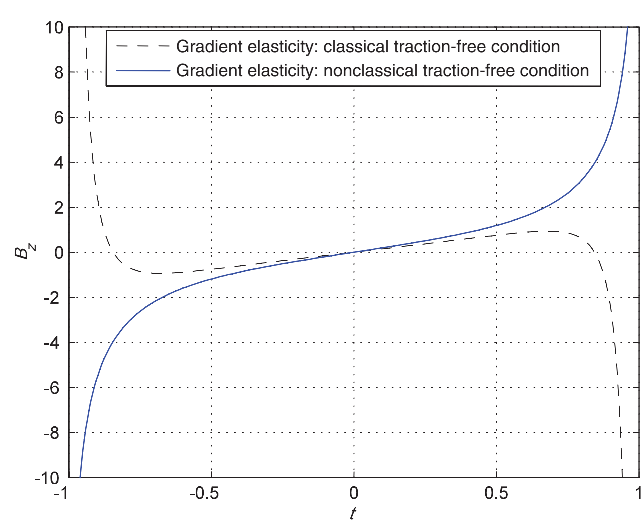

Figure 5: Dislocation density of crack for classical and nonclassical traction-free conditions in gradient theory.

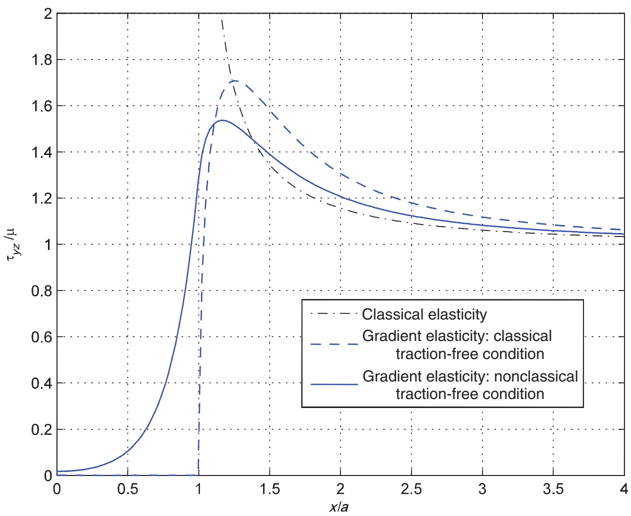

Figure 6: Stress τ_{yz} for classical and nonclassical traction-free conditions in gradient theory compared to the one in classical elasticity, ℓ=0.2a.

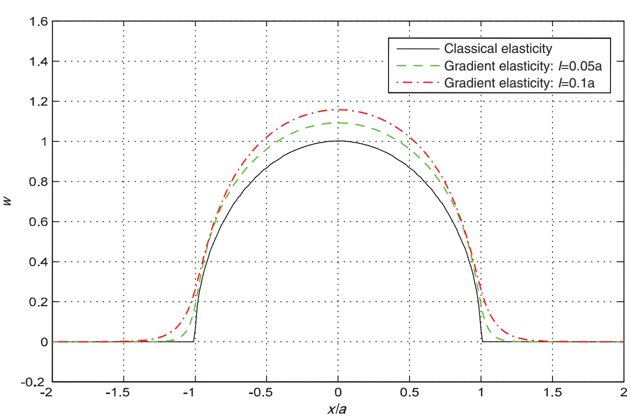

Figure 7: Crack opening displacement for classical and gradient elasticity.

for the nonclassical traction-free condition approaches a positive (negative) singularity, whereas the situation for the classical traction-free condition is vice versa. Such trend has also been reported recently for the dislocation density of cracks within a nonlocal elasticity framework [15].

To investigate the effect of the boundary condition, the stress components resulting from classical and nonclassical traction-free conditions are compared in Figure 6. It is observed that once considering the classical traction-free condition in GradEla, the stress field τ_{yz} is zero along the crack faces.

In addition to regularizing the stress field, GradEla also provides nonsingular strain field. Consequently, it offers a modified crack opening displacement (COD). In contrast, because no nonlocal strain appears in nonlocal elasticity, the COD for nonlocal elasticity is identical to the one in classical elasticity. Figure 7 compares the COD within classical and GradEla.

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