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Wave propagation in an initially stressed transversely isotropic thermoelastic half-space

Abstract: The present paper deals with the study of reflection waves in an initially stressed transversely isotropic medium, in the context of Green and Naghdi (GN) thermoelasticity theory type II and III. The components of displacement, stresses and temperature distributions are determined through the solution of the wave equation by imposing the appropriate boundary conditions. Numerically simulated results are plotted graphically with respect to frequency in order to show the effect of anisotropy.

Keywords: anisotropy; Green and Naghdi theory; reflection waves; thermoelasticity.

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1 Introduction

Several efforts are made to remove the "so-called paradox", inherent in the classical coupled dynamical theory of thermoelasticity [1]: that the thermal signal propagates with an infinite speed. An extended thermoelasticity theory introducing one thermal relaxation time in the thermoelastic process was proposed by Lord and Shulman [2] and the temperature-rate dependent theory of thermoelasticity – which takes into account two relaxation times – was developed by Green and Lindsay [3]. Chandrasekharaiah [4], Hetnarski and Iganazack [5] in their recent surveys, reviewed the theory proposed by Green and Naghdi [6–9] as an alternate way for formulating the propagation of heat. This theory is capable of incorporating thermal pulse transmission in a consistent manner and makes use of general entropy law rather than the usual entropy inequality. The characterization of thermoelastic material response is based on three types of constitutive functions: type I, type II, and type III. When the theory of type I is linearized, a parabolic equation of heat conduction arises. Here, we focus on the theory of type II (a limiting case of type III), which does not admit energy dissipation. Following the Green-Naghdi (GN) theory of thermoelasticity without energy dissipation, further research work was conducted on the wave propagation in isotropic generalized thermoelastic solids (e.g., Quintanilla [10]; Taheri et al. [11]; Puri and Jordan [12]; Roychoudhuri and Byopadhyay [13]; Lazzari and Nibbi [14]; Quintanilla [15]).

Initial stresses may develop in a medium for several reasons, e.g., temperature variation during processing, rapid quenching, slow creep processes, differential external forces, gravity variations, etc. The Earth, in particular, is assumed to be under high initial stresses. Dey et al. [16, 17] studied the propagation of waves in a thermoelastic medium under initial stresses. Gupta and Gupta [18] discussed the reflection of waves in an initially stressed fiber-reinforced transversely isotropic medium. Based on this, the present paper deals with the propagation of waves in an initially stressed transversely isotropic medium in the context of the GN thermoelasticity theory of types II and III. This study may have applications in various fields of science and technology, including atomic physics, aerospace and industrial engineering (thermal power plants, submarine structures, pressure vessels, chemical pipes).

2 Basic equations

The constitutive relations and balance laws for a general anisotropic (with a center of symmetry) initially stressed thermoelastic medium, in the absence of body forces, are given by Green and Naghdi [9] and Montanaro [19] as follows:

– **Constitutive relation:**

$$t_{ij}=C_{ijkl}e_{kl}-\beta_{ij}T, \tag{1}$$

– **Balance law:**

$$t_{ij,j}-P\omega_{ij,j}=\rho\ddot{u}_i \tag{2}$$

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– **Equation of heat conduction:**

$$K_{ij}\dot{T}_{,ij}+K^{*}_{ij}T_{,ij}=(T_0\beta_{ij}\ddot{u}_{i,j}+\rho c^{*}\ddot{T}),\quad i, j=1,\ 2,\ 3, \tag{3}$$

where ρ is the mass density, t_{ij} are the components of stress tensor, u_i is the mechanical displacement vector, $e_{ij}=\frac{(u_{i,j}+u_{j,i})}{2}$ are the components of the infinitesimal strain, T is the temperature change of a material particle, T_0 is the reference uniform temperature of the body, P=-t_{11} is the normal initial stress and $\omega_{ij}=(u_{j,i}-u_{i,j})/2$ is the rotation tensor. Moreover, K_{ij} is the thermal conductivity tensor, K^{*}_{ij} denotes a characteristic material constant tensor, $\beta_{ij}=C_{ijkl}\alpha_{kl}$ denotes the thermal elastic coupling tensor, α_{kl} denotes the coefficient of linear thermal expansion, c^{*} is the specific heat at constant strain and C_{ijkl} is the elasticity matrix. The various material tensors introduced obey the following symmetry properties $C_{ijkl}=C_{klij}=C_{jikl}$, $K^{*}_{ij}=K^{*}_{ji}$, $K_{ij}=K_{ji}$, $\beta_{ij}=\beta_{ji}$. A comma notation is used for spatial derivatives and a superimposed dot denotes time differentiation.

3 Problem formulation

Following Slaughter [20], an appropriate transformation is applied to Eq. (1), in order to derive the governing equations for an initially stressed transversely isotropic medium, when our analysis is restricted to two dimensions. The origin of the coordinate system (x_1, x_2, x_3) is taken at the free surface of the half-space. The x_1–x_2 plane is chosen to coincide with the free surface and the x_3 axis is then normal to the half-space ($x_3\geq 0$). We consider plane waves such that all particles on a line parallel to x_2-axis are equally displaced. Therefore, all field quantities will be independent of the x_2 coordinate. Then, the component of the displacement vector is of the form

$$\vec{u}=(u_1,\ 0,\ u_3), \tag{4}$$

and the solutions are independent of x_2, i.e., $\partial/\partial x_2\equiv 0$. Thus, the governing differential equations for such a medium reduce to:

$$C_{11}\frac{\partial^2 u_1}{\partial x_1{}^2}+\frac{C_{55}}{2}\frac{\partial^2 u_1}{\partial x_3{}^2}+\left(C_{13}+\frac{C_{55}}{2}\right)\frac{\partial^2 u_3}{\partial x_1\partial x_3}-\beta_1\frac{\partial T}{\partial x_1}$$
$$-\frac{P}{2}\left(\frac{\partial^2 u_3}{\partial x_1\partial x_3}-\frac{\partial^2 u_1}{\partial x_3^2}\right)=\rho\frac{\partial^2 u_1}{\partial t^2}, \tag{5}$$

$$\frac{C_{55}}{2}\frac{\partial^2 u_3}{\partial x_1{}^2}+C_{33}\frac{\partial^2 u_3}{\partial x_3{}^2}+\left(C_{13}+\frac{C_{55}}{2}\right)\frac{\partial^2 u_1}{\partial x_1\partial x_3}-\beta_3\frac{\partial T}{\partial x_3}$$
$$-\frac{P}{2}\left(\frac{\partial^2 u_1}{\partial x_1\partial x_3}-\frac{\partial^2 u_3}{\partial x_1^2}\right)=\rho\frac{\partial^2 u_3}{\partial t^2}, \tag{6}$$

$$K_1\frac{\partial^2\dot{T}}{\partial x_1{}^2}+K_3\frac{\partial^2\dot{T}}{\partial x_3{}^2}+K^{*}_1\frac{\partial^2 T}{\partial x_1{}^2}+K^{*}_3\frac{\partial^2 T}{\partial x_3{}^2}$$
$$=\rho c^{*}\frac{\partial^2 T}{\partial t^2}+T_o\left(\beta_1\frac{\partial\ddot{u}_1}{\partial x_3}+\beta_3\frac{\partial\ddot{u}_3}{\partial x_1}\right), \tag{7}$$

where $\beta_1=C_{11}\alpha_1+C_{13}\alpha_3$, $\beta_3=C_{31}\alpha_1+C_{33}\alpha_3$ and we have also used the notation $11\rightarrow 1, 13\rightarrow 5, 33\rightarrow 3$ for the material constants.

To proceed further, it is convenient to introduce the non-dimensional quantities defined by

$$x'_i=\frac{x_i}{L},\ u'_i=\frac{u_i}{L},\ t'_{ij}=\frac{t_{ij}}{C_{11}},\ t'=\frac{t}{t_o},\ T'=\frac{T}{T_o}, \tag{8}$$

where L, t_o, T_o are parameters having dimension of length, time and temperature, respectively.

4 Solution of the problem

Let $\vec{p}=(p_1,\ 0,\ p_3)$ denote the unit propagation vector, with c and ξ denoting respectively the phase velocity and the wave number of plane waves propagating in the x_1–x_3 plane. Then by seeking for plane wave solutions of the equations of motion of the form

$$(u_1,\ u_3,\ T)=(\bar{u}_1,\ \bar{u}_3,\ \bar{T})e^{i\xi(p_1x_1+p_3x_3-ct)}. \tag{9}$$

We introduced Eqs. (8) and (9) into Eqs. (5)–(7) to obtain three homogeneous equations in three unknowns. Non-trivial solutions of the resulting system of equations can be derived when the following condition is fulfilled

$$Ac^6+Bc^4+Cc^2+D=0, \tag{10}$$

where

$$A=f_{10},\ B=-f_5f_{10}-f_1f_{10}+f_6f_8+f_3f_7+f_9,\quad D=f_1f_5f_9-f_2f_4f_9,$$

$$C=-f_5(f_9+f_1f_{10}-f_3f_7)-f_1(f_9-f_6f_8)-f_2(f_4f_{10}+f_6f_7)+f_3f_4f_8,$$

$$f_1=p_1^2d_1+p_3^2d_3-d_{13}Pp_3^2/2,\ f_2=p_1p_3d_3-d_{13}Pp_1p_3/2,$$
$$f_3=ip_1d_4,\ f_4=p_1p_3d_2-d_{13}Pp_1p_3/2,$$

$$f_5=p_1^2d_3+p_3^2d_5+d_{13}Pp_1^2/2,\ f_6=ip_3d_6,\ f_7=ip_7d_{11},\ f_8=ip_3d_{12},$$
$$f_9=i\omega p_1^2-d_8p_1^2+i\omega\bar{k}p_3^2-d_9p_3^2$$

$$f_{10}=\varepsilon_1,\ d_1=\frac{C_{11}t_o^2}{\rho L^2},\ d_2=\frac{(C_{13}+C_{55}/2)t_o^2}{\rho L^2},\ d_3=\frac{C_{55}t_o^2}{2\rho L^2},$$
$$d_4=\frac{\beta_1 T_o t_o^2}{\rho L^2},\ d_5=\frac{C_{33}t_o^2}{\rho L^2},\ d_6=\frac{\beta_3 T_o t_o^2}{\rho L^2},$$

$$\bar{k}=d_7=\frac{k_3}{k_1},\ d_8=\frac{k_1^* t_o}{k_1},\ d_9=\frac{k_3^* t_o}{k_1},\ \varepsilon_1=d_{10}=\frac{\rho C^* L^2}{k_1 t_o},$$
$$d_{11}=\frac{\beta_1 L^2}{k_1 t_0},\ d_{12}=\frac{\beta_3 L^2}{k_1 t_0},\ d_{13}=\frac{t_0^2}{\rho L^2}.$$

The roots of this equation give three values of c^2. The corresponding three positive values of c will then be the propagation velocities of the three possible waves. The waves with velocities c_1, c_2, c_3 correspond to three types of quasi waves. We name these waves as quasi-longitudinal displacement (qLD) wave, quasi thermal wave (qT), and quasi transverse displacement (qTD) wave.

5 Reflection waves

Consider a homogeneous initially stressed transversely isotropic half-space, in the context of G-N thermoelasticitytheory of types II and III, occupying the region $x_3>0$. Incident qLD or qT or qTD waves at the interface will generate reflected qLD, qT and qTD waves in the half-space $x_3>0$. The displacements and temperature distributions are given by

$$(u_1,u_3,T)=\sum_{j=1}^{6}A_j(1,r_j,s_j)e^{iB_j},\tag{11}$$

where

$$B_j=\begin{cases}\omega(t\text{-}x_1\sin e_j\text{-}x_3\cos e_j)/c_j, & j=1,2,3,\\ \omega(t\text{-}x_1\sin e_j+x_3\cos e_j)/c_j, & j=4,5,6,\end{cases}\tag{12}$$

with ω denoting the angular frequency. Here the subscripts 1, 2, 3 denote, respectively, the quantities corresponding to incident qLD, qT and qTD waves, whereas the subscripts 4, 5 and 6 denote, respectively, the corresponding reflected waves, with

$$r_j=\frac{\wedge_{1j}}{\wedge_j},\quad s_j=\frac{\wedge_{2j}}{\wedge_j},$$

For incident waves:

– qLD-wave: $p_1=\sin e_1$, $p_3=\cos e_1$,

– qT-wave: $p_1=\sin e_2$, $p_3=\cos e_2$,

– qTD-wave: $p_1=\sin e_3$, $p_3=\cos e_3$,

For reflected waves:

– qLD-wave: $p_1=\sin e_4$, $p_3=\cos e_4$,

– qT-wave: $p_1=\sin e_5$, $p_3=\cos e_5$,

– qTD-wave: $p_1=\sin e_6$, $p_3=\cos e_6$.

It is further noted that $e_1=e_4$, $e_2=e_5$, $e_3=e_6$, that is, the angle of incidence is equal to the angle of reflection in generalized thermoelastic transversely isotropic media, so that the velocities of reflected waves are equal to their corresponding incident waves, i.e., $c_1=c_4$, $c_2=c_5$, $c_3=c_6$.

6 Boundary conditions

The boundary conditions at the thermally insulated surface $x_3=0$ are given by

$$t_{33}=0,\quad t_{31}=0,\quad \frac{\partial T}{\partial x_3}=0,\tag{13}$$

where

$$t_{33}=C_{13}\frac{\partial u_1}{\partial x_1}+C_{33}\frac{\partial u_3}{\partial x_3}\text{-}\beta_3 T,\quad t_{31}=\frac{C_{55}}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right).\tag{14}$$

The wave numbers ξ_j, $j=1,2,\ldots,6$, and the apparent velocity c_j, $j=1,2,\ldots,6$, are connected through the relation

$$c_1\xi_1=c_2\xi_2=\ldots=c_6\xi_6=\omega,\tag{15}$$

at the surface $x_3=0$. In order to satisfy the boundary conditions given by Eqs. (3), (15) may also be re-written as

$$\frac{\sin e_1}{c_1}=\frac{\sin e_2}{c_2}=\ldots=\frac{\sin e_6}{c_6}=\frac{1}{c}.\tag{16}$$

Making use of Eqs. (8), (14), (15) and (16), along with the thermally insulated boundary conditions given by Eq. (3), we obtain

$$\wedge_j=\begin{vmatrix}\xi^2(f_5\text{-}c_j^2) & \xi f_6\\ c_j^2\xi^3 f_8 & \xi^2(f_9+f_{10}c_j^2)\end{vmatrix},\quad \wedge_{1j}=\begin{vmatrix}\xi^2 f_4 & \xi f_6\\ c_j^2\xi^3 f_7 & \xi^2(f_9+f_{10}c_j^2)\end{vmatrix},\quad \wedge_{2j}=\begin{vmatrix}\xi^2 f_4 & \xi^2(f_5\text{-}c_j^2)\\ c_j^2\xi^3 f_7 & c_j^2\xi^3 f_8\end{vmatrix}.$$

$$\sum_{j=1}^{6} A_{ij} A_j = 0, \quad i=1,\ 2,\ 3, \tag{17}$$

where

$$A_{1j}=\begin{cases} a_j^1+r_j a_j^2-t_j a_j^3, & j=1,2,3,\\ a_j^1-r_j a_j^2-t_j a_j^3, & j=4,5,6,\end{cases} \quad A_{2j}=\begin{cases} b_j^1+r_j b_j^2, & j=1,2,3,\\ -b_j^1+r_j b_j^2, & j=4,5,6,\end{cases}$$

$$A_{3j}=\begin{cases} t_j c_j^1, & j=1,2,3,\\ -t_j c_j^1, & j=4,5,6,\end{cases}$$

where

$$a_j^1=-\frac{i\omega C_{13}}{C_{11}}\frac{\sin e_j}{c_j},\quad a_j^2=\frac{i\omega C_{33}}{C_{11}}\frac{\cos e_j}{c_j},\quad a_j^3=\frac{i\beta_3 T_0 s_j}{C_{11}},$$
$$b_j^1=\frac{i\omega C_{55}\cos e_j}{2C_{11}c_j},\quad b_j^2=\frac{i\omega C_{55}\sin e_j}{2C_{11}c_j},\quad c_j^1=\frac{\omega\cos e_j}{c_j}.$$

– **Incident qLD-wave:**

In the case of incident qLD-wave, $A_2=A_3=0$. Dividing the set of Eqs. (17) throughout by A_1, we obtain a system of three non-homogeneous equations in three unknowns which can be solved by using the Gauss elimination method to obtain

$$Z_i=\frac{A_{i+3}}{A_1}=\frac{\Delta_i^1}{\Delta}, \quad i=1,\ 2,\ 3. \tag{18}$$

– **Incident qT-wave:**

In the case of incident qT-wave, $A_1=A_2=0$, and thus we have

$$Z_i=\frac{A_{i+3}}{A_2}=\frac{\Delta_i^2}{\Delta}, \quad i=1,\ 2,\ 3. \tag{19}$$

– **Incident qTD-wave:**

In the case of incident qTD-wave, $A_1=A_2=0$, and thus we have

$$Z_i=\frac{A_{i+3}}{A_3}=\frac{\Delta_i^3}{\Delta}, \quad i=1,\ 2,\ 3, \tag{20}$$

where $\Delta=|A_{i\,i+3}|_{3\times3}$ and $\Delta_i^p\,(i=1,2,3,\ p=1,2,3)$ can be obtained by replacing, respectively, the first, second and third column of Δ by $[-A_{1p}\,-A_{2p}\,-A_{3p}]^T$, where $[\ ]^T$ denotes the transpose of the matrix.

7 Numerical results and discussion

In order to illustrate the theoretical analysis given in the preceding sections, we now present some numerical results. The following relevant physical constants for Cobalt material are taken from Dhaliwal and Singh [21] for a thermoelastic transversely isotopic material:

$$\begin{aligned}
&C_{11}=3.071\times10^{11}\,\mathrm{Nm^{-2}},\ C_{12}=1.650\times10^{11}\,\mathrm{Nm^{-2}},\\
&C_{13}=1.027\times10^{11}\,\mathrm{Nm^{-2}},\ C_{33}=3.581\times10^{11}\,\mathrm{Nm^{-2}},\\
&C_{55}=1.51\times10^{11}\,\mathrm{Nm^{-2}},\ \beta_1=7.04\times10^{6}\,\mathrm{Nm^{-2}K},\\
&\beta_1=6.98\times10^{6}\,\mathrm{Nm^{-2}K},\ \rho=8.836\times10^{3}\,\mathrm{Kgm^{-3}},\\
&K_1=6.90\times10^{2}\,\mathrm{Wm^{-1}K},\ K_3=7.01\times10^{2}\,\mathrm{Wm^{-1}K},\\
&K_1^*=1.313\times10^{2}\,\mathrm{W\ sec},\ K_3^*=1.54\times10^{2}\,\mathrm{W\ sec},\\
&c^*=4.27\times10^{2}\,\mathrm{J\ \ Kg\ \ K},\ T=298\ \mathrm{K}.
\end{aligned}$$

The variations of amplitude ratio of the reflected qLD, qTD and qT waves, for incident qLD, qTD and qT waves at the free surface are shown graphically in order to compare the results obtained in the two cases: i.e., incident waves for a transversely isotropic medium in the context of thermoelasticity with energy dissipation (ISTIWED) and the standard case for isotropic thermoelastic (ISIWED) waves. In Figure 1, the graphical representation is given for the variation of amplitude ratios $|Z_1|$, $|Z_2|$ and $|Z_3|$ for incident

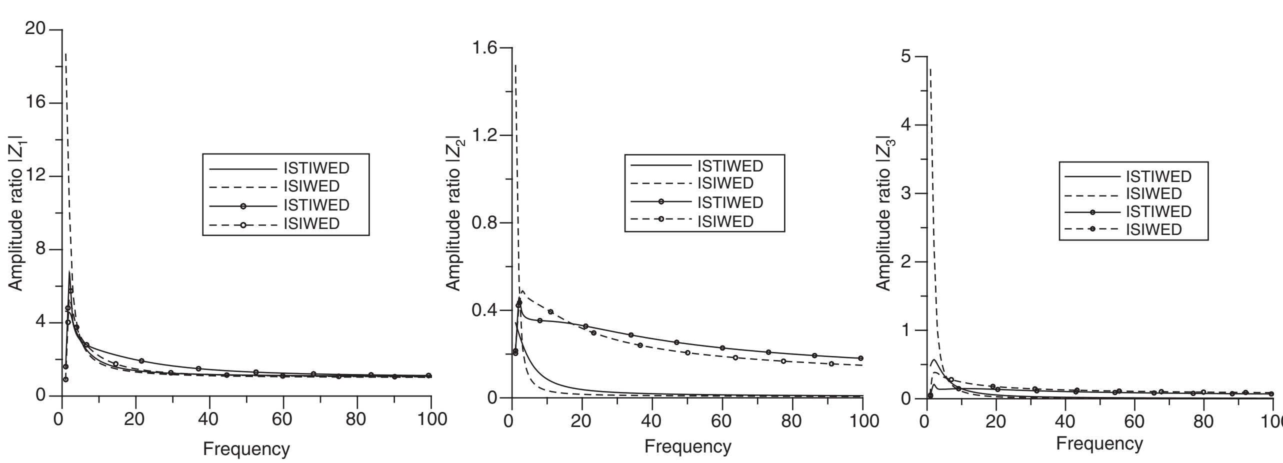

Figure 1 The variation of amplitude ratios of $|Z_i|$ (i=1, 2, 3) with frequency for incident qLD-wave.

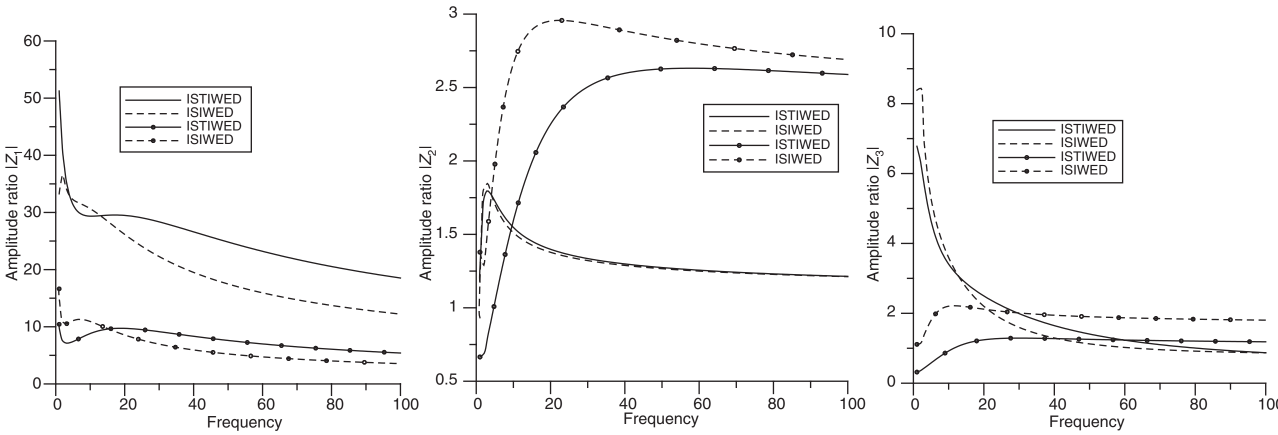

Figure 2 The variation of amplitude ratios of $|Z_i|$ (i=1, 2, 3) with frequency for incident qT-wave.

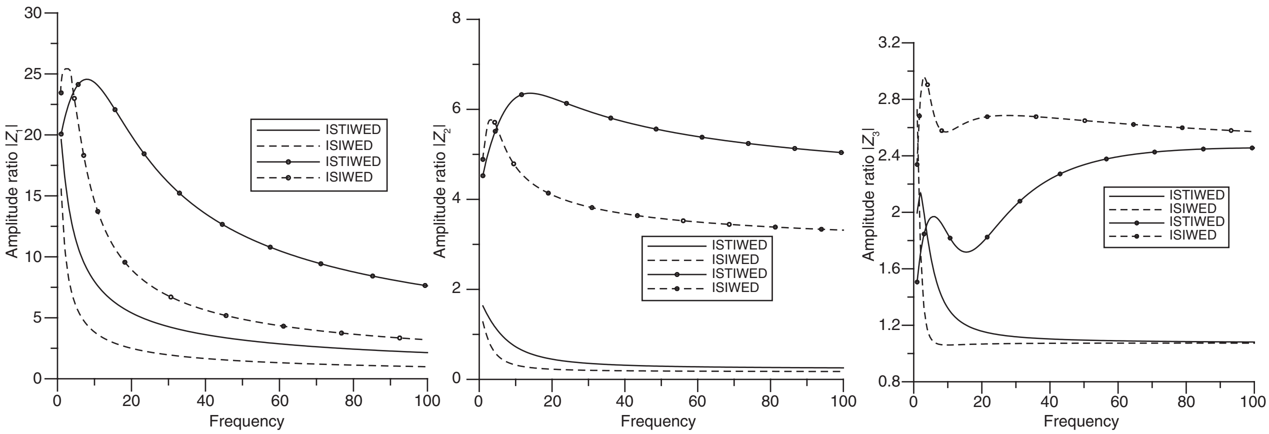

Figure 3 The variation of amplitude ratios of $|Z_i|$ (i=1, 2, 3) with frequency for incident qTD-wave.

qLD-wave. Figures 2 and 3, respectively, represent similar situations, when qTD and qT waves are incident.

Here $|Z_1|$, $|Z_2|$ and $|Z_3|$ are the amplitude ratios of reflected qLD, qTD and qT waves, respectively. These variations are shown for two angles of incidence viz, θ=30°, 45°. In these figures the solid curves lines correspond to the case of ISTIWED, while broken curves correspond to the case of ISIWED. Moreover, the curves without the center symbol correspond to the case when θ=30°, and those with the center symbol (–*o*–*o*–) correspond to the case of θ=45°.

8 Conclusion

Reflection of waves from the free surface of an initially stressed transversely isotropic medium in the context of the G-N thermoelasticity theory of types II and III has been discussed. The appreciable effect of anisotropy and angle of incidence is depicted on amplitude ratios for various reflected waves. It can be concluded from the graphs that the amplitude ratio $|Z_1|$ exhibits higher values because of anisotropy for all three types of incident waves (viz., qLD, qTD, qT), whereas the amplitude ratios $|Z_2|$, $|Z_3|$ shows oscillating behavior.

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