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Research Article

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Characterization of neighborhood operators based on neighborhood relationships

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Abstract: Neighborhood relationships play a pivotal role in rough set theory, addressing the limitations of equivalence relations. This article focuses on defining upper and lower approximation operators using neighborhood relationships and exploring their properties in terms of serialization, inverse serialization, reflexivity, symmetry, transitivity, and Euclidean relations. Furthermore, a necessary and sufficient condition for the upper approximation operator to function as a topological closure operator is derived. Overall, this research sheds light on the significance of neighborhood relationships and their implications within rough set theory.

Keywords: neighborhood relationship, topology, closure operator

1 Introduction

Rough set theory was proposed by Professor Pawlak in 1982 as a new and effective tool for handling imprecise, incomplete, and inconsistent data [1–3]. In recent years, with the development of technology, rough set theory has been widely applied in various fields such as artificial intelligence, machine learning, data mining, and pattern recognition. It has been continuously optimized, leading to the definition of many effective models. However, scholars found that the rough set of equivalence relations has its limitations. To solve these problems, scholars have defined rough sets based on neighborhood relationships. Neighborhood rough set has become an important branch of rough set. For information systems, a neighborhood relationship is to classify and distinguish the range while keeping its neighborhood range unchanged. The purpose of neighborhood relationship is to standardize the operation rules of neighborhood operators under various constraints and obtain corresponding results.

The process of human understanding various things in the world begins with establishing concepts, from which continuous thinking approximation is conducted. This process ultimately leads to rational judgments and decisions. Pawlak's rough set has expanded from two aspects: granulation and approximation. Based on human cognitive methods, it has further deepened our understanding of the profound meaning of rough sets over the course of its 40-year development. In 1990, Lin [4] proposed the concept of neighborhood rough sets, using neighborhood relationships as a new granulation method to implement domain approximation. Lin [5]

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put forward the concept of neighborhood relationship and neighborhood model based on interior point and closure in topology. In decision analysis, relevant scholars introduced the dominance relation rough set model for the problem of ordered classification, which is used for the case that the decision attributes contain ordered structure. Mao et al. [6] researched on attribute reduction under co-occurrence neighborhood relationships. Li and Yang [7] investigated three-way decisions with fuzzy probabilistic covering-based rough sets and their applications in credit evaluation. Qi et al. [8] obtained some neighborhood-related fuzzy covering-based rough set models and their applications for decision-making.

Topology [9] is a very important basic subject in mathematics, which provides a strong foundation in mathematics for the study of other theories. In this article, we discuss some properties of approximation operators with the help of neighborhood relations and prove the conditions under which the topological closure operator holds. From the perspective of closure operator, we prove the relationship between closure operators and neighborhood relations.

Although many generalized rough set models have been proposed. However, the equivalence relation on the universe plays a crucial role in the Pawlak rough set model, but in practical problems, the neighborhood relationship on the universe is not equivalent. Because the application of Pawlak rough set model is limited, it must be generalized. This article studies the rough set model of the general neighborhood relationship.

This article is organized as follows: Section 2 reviews some elementary concepts on relations, defines neighborhood relation, and obtains some interesting results; Section 3 is the core of this article, wherein we introduce the properties of $\overline{\delta}$ under neighborhoods relationships and obtain the necessity and sufficiency conditions for neighborhood operators to become topological closure operator under condition attribute B; and Section 4 summarizes this article.

2 Preliminaries

A decision information system is a binary group $DS = \langle U, AT \cup D\rangle$, where U is a non-empty domain, AT is a condition attribute, D is a decision attributes, and $AT \cap D = \varnothing$.

$U/\mathrm{IND}(D) = \{X_1, X_2, \dots, X_N\}$ represents the division on the domain of discourse induced by the decision attribute D, $\forall X_i \in U/\mathrm{IND}(D)$, and X_i represents the ith decision, $1 \le i \le N$.

Definition 2.1. Given a decision information system (DIS) $\forall B \subseteq AT$, and $r \ge 0$, we define

$$N_B = \{(x, y) \in U \times U | \triangle_B(x, y) \le r\}, \tag{1}$$

where $\forall x, y \in U$, $\triangle_B(x, y)$ represents the distance between x and y under the condition attribute B, and $\triangle_B(x, y) = \sqrt{\sum_{a\in B}(a(x)^2 + a(y)^2)}$, where $a(x)$ represents the value of x in attribute a.

$\forall x \in U$ and according to equation (1), we obtain the neighborhood of sample x with respect to B:

$$N_B(x) = \{y \in U | (x, y) \in N_B\}, \tag{2}$$

where $\{N_B(x) : \forall x \in U\}$ represents a neighborhood granulation result induced by the conditional attribute B, which constitutes the covering of domain U.

Definition 2.2. [10] U is a non-empty set, $T \subseteq 2^U$. T is called a topology, if it satisfies the following conditions:
(1) $\varnothing, U \in T$,
(2) if $A, B \in T$, then $A \cap B \in T$,
(3) if $T_1 \subseteq T$, then $\bigcup_{A\in T_1} A \in T$.

(U, T) is called a topological space, and each element of T is called an open set of a topological space (U, T). If $X \subseteq U$ and $x \in X$, there exists an open set $A \in T$ such that $x \in A \subseteq X$, then X is called the open neighborhood of point x. If X is the open neighborhood of point x, then the point x is an inner point of the set X.

If the set X is an open set of topological space, then the complement X of U is a closed set. If the set X is both an open set and a closed set, then X is said to be an open close set.

In a topological space, the closure of the set X is defined as follows:

$$Cl(X) = \bigcap\{F|\forall F \subseteq U, X \subseteq F\}.$$

Definition 2.3. [11] Let $N_B : U \longrightarrow P(U)$ be a neighborhood operator.
(1) For any $x \in U$, there exists $y \in U$ such that $y \in N_B(x)$. That is to say, $\forall x \in U, N_B(x) \neq \emptyset$. Then, N_B is called serial.
(2) For any $x \in U$, there exists $y \in U$ such that $x \in N_B(y)$. That is to say, $N_B(U) = U$. Then, N_B is called inverse serial.
(3) For any $x \in U$, then $x \in N_B(x)$, and N_B is called reflexive.
(4) For any $x, y \in U$, $x \in N_B(y)$ implies $y \in N_B(x)$, and N_B is called symmetric.
(5) For any $x, y, z \in U$, $y \in N_B(x)$ and $z \in N_B(y)$ imply $z \in N_B(x)$. Then, N_B is called transitive.
(6) For any $x, y, z \in U$, $y \in N_B(x)$ and $z \in N_B(x)$ imply $z \in N_B(x)$. Then, N_B is called Euclidean.

Definition 2.4. Let N_B and N_S be two neighborhood relations on U.

$$N_B \cup N_S = \{(x, y)|xN_By \vee xN_Sy\},$$
$$N_B \cap N_S = \{(x, y)|xN_By \wedge xN_Sy\},$$
$$N_B^{-1} = \{(y, x)|xN_By\},$$
$$\sim N_B = \{(x, y)|(x, y) \notin N_B\}.$$

They are called the union, intersection, inverse, and complement of N_B and N_S, respectively.

Obviously, N_B is reflexive $\Leftrightarrow N_B^{-1}$ is reflexive. N_B is transitive $\Leftrightarrow N_B^{-1}$ is transitive. N_B is symmetrical $\Leftrightarrow N_B = N_B^{-1}$. For any neighborhood relation N_B on U, $N_B \cap N_B^{-1}$and $N_B \cup N_B^{-1}$ are symmetric.

Let N_B be a neighborhood relation over U, $\forall x, y \in U$, if $(x, y) \in N_B$, then x is the predecessor of y, and y is the successor of x. Record separately as $N_{B_p}(y)$ and $N_{B_s}(x)$. Define:

$$N_B(x) = N_{Bs}(x) \cap N_{B_p}(x).$$
$$N_{B_s}(x) = y \in U, (x, y) \in N_B.$$
$$N_{B_p}(x) = y \in U, (x, y) \in N_B.$$
$$N_{B_{p\wedge s}}(x) = N_{B_s} \cap N_{B_p}(x).$$
$$N_{B_{p\vee s}}(x) = N_{B_s}(x) \cup N_{B_p}(x).$$

Obviously, these neighborhoods have the following relationships:

$$N_{B_{p\wedge s}}(x) \subseteq N_{B_p}(x) \subseteq N_{B_{p\vee s}}(x).$$
$$N_{B_{p\wedge s}}(x) \subseteq N_{B_s}(x) \subseteq N_{B_{p\vee s}}(x).$$
$$N_{B_p}(x) = N_{B_s}^{-1}(x).$$

The neighborhood relation N_B and the neighborhood operators N_{B_s} and N_{B_p} can be mutually uniquely determined as follows:

$$(x, y) \in N_B \Leftrightarrow x \in N_{B_p}(y) \Leftrightarrow y \in N_{B_s}(x).$$

Theorem 2.5. *Let N_B and M_B be two neighborhood relations on U, then*
(1) $N_B \subseteq M_B \Leftrightarrow \forall x \in U, N_{B_s}(x) \subseteq M_{B_s}(x)$.
(2) $(\sim N_{B_s})(x) = \sim N_{Bs}(x)$.

(3) $(N_{B_s} \cap M_{B_s})(x) = N_{B_s}(x) \cap M_{B_s}(x)$, *especially* $N_{B_{p\wedge s}}(x) = (N_{B_s} \cap N_{B_s}^{-1})(x)$.

(4) $(N_{B_s} \cup M_{B_s})(x) = N_{Bs}(x) \cup M_{Bs}(x)$, *especially* $N_{B_{p\vee s}}(x) = (N_{B_s} \cup N_{B_s}^{-1})(x)$.

Proof. (1) $\forall y \in N_{Bs}(x)$, we have $(x, y) \in N_B$. Since $N_B \subseteq M_B$, thus $(x, y) \in M_B$, we obtain $y \in M_{B_s}(x)$, therefore $N_{B_s}(x) \subseteq M_{B_s}(x)$. It is similar to prove the converse.

Similarly, (2)–(4) can be obtained.

The properties of neighborhood relations can be described by the following neighborhood operators:

N_B is serial $\Leftrightarrow \forall x \in U, N_{B_s}(x) \neq \varnothing \Leftrightarrow N_{B_p}(U) = U$.

N_B is inverse-serial $\Leftrightarrow \forall x \in U, N_{B_p}(x) \neq \varnothing \Leftrightarrow N_{B_s}(U) = U$.

N_B is reflexive $\Leftrightarrow \forall x \in U, x \in N_{B_s}(x)$.

N_B is symmetric $\Leftrightarrow \forall x, y \in U, x \in N_{Bs}(y)$ implies $y \in N_{Bs}(x)$.

N_B is transitive $\Leftrightarrow \forall x, y, z \in U, y \in N_{B_s}(x)$ and $z \in N_{B_s}(y)$ imply $z \in N_{B_s}(x)$

$$\Leftrightarrow \forall x, y \in U, y \in N_{B_s}(x) \text{ implies } N_{Bs}(y) \subseteq N_{B_s}(x).$$

N_B is Euclid $\Leftrightarrow \forall x, y, z \in U, y \in N_{B_s}(x)$ and $z \in N_{B_s}(x)$ imply $z \in N_{B_s}(x)$.

$$\Leftrightarrow \forall x, y \in U, y \in N_{B_s}(x) \text{ implies } N_{B_s}(x) \subseteq N_{Bs}(y).$$ □

3 Main conclusions

Definition 3.1. Let U be a finite non-empty domain and N_B be an arbitrary neighborhood relation over U, then (U, N_B) is called a generalized approximation space by condition attribute B. For any $X \subseteq U$, the lower neighborhood operator $\underline{\delta}_B$ and the upper neighborhood operator $\overline{\delta}_B$ of X with respect to B are defined as follows:

$$\underline{\delta}_B(X) = \{y \in U | N_B(y) \subseteq X\},$$
$$\overline{\delta}_B(X) = \{y \in U | N_B(y) \cap X \neq \varnothing\}.$$

The positive neighborhood, the negative neighborhood, and the boundary of X with respect to the approximation space (U, N_B) are defined as follows:

$$\mathrm{pos}_B(X) = \underline{\delta}_B(X),$$
$$\mathrm{neg}_B(X) = U - \overline{\delta}_B(X),$$
$$bn_B(X) = \overline{\delta}_B(X) - \underline{\delta}_B(X).$$

When $\underline{\delta}_B(X) = \overline{\delta}_B(X)$, X is said to be definable with respect to the approximation space (U, N_B); otherwise, X is said to be rough.

Remark 3.2. If N_B is an equivalence relation, then $N_{B_s}(x)$ is the N_B equivalence class and is abbreviated as $[x]_{N_B}$, i.e., the equivalence class of x can be regarded as the neighborhood of x, then the obtained lower approximation $\underline{\delta}_B(X)$ and upper approximation $\overline{\delta}_B(X)$ are the lower approximation $\underline{R}(X)$ and upper approximation $\overline{R}(X)$ in the sense of Pawlak.

Obviously, $\underline{\delta}$ and $\overline{\delta}$ are operators of $P(U) \to P(U)$.

Theorem 3.3. *Let N_B be an arbitrary neighborhood relation over the domain U, then the lower and upper approximation operators given by Definition* 3.1 *have the following properties*:

(1) $\underline{\delta}_B(X) = \sim(\overline{\delta}_B(\sim X)), \overline{\delta}_B(X) = \sim(\underline{\delta}_B(\sim X))$.

(2) $\underline{\delta}_B(U) = U, \overline{\delta}_B(\varnothing) = \varnothing$.

(3) $\underline{\delta}_B(X \cap Y) = \underline{\delta}_B(X) \cap \underline{\delta}_B(Y), \overline{\delta}_B(X \cup Y) = \overline{\delta}_B(X) \cup \overline{\delta}_B(Y)$.

(4) $X \subseteq Y \Rightarrow \underline{\delta}_B(X) \subseteq \underline{\delta}_B(Y), \overline{\delta}_B(X) \subseteq \overline{\delta}_B(Y)$.
(5) $\forall X, Y \subseteq U$, $\underline{\delta}_B(X \cup Y) \supseteq \underline{\delta}_B(X) \cup \underline{\delta}_B(Y)$, $\overline{\delta}_B(X \cap Y) \subseteq \overline{\delta}_B(X) \cap \overline{\delta}_B(Y)$.

Proof. (1) Since $x \in \underline{\delta}_B(X) \Leftrightarrow N_B(x) \subseteq X \Leftrightarrow N_B(x) \cap (\sim X) = \varnothing$

$\Leftrightarrow x \in \sim(\overline{\delta}_B(\sim X))$.

Therefore, $\underline{\delta}_B(X) = \sim(\overline{\delta}_B(\sim X))$. It is similar to prove $\overline{\delta}_B(X) = \sim(\underline{\delta}_B(\sim X))$.

(2) Since $x \in \underline{\delta}_B(U) \Leftrightarrow N_B(x) \subseteq U \Leftrightarrow N_B(x) \cap (\sim U) = \varnothing \Leftrightarrow x \in U$.

Therefore, $\underline{\delta}_B(U) = U$. It is similar to prove $\overline{\delta}_B(\varnothing) = \varnothing$,

(3) Since $x \in \underline{\delta}_B(X \cap Y) \Leftrightarrow N_B(x) \subseteq X \cap Y \Leftrightarrow N_B(x) \subseteq X$ and $N_B(x) \subseteq Y$.

$\Leftrightarrow x \in \underline{\delta}_B(X)$ and $x \in \underline{\delta}_B(Y)$.

$\Leftrightarrow x \in \underline{\delta}_B(X) \cap \underline{\delta}_B(Y)$.

Therefore, $\underline{\delta}_B(X \cap Y) = \underline{\delta}_B(X) \cap \underline{\delta}_B(Y)$. It is similar to prove $\overline{\delta}_B(X \cup Y) = \overline{\delta}_B(X) \cup \overline{\delta}_B(Y)$.

(4) It can be directly proven by Definition 3.1.

(5) Derived from property (4). □

Remark 3.4. The following example shows that $\forall X, Y \subseteq U$, $\underline{\delta}_B(X \cup Y) \subseteq \underline{\delta}_B(X) \cup \underline{\delta}_B(Y)$, and $\overline{\delta}_B(X \cap Y) \supseteq \overline{\delta}_B(X) \cap \overline{\delta}_B(Y)$ may not true.

Example 3.5. Let $U = \{a, b, c\}$, $N_B = \{(a, a), (b, b), (b, a), (c, b)\}$, $X = \{a\}$, and $Y = \{b, c\}$. It is easy to obtain $\underline{\delta}_B(X) \cup \underline{\delta}_B(Y) = \{c\}$. But $\underline{\delta}_B(X \cup Y) = U$.

Take $A = \{a, c\}, B = \{b, c\}$, it is not difficult to obtain $\overline{\delta}_B(A \cap B) = \overline{\delta}_B\{c\} \neq \varnothing$. However, $\overline{\delta}_B(A) \cap \overline{\delta}_B(B) = \{a, b\} \cap U = \{a, b\}$.

Theorem 3.6. *Let N_B be a neighborhood relation over U, the following conditions are equivalent*:
(1) *N_B is serial.*
(2) $\underline{\delta}_B(X) \subseteq \overline{\delta}_B(X), \forall X \subseteq U$.
(3) $\underline{\delta}_B(\varnothing) = \varnothing$.
(4) $\overline{\delta}_B(U) = U$.

Proof. (1) ⇒ (2) $\forall x \in \underline{\delta}_B(X)$, we have $N_B(X) \subseteq X$ by Definition 3.1; since N_B is serial, we obtain $N_B(X) \neq \varnothing$, hence $N_B(X) \cap X \neq \varnothing$, and thus $x \in \overline{\delta}_B(X)$, therefore $\underline{\delta}_B(X) \subseteq \overline{\delta}_B(X)$.

(2) ⇒ (1) Suppose $\exists x \in U$ such that $N_B(X) \cap X = \varnothing$, then $\forall X \subseteq U, N_B(X) \subseteq X$, and from Definition 3.1, we obtain $x \in \underline{\delta}_B(X)$ but $N_B(X) \cap X = \varnothing$, and $x \notin \overline{\delta}_B(X)$ is contradictory to (2).

(2) ⇔ (3) It is obtained by (1) of Theorem 3.3:

$$\underline{\delta}_B(X) \subseteq \overline{\delta}_B(X) \Leftrightarrow \underline{\delta}_B(X) \cap \sim\overline{\delta}_B(X) = \varnothing.$$

$$\Leftrightarrow \underline{\delta}_B(X) \cap \sim\overline{\delta}_B(X) = \varnothing.$$

$$\Leftrightarrow \underline{\delta}_B(X \cap \sim X) = \varnothing.$$

$$\Leftrightarrow \underline{\delta}_B(\varnothing) = \varnothing.$$

(3) ⇔ (4) It is obtained by the dual property (1) of Theorem 3.3:

$$\underline{\delta}_B(\varnothing) = \varnothing \Leftrightarrow \sim\underline{\delta}_B(\varnothing) = \sim\underline{\delta}_B(\sim U) = \sim\varnothing = U.$$

$$\Leftrightarrow \overline{\delta}_B(U) = U.$$ □

Lemma 3.7. *Let N_B be any neighborhood relation on U,*

$\overline{\delta}_B(\{x\}) = N_B^{-1}(\{x\}) = \{y \in U | x \in N_B(y)\}$.

Theorem 3.8. *Let N_B be a neighborhood relation over U, then the following conditions are equivalent*:
(1) N_B *is inverse serial.*
(2) $\overline{\delta}_B(\{x\}) \neq \emptyset \ \forall x \in U$.
(3) $N_B(U) = U$.

Proof. Since N_B is inverse serial if and only if $\forall x \in U$, we have $N_B^{-1}(x) = N_B(x) \neq \emptyset$. From Lemma 3.7, and Definition 3.1, we can straight to prove the following theorem. □

Theorem 3.9. *Let N_B be a neighborhood relation over U, then the following conditions are equivalent*:
(1) N_B *is reflexive.*
(2) $\underline{\delta}_B(X) \subseteq X \ \forall X \subseteq U$.
(3) $X \subseteq \overline{\delta}_B(X) \ \forall X \subseteq U$.

Proof. (1) ⇒ (2) Let $x \in \underline{\delta}_B(X)$, then we have $N_B(x) \subseteq X$ by Definition 3.1, and since N_B is reflexive, we can obtain $x \in N_B(x)$, and thus $x \in X$, therefore $\underline{\delta}_B(X) \subseteq X$.

(2) ⇒ (3) $\forall X \subseteq U$, $\underline{\delta}_B(\sim X) \subseteq (\sim X)$ is obtained by Definition 3.1, and thus $X = \sim(\sim X) \subseteq \sim\underline{\delta}_B(\sim X) = \overline{\delta}_B(X)$ is known by (1) of Theorem 3.3.

(3) ⇒ (1) If (3) holds, then for any $x \in U$, there exists a $x \in \overline{\delta}_B(\{x\})$, we obtain $x \in N_B(x)$ by Definition 3.1, thus N_B is reflexive. □

Theorem 3.10. *Let N_B be the neighborhood relation on U, then the following conditions are equivalent*:
(1) N_B *is transitive,*
(2) $\underline{\delta}_B(X) \subseteq \underline{\delta}_B(\underline{\delta}_B(X))$, $\forall X \subseteq U$.
(3) $\overline{\delta}_B(\overline{\delta}_B(X)) \subseteq \overline{\delta}_B(X)$, $\forall X \subseteq U$.

Proof. (1) ⇒ (2) $\forall x \in \underline{\delta}_B(X)$, we obtain $N_B(x) \subseteq X$ by Definition 3.1, $\forall y \in N_B(x)$, thus $(x, y) \in N_B$, we have $y \in N_B(x)$, since N_B is transitive, we obtain $N_B(y) \subseteq N_B(x) \subseteq X$, therefore $N_B(y) \subseteq X$. From Definition 3.1, we know that $y \in \underline{\delta}_B(X)$, and by the arbitrariness of $y \in \underline{\delta}_B(X)$, we obtain $N_B(x) \subseteq \underline{\delta}_B(X)$, so we have $x \in \underline{\delta}_B(\underline{\delta}_B(X))$, therefore (2) is true.

(2) ⇒ (3) It is easy to prove by Theorem 3.3:

$$\underline{\delta}_B(X) = \sim\overline{\delta}_B(\sim X), \overline{\delta}_B(X) = \sim\underline{\delta}_B(\sim X)$$

$$\underline{\delta}_B(X) \subseteq \underline{\delta}_B(\underline{\delta}_B(X))$$

$$\Rightarrow \sim\overline{\delta}_B(\sim X) \subseteq \sim(\overline{\delta}_B(\sim\overline{\delta}_B(X)))$$

$$\Rightarrow \overline{\delta}_B(\overline{\delta}_B(X)) \subseteq \overline{\delta}_B(X)$$

(3) ⇒ (1) Let $y \in N_B(x)$ and $z \in N_B(y)$, NB is transitive, then $y \in \delta(\{x\})B$ and $z \in N(\{y\})B$ and Lemma 3.7.

This shows that $N_B(x) \cap \overline{\delta}_B(\{z\}) \neq \emptyset$, thus $x \in \overline{\delta}_B(\overline{\delta}_B(\{z\}))$, and from (3), we have $x \in \overline{\delta}_B(\{x\})$, thus $z \in N_B(x)$, so we obtain $z \in N_B(x)$ from $y \in N_B(x)$ and $z \in N_B(y)$, i.e., N_B is transitive. □

Theorem 3.11. *Let N_B be any neighborhood relation on U, then the following conditions are equivalent*:
(1) N_B *is symmetrical.*
(2) $X \subseteq \underline{\delta}_B(\overline{\delta}(X)) \ \forall X \subseteq U$.
(3) $\overline{\delta}_B(\underline{\delta}_B(X)) \subseteq X \ \forall X \subseteq U$.

Proof. (1) ⇒ (2) $\forall x \in X$ and $y \in N_B(x)$, since N_B is symmetric, $x \in N_B(y)$, therefore $x \in N_B(y) \cap X$. This means that $N_B(y) \cap X \neq \emptyset$, thus $y \in \overline{\delta}_B(X)$. By the arbitrariness of $x \in N_B(x)$ and $N_B(x) \subseteq \overline{\delta}_B(X)$, we have $X \subseteq \underline{\delta}_B(\overline{\delta}_B(X))$.

(2) ⇒ (1) For any $x, y \in U$ and $y \in N_B(x)$, from (2), we obtain $x \in \underline{\delta}_B(\overline{\delta}_B(\{x\}))$ and $N_B(x) \subseteq \overline{\delta}_B(\{x\})$ by Definition 3.1, since $y \in N_B(x)$, we have $y \in \overline{\delta}_B(\{x\})$, thus $N_B(y) \cap \{x\} \neq \varnothing$ by definition of the upper approximation, so we have $x \in N_B(y)$, therefore N_B is symmetric.

(2) ⇔ (3) It is obtained by (1) of Theorem 3.3. □

Theorem 3.12. *Let N_B be any domain relation on U, then the following conditions are equivalent*:
(1) N_B *is Euclidean.*
(2) $\overline{\delta}_B(X) \subseteq \underline{\delta}_B(\overline{\delta}_B(X))\ \forall X \subseteq U$.
(3) $\overline{\delta}_B(\underline{\delta}_B(X)) \subseteq \underline{\delta}_B(X)\ \forall X \subseteq U$.

Proof. (1) ⇒ (2) $\forall x \in \overline{\delta}_B(X)$ and from the definition of above approximation, we have $N_B(x) \cap X \neq \varnothing$; and $\forall y \in N_B(x)$, since N_B is Euclidean, we have $N_B(x) \subseteq N_B(y)$. Combined with $N_B(x) \cap X \neq \varnothing$, it is obtained $N_B(y) \cap X \neq \varnothing$, and then $y \in \overline{\delta}_B(X)$. From the arbitrariness of $y \in N_B(x)$, we obtain $N_B(x) \subseteq \overline{\delta}_B(X)$ and $x \in \underline{\delta}_B(\overline{\delta}_B(X))$, so $\overline{\delta}_B(X) \subseteq \underline{\delta}_B(\overline{\delta}_B(X))$.

(2) ⇒ (1) Let $y \in N_B(x)$ and $z \in N_B(x)$, and from $z \in N_B(x)$ and Lemma 3.7, we obtain $x \in \overline{\delta}_B(\{z\})$. Since (2) holds, therefore $x \in \underline{\delta}_B\overline{\delta}_B(\{z\})$, we have $N_B(x) \subseteq \overline{\delta}_B(\{z\})$. Since $y \in N_B(x)$, then $y \in \overline{\delta}_B(\{z\})$, from Lemma 3.7, we obtain $z \in N_B(y)$, thus we obtain $z \in N_B(y)$ from $y \in N_B(x)$ and $z \in N_B(x)$, i.e., N_B is Euclidean.

(2) ⇔ (3) It is obvious by Theorem 3.3. □

Theorem 3.13. *$\overline{\delta}_B$ is a topological closure operator if and only if N_B is serial, reflexive, and transitive relation.*

Proof. (1) We prove that if N_B is reflexive and transitive relation, then $\overline{\delta}_B$ is a topological closure operator.

$\forall X \subseteq U$, if N_B is reflexive, we have $X \subseteq \overline{\delta}_B(X)$ by Theorem 3.9. We also obtain $\overline{\delta}_B(X \cup Y) = \overline{\delta}_B(X) \cup \overline{\delta}_B(Y)$ by (3) of Theorem 3.3. $\overline{\delta}_B(\varnothing) = \varnothing$ by Definition 3.1. If N_B is serial and transitive, then $\overline{\delta}_B(\overline{\delta}_B(X)) \subseteq \overline{\delta}_B(X)$ by Theorem 3.10, $\overline{\delta}_B(\overline{\delta}_B(X)) \supseteq \overline{\delta}_B(X)$ is obvious. Thus, $\overline{\delta}_B(\overline{\delta}_B(X)) = \overline{\delta}_B(X)$. Therefore, $\overline{\delta}_B$ is a topological closure operator.

(2) We shall prove that if $\overline{\delta}_B$ is a topological closure operator, then ⇒N_B is serial, reflexive, and transitive relation.

$\forall X \subseteq U$, $X \subseteq \overline{\delta}_B(X)$ by Theorem 3.9, so N_B is reflexive. If $\overline{\delta}_B(\overline{\delta}_B(X) = \overline{\delta}_B(X))$, then N_B is transitive by Theorem 3.10. $\overline{\delta}_B(\varnothing) = \varnothing \Leftrightarrow \underline{\delta}_B(\varnothing) = \varnothing$, therefore N_B is serial by Theorem 3.6. □

4 Conclusion

Topology and neighborhood relation play a very important role in the field of modern intelligence. Therefore, this article gives the definition of neighborhood relation under various conditions. Although we have defined information granules through neighborhood relationships, we do not discuss the base and topology of information particle construction. In addition, what are the connections and differences between the topology content in future work and Theorem 3.13? we will discuss these questions in the future.

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References

[1] Pawlak Z. Rough sets. Int J Inform Comput Sci. 1982;11(1):341–56.
[2] Pawlak Z. Rough set approach to multi-attriute decision analysis. Eur J Operat Res. 1994;77:443–59.
[3] Kryszkiewicz M. Comparative studies of alternative type of knowledge reduction in inconsistent systems. Int J Intell Sys. 2001;16(1):105–20.
[4] Lin TY. Rough sets, neighborhood systems and approximation. ISMIS 1990: Proceedings of the 5th International Symposium on Methodologies of Intelligent System. Charlotte: Elsevier Science; 1990. p. 19–90.
[5] Lin TY. Rough sets, neighborhood systems and approximation. World J Surgery. 1986;10(2):189–94.
[6] Mao ZY, Dou HL, Song JJ, Jiang ZH, Wang PX. Research on attribute reduction under co-occurrence neighborhood relationship. J Nanjing Univ (Natural Science). 2021;57(1):150–9. doi: 10.13232/j.cnki.jnju.2021.01.016.
[7] Li W, Yang B. Three-way decisions with fuzzy probabilistic covering-based rough sets and their applications in credit evaluation. Appl Soft Comput. 2023;136:110–44.
[8] Qi G, Yang B, Li W. Some neighborhood-related fuzzy covering-based rough set models and their applications for decision making. Inform Sci. 2023;621:799–843.
[9] Engeiking R. General topology. Berline: Heldermann Verlag; 1989.
[10] Huang J, Jiang DH. Separation of matroid closure operator induced topological space. Cognition Math Practice. 2016;46(9):262–8.
[11] Zhang WX, Wu WZ, Liang JY, Li DY. Rough set theory and methods. Beijing: Science Press; 2001. p. 41–53.