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Power Average Operators of Trapezoidal Cubic Fuzzy Numbers and Application to Multi-attribute Group Decision Making

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Abstract: Trapezoidal cubic fuzzy numbers (TzCFNs) are an extraordinary cubic fuzzy set on a real number set. TzCFNs are useful for dealing with well-known quantities in decision data and decision making problems themselves. This paper is about multi-attribute group decision making problems in which the attribute values are stated with TzCFNs, which are solved by developing a new decision method based on power average operators of TzCFNs. The new operation laws for TzCFNs are given. Hereby, the power average operator of real numbers is extended to four kinds of power average operators of TzCFNs, involving the power average operator of TzCFNs, the weighted power average operator of TzCFNs, the power ordered weighted average operator of TzCFNs, and the power hybrid average operator of TzCFNs. In the proposed group decision method, the individual overall evaluation values of alternatives are generated by using the power average operator of TzCFNs. Applying the hybrid average operator of TzCFNs, the specific general evaluation standards of alternatives are then combined into the collective ones, which are used to rank the alternatives. The example analysis shows the practicality and effectiveness of the proposed method.

Keywords: Multi-attribute group decision making; trapezoidal cubic fuzzy number; Hausdorff metric; power average operator.

1 Introduction

Fuzzy sets were presented by Zadeh [60] to describe fuzzy problems with the membership function. The drawback of using the single membership value in the fuzzy set theory is that the evidence for $x \in X$ and the evidence against $x \in X$ are in fact mixed together (here, X is the universe of discourse). Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFSs) characterized by a membership function and a non-membership function, which is more suitable for dealing with fuzziness and uncertainty than the fuzzy set. The IFS is highly useful in depicting the uncertainty and vagueness of an object, and thus can be used as a powerful tool to express data information under various different fuzzy environments, which has attracted great attention. The IFS has received more and more attention since its appearance, because the information about attribute values is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human thinking.

IFS has been commonly useful to multi-attribute decision making (MADM) and multi-attribute group decision making (MAGDM) [6, 17, 20, 22, 34, 38, 39, 41, 46, 48, 50, 52, 55, 57, 61, 62]. These investigates container be incompletely classified into four types: aggregation operators [3, 6, 17, 20, 22, 32, 34, 38, 39, 41, 46, 48, 50, 52, 56, 57, 61, 62], similarity (or distance) measures and entropy [15, 21, 45, 47, 49, 53], extension of classic decision making methods [19, 37, 40, 42, 43, 51, 54], new decision making methods [31, 35, 56], and judgment matrix [53].

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In the characteristic of aggregation operators, Li et al. [17, 18, 20] introduced the generalized ordered weighted average operators with IFSs. Zhao et al. [62] established some new generalized aggregation operators, such as generalized intuitionistic fuzzy (IF) weighted averaging operator, generalized IF ordered weighted averaging operator, generalized IF hybrid averaging operator, and practical to MADM with IF data. Xu and Yager [52], Xu [48], and Wei [38] developed some geometric aggregation operators based on IFS, such as the IF weighted geometric operator, the IF ordered weighted geometric operator, and the IF hybrid geometric operator. Wei [39] proposed some induced geometric aggregation operators with IF information. Xu and Wang [55] developed induced generalized aggregation operators for IFSs. Wei and Zhao [41] developed some induced correlated aggregating operators with IF information. Su et al. [34] proposed the induced generalized IF ordered weighted average operator. Liu and Wang [22] proposed the IF point operators. Xu [50] developed the IF power aggregation operators. Yang and Chen [58] defined the quasi-arithmetic IF ordered weighted average operators. Zeng and Su [61] proposed the IF ordered weighted distance operator. Yu and Xu [59] proposed the prioritized IF aggregation operators. These aggregation operators for IF numbers (IFNs) may be observed as the extension of the ones for real numbers, which mainly involve the arithmetic aggregation operators, geometric aggregation operators, power aggregation operators, generalized average operators, and induced aggregation operators. Yager [57] defined the idea of a power median. We introduced some possible formulations for the support function used in the power average.

For the resemblance (or distance) measures and entropy, Li [15] discussed some measures of dissimilarity in IF structures. Xu [49] defined the normalized Hamming distance between two IFNs and proposed the IF MADM method. Xu [47], Xu and Yager [53], and Huang et al. [13] defined similarity measures of IFSs and applied the similarity measures to the MADM or MAGDM under IF environment. These similarity (or distance) measures of IFSs can also be applied to pattern recognitions and approximate reasoning. Xia and Xu [45] developed the entropy/cross entropy for IFSs. Li et al. [21] researched the relationship between similarity measure and entropy of IFSs.

In addition of classic decision making methods, Li [16] and Li et al. [19] extended the classic LINMAP method to the IF environments. Wu and Chen [43] proposed the ELECTRE multi-criteria analysis approach based on IFSs. Xu and Hu [54] constructed the projection models for IF MADM. Dymova and Sevastjanov [6] investigated the operations on IF values in the framework of the Dempster-Shafer theory. Wei et al. [42] and Wei [40] proposed the gray relational analysis method for IF MADM. These decision making methods under IF environments generalize the classic decision making methods, such as TOPSIS, ELECTRE, LINMAP, and gray relational analysis. Wei [37] proposed the maximizing deviation method for IF MADM. Xu [51] provided an error-analysis-based method for the priority of an intuitionistic preference relation in decision making. Xu et al. [56] defined the interactive method for eliminating any dominated alternatives by updating the decision maker's preferences gradually so as to find out the optimal one eventually. Wu et al. [44] proposed score functions; a method is developed to construct Fuzzy preference relations from a given intuitionistic fuzzy preference relation and interval-valued intuitionistic fuzzy preference relation, respectively. Vahdani et al. [35] proposed IF ELECTRE, which utilizes the truth-membership function and non-truth-membership function to indicate the degrees of satisfiability and non-satisfiability of each alternative with respect to each criterion and the relative importance of each criterion, respectively. Pei and Zeng [31] proposed a new approach in which the degree of membership, the degree of non-membership, and the degree of hesitation are considered with various importance in reflecting the true image of the respective alternative. Chen et al. [5] proposed an approach that can be easily extended to deal with problems in an interval-valued IF environment. Chen [4] defined an experimental analysis conducted to examine the relationship between the results yielded by different score functions, considering the average Spearman correlation coefficients and contradiction rates. Xu and Yager [53] investigated some IF preference relations and their measures of similarity for the evaluation of agreement within a group. These IF preference relations enrich the research contents of the IF decision making theory.

Liu and Tang [23] defined the interval neutrosophic uncertain linguistic variables handling the uncertainty of the decision makers' cognition in multi-criteria group decision making problems. Liu et al. [28] proposed the multi-valued neutrosophic weighted Bonferroni mean operator and the multi-valued neutrosophic weighted geometric Bonferroni mean operator, and some of their properties are also investigated. Liu and Shi [24] proposed the single-valued neutrosophic uncertain linguistic set by combining the uncertain linguistic numbers and the single-valued neutrosophic set. Further, the neutrosophic uncertain linguistic number improved generalized weighted Heronian mean operator and the neutrosophic uncertain linguistic number improved generalized geometric weighted Heronian mean operator were developed. Liu et al. [29] extended the Bonferroni mean operator based on the Dombi operations to propose the IF Dombi Bonferroni mean operator, the IF weighted Dombi Bonferroni mean operator, the IF Dombi geometric Bonferroni mean operator, and the IF weighted Dombi geometric Bonferroni mean operator for dealing with the aggregation of IFNs, and proposed some MAGDM methods. Liu et al. [30] extended the partitioned Heronian mean operator to linguistic IFNs based on new operational rules, and proposed the linguistic IF partitioned Heronian mean operator, the linguistic IF weighted partitioned Heronian mean operator, the linguistic IF partitioned geometric Heronian mean operator, and the linguistic IF weighted partitioned geometric Heronian mean operator. Liu and Liu [26] combined the Bonferroni mean operator with q-rung orthopair fuzzy numbers to propose the q-rung orthopair fuzzy Bonferroni mean operator, the q-rung orthopair fuzzy weighted Bonferroni mean operator, the q-rung orthopair fuzzy geometric Bonferroni mean operator, and the q-rung orthopair fuzzy weighted geometric Bonferroni mean operator, as well as developed MAGDM methods based on these operators. Liu and Wang [27] proposed the q-rung orthopair fuzzy weighted averaging operator and the q-rung orthopair fuzzy weighted geometric operator to deal with decision information, and some of their properties are well proved. Liu and Chen [25] proposed a new MAGDM method with the I2LI based on the proposed I2LGA operator.

The cubic sets introduced by Jun et al. [14] are the generalizations of fuzzy sets and IFSs, in which there are two representations: one is used for the degree of membership and other is used for the degree of non-membership. The membership function is held in the form of interval, while non-membership is thought over the normal fuzzy set.

Fahmi et al. [9] developed the Hamming distance for triangular cubic fuzzy number and weighted averaging operator. Fahmi et al. [8] proposed the cubic TOPSIS method and gray relational analysis set. Fahmi et al. [11] defined the triangular cubic fuzzy number and operational laws. The authors developed the triangular cubic fuzzy hybrid aggregation (TCFHA) administrator to total all individual fuzzy choice structures provided by the decision makers into the aggregate cubic fuzzy decision matrix. Fahmi et al. [1] defined the generalized triangular cubic linguistic hesitant fuzzy weighted geometric operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted average operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted geometric operator, generalized triangular cubic linguistic hesitant fuzzy hybrid averaging operator, and generalized triangular cubic linguistic hesitant fuzzy hybrid geometric operator. Fahmi et al. [7] developed the trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the multi-criteria decision making (MCDM) method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method. Fahmi et al. [12] defined aggregation operators for triangular cubic linguistic hesitant fuzzy sets, which include the cubic linguistic fuzzy (geometric) operator, triangular cubic linguistic hesitant fuzzy weighted geometric operator, triangular cubic linguistic hesitant fuzzy ordered weighted geometric operator, and triangular cubic linguistic hesitant fuzzy hybrid geometric operator. Fahmi et al. [10] defined the trapezoidal cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator. The expected values, score function, and accuracy function of trapezoidal cubic fuzzy numbers (TzCFNs) are defined.

Due to the motivation and inspiration of the above discussion in this paper, we generalized the concept of trapezoidal fuzzy sets, trapezoidal IFSs, interval-valued trapezoidal IFN, trapezoidal fuzzy power aggregation method, interval-valued trapezoidal power aggregation operator, and interval-valued trapezoidal IF power aggregation operator, and introduce the concept of trapezoidal cubic fuzzy sets. If we take only one element in the membership degree of the TzCFN (i.e. instead of interval, we take a fuzzy number), we get trapezoidal IFNs. Similarly, if we take membership degree as fuzzy number and non-membership degree equal to zero, than we get trapezoidal fuzzy numbers.

The rest of this paper is organized as follows. In Section 2, we provide basic definitions of fuzzy set and cubic set. Section 3 presents the distances and operation laws of TzCFNs. Four kinds of power average operators of TzCFNs are exhibited in Section 4. Section 5 shows the MAGDM model and technique utilizing

TzCFNs. In Section 6, a numerical example is used to verify the proposed method. In Section 7, we propose the comparison method. The paper is concluded in Section 8.

2 Preliminaries

In this section, we present a few basic definitions of fuzzy set theory, IFS, and cubic set theory.

Definition 1 ([60]): Let H be a universe of discourse. The idea of fuzzy set was presented by Zadeh and defined as follows: $J = \{\ddot{h}, \Gamma_J(\ddot{h}) | \ddot{h} \in H\}$. A fuzzy set in a set $\ddot{h}$ is defined $\Gamma_J : H \to I$, which is a membership function. $\Gamma_J(\ddot{h})$ denotes the degree of membership of the element $\ddot{h}$ to the set H, where $I = [0, 1]$. The collection of all fuzzy subsets of H is denoted by I^H. Define a relation on I^H as follows: $(\forall \Gamma, \eta \in I^H)(\Gamma \leq \eta \Leftrightarrow (\forall \ddot{h} \in \ddot{h})(\Gamma(\ddot{h}) \leq \eta(\ddot{h})))$.

Definition 2 ([2]): An Atanassov IFS on H is a set $A = \cup\{a_{\ddot{h}} | a_{\ddot{h}} \in H\}$. The membership and non-membership function, Γ_J and η_j, are, respectively given by the following: $\Gamma_J(\ddot{h}) : \ddot{h} \to [0, 1], \ddot{h} \in H \to \Gamma_J(\ddot{h}) \in [0, 1]$; $\eta_J(\ddot{h}) : \ddot{h} \to [0, 1], \ddot{h} \in H \to \eta_J(\ddot{h}) \in [0, 1]$ and $0 \leq \Gamma_J(\ddot{h}) + \eta_J(\ddot{h}) \leq 1$ for all $\ddot{h} \in H$; $\pi_J(\ddot{h}) = 1 - \Gamma_J(\ddot{h}) - \eta_J(\ddot{h})$.

Definition 3 ([14]): Let H be a non-empty set. By a cubic set in H we mean a structure $F = \{\ddot{h}, \alpha(\ddot{h}), \beta(\ddot{h}) : \ddot{h} \in H\}$ in which α is an interval-valued fuzzy set in H and β is a fuzzy set in H. A cubic set $F = \{\ddot{h}, \alpha(\ddot{h}), \beta(\ddot{h}) : \ddot{h} \in H\}$ is simply denoted by $F = \langle \alpha, \beta \rangle$. The collection of all cubic sets in $\ddot{h}$ is denoted by C^H. A cubic set $F = \langle \alpha, \beta \rangle$ in which $\alpha(\ddot{h}) = 0$ and $\beta(\ddot{h}) = 1$ [resp. $\alpha(\ddot{h}) = 1$ and $\beta(\ddot{h}) = 0$] for all $\ddot{h} \in H$ is denoted by 0 (resp. 1). A cubic set $D = \langle \lambda, \xi \rangle$ in which $\lambda(\ddot{h}) = 0$ and $\xi(\ddot{h}) = 0$ [resp. $\lambda(\ddot{h}) = 1$ and $\xi(\ddot{h}) = 1$] for all $\ddot{h} \in H$ is denoted by 0 (resp. 1).

Definition 4 ([14]): Let H be a non-empty set. A cubic set $F = (C, \lambda)$ in H is said to be an internal cubic set if $C^-(\ddot{h}) \leq \lambda(\ddot{h}) \leq C^+(\ddot{h})$ for all $\ddot{h} \in H$.

Definition 5 ([14]): Let H be a non-empty set. A cubic set $F = (C, \lambda)$ in H is said to be an external cubic set if $\lambda(\ddot{h}) \notin (C^-(\ddot{h}), C^+(\ddot{h}))$ for all $\ddot{h} \in H$.

3 Distances for TzCFNs

Definition 6: For any two subsets U and W of a Banach space Z, the Hausdorff metric is $d(U, W) = \max\{\sup_{u \in U} \inf_{w \in W} |u - w|, \sup_{w \in W} \inf_{u \in U} |u - w|\}$. If $Z = R$, $U = [u_1, u_2]$ and $W = [w_1, w_2]$ are intervals, then the Hausdorff metric reduces to $d(U, W) = \max\{|u_1 - w_1|, |u_2 - w_2|\}$.

Definition 7: Let $a_1 = \left\{ \begin{array}{l} [a_1, b_1, c_1, d_1]; \\ \langle [w_1^-, w_1^+], w_1 \rangle \end{array} \right\}$ and $a_2 = \left\{ \begin{array}{l} [a_2, b_2, c_2, d_2]; \\ \langle [w_2^-, w_2^+], w_2 \rangle \end{array} \right\}$ be two TzCFNs. The Hamming distance and Euclidean distance between them are, respectively defined as

$$d_H(a_1, a_2) = \left\{ \begin{array}{c} \frac{1}{12}[|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2| + \\ \max\left\{ \left[\left| w_1^- - w_2^- \right|, \left| w_1^+ - w_2^+ \right| \right], |w_1 - w_2| \right\} \end{array} \right\} \tag{1}$$

and

$$d_e(a_1, a_2) = \frac{1}{6} \sqrt{ \begin{array}{c} [(a_1 - a_2) + (b_1 - b_2) + (c_1 - c_2) + (d_1 - d_2) + \\ \max\{[(w_1^- - w_2^-), (w_1^+ - w_2^+)], w_1 - w_2\}. \end{array} } \tag{2}$$

Example 1: Let $a_1 = \begin{Bmatrix} [0.5, 0.6, 0.7, 0.8]; \\ \langle [0.35, 0.37], 0.36 \rangle \end{Bmatrix}$ and $a_2 = \begin{Bmatrix} [0.1, 0.2, 0.3, 0.4]; \\ \langle [0.20, 0.22], 0.21 \rangle \end{Bmatrix}$ be two TzCFNs

$$d_H(a_1, a_2) = \begin{Bmatrix} \frac{1}{12}[|0.5-0.1| + |0.6-0.2| + |0.7-0.3| + |0.8-0.4| + \\ \max\{[|0.35-0.20|, |0.37-0.22|], |0.36-0.21|\} \\ = \frac{1}{12}[0.4+0.4+0.4+0.4+\max\{[0.15, 0.15], 0.15\} = \\ \frac{1.75}{12} = 0.1458 \end{Bmatrix}$$

and

$$d_e(a_1, a_2) = \begin{Bmatrix} \frac{1}{6}\sqrt{\begin{matrix}[(0.5-0.1)+(0.6-0.2)+(0.7-0.3)+(0.8-0.4)+ \\ \max\{[(0.35-0.20), (0.37-0.22)], 0.36-0.21\}\end{matrix}} \\ = \frac{1}{6}\sqrt{[0.4+0.4+0.4+0.4+\max\{[0.15, 0.15], 0.15\}} \\ = \frac{\sqrt{1.75}}{6} = \frac{1.3228}{6} = 0.2204. \end{Bmatrix}$$

Theorem 1: *Eqs.* (1) *and* (2) *meet the non-negative symmetric and triangle inequality.*

3.1 Operational Laws and Properties for TzCFNs

Definition 8: If $a \geq 0$ and one of the four values a, b, c, and d is not always identical to zero, then the TzCFN $a = [a, b, c, d]; \langle [w^-, w^+], w \rangle$ is known as a positive TzCFN, denoted by $\tilde{a} > 0$. The TzCFNs mentioned in the following are all positive TzCFNs.

Definition 9: Let $a_1 = \begin{Bmatrix} [a_1, b_1, c_1, d_1]; \\ \langle [w_1^-, w_1^+], w_1 \rangle \end{Bmatrix}$ and $a_2 = \begin{Bmatrix} [a_2, b_2, c_2, d_2]; \\ \langle [w_2^-, w_2^+], w_2 \rangle \end{Bmatrix}$ be two TzCFNs and $\lambda \geq 0$. Then, the operational laws for TzCFNs are defined as follows:

(1) $a_1 + a_2 = [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2], \langle [w_1^- \wedge w_2^-, w_1^+ \wedge w_2^+], w_1 \vee w_2 \rangle$.
(2) $a_1 - a_2 = [a_1 - a_2, b_1 - b_2, c_1 - c_2, d_1 - d_2], \langle [w_1^- \vee w_2^-, w_1^+ \vee w_2^+], w_1 \wedge w_2 \rangle$.
(3) $a_1 a_2 = [a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2], \langle [w_1^- \wedge w_2^-, w_1^+ \wedge w_2^+], w_1 \vee w_2 \rangle$.
(4) $\lambda a_1 = [\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1];\ \langle [\lambda w_1^-, \lambda w_1^+], \lambda w_1 \rangle$.
(5) $a_1^\lambda = [a_1^\lambda, b_1^\lambda, c_1^\lambda, d_1^\lambda];\ \langle [w_1^-, w_1^+], w_1 \rangle$.

4 Power Average Operators of TzCFNs

Definition 10: For real numbers $\{a_1, a_2, \ldots, a_m\}$, the power average operator is defined as $PA(a_1, a_2, \ldots, a_m) = \frac{\sum_{k=1}^m [1+T(a_k)]}{\sum_{k=1}^m [1+T(a_k)]}$, where $T(a_k) = \sum_{j=1}^m \operatorname{Sup}(a_k, a_j)$ and $\operatorname{Sup}(a, b)$ is the support for a from b, satisfying the following properties:

(1) $\operatorname{Sup}(a, b) \in [0, 1]$;
(2) $\operatorname{Sup}(a, b) = \operatorname{Sup}(b, a)$;
(3) If $|a - b| < |x - y|$, then $\operatorname{Sup}(a, b) > \operatorname{Sup}(x, y)$.

4.1 Four Kinds of Power Average Operators of TzCFNs

Definition 11: Let $\{a_1, a_2, \ldots, a_m\}$ be TzCFNs. The power average operator of TzCFNs is defined as

$$\mathrm{TzCFPA}(a_1, a_2, \ldots, a_m) = \frac{\sum_{k=1}^m [1 + T(a_k)]}{\sum_{k=1}^m [1 + T(a_k)]}, \tag{3}$$

where $T(a_k) = \sum_{j=1}^{m} \mathrm{Sup}(a_k, a_j)$ and $\mathrm{Sup}(a, b)$ is the support for TzCFN a from TzCFN b, satisfying the following properties:

(1) $\mathrm{Sup}(a, b) \in [0, 1]$;
(2) $\mathrm{Sup}(a, b) = \mathrm{Sup}(b, a)$;
(3) If $d(a, b) < d(x, y)$, then $\mathrm{Sup}(a, b) > \mathrm{Sup}(x, y)$, where d is Hamming or Euclidean distance defined in Definition 7.

Thus, we see that the more similar and the closer two values, the more they support each other. $T(a_k)$ can be considered as the support of a_k by all the other TzCFNs. The power average operator of TzCFNs exhibits a number of properties desirable for an aggregation operator.

Theorem 2: *If $Sup(a_k, a_j) = c(c \in [0, 1], k \neq j)$, then the power average operator of TzCFNs reduces to the arithmetic average operator of TzCFNs as follows: TzCFPA $(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m} \frac{a_k}{m}$.*

Theorem 3 (Boundedness): *The power average operator of TzCFNs satisfies $min\{a_k | k = 1, 2, \ldots, m\} \leq TzCFPA(a_1, a_2, \ldots, a_m) \leq \max\{a_k | k = 1, 2, \ldots, m\}$.*

Theorem 4 (Commutativity): *Let $a_j^{'}$ $(j = 1, 2, \ldots, m)$ is any permutation of $(a_1, a_2, \ldots, a_m)$. Then $TzCFPA(a_1, a_2, \ldots, a_m) = TzCFPA(a_1^{'}, a_2^{'}, \ldots, a_m^{'})$.*

Theorem 5 (Idempotency): *If $a_j = a(j = 1, 2, \ldots, m)$, then TzCFPA $(a_1, a_2, \ldots, a_m) = a$.*

Definition 12: The weighted power average operator of TzCFNs is defined as

$$\mathrm{TzCFWPA}(a_1, a_2, \ldots, a_m) = \frac{\sum_{k=1}^{m}[(1 + T^{'}(a_k))\omega_k a_k]}{\sum_{k=1}^{m}[(1 + T^{'}(a_k))\omega_k]}, \tag{4}$$

where $T^{'}(a_k) = \sum_{j=1}^{m} \mathrm{Sup}(a_k, a_j)$ and $\mathrm{Sup}(a, b)$ and $w = (w_1, w_2, \ldots, w_m)^T$ is the weight vector of TzCFNs $(a_1, a_2, \ldots, a_m)$, satisfying that $0 \leq w_k \leq 1$ $(k = 1, 2, \ldots, m)$ and $\sum_{k=1}^{m} w_k = 1$.

Theorem 6: *If $Sup(a_k, a_j) = c(c \in [0, 1], k \neq j)$, then the weighted power average operator of TzCFNs reduces to the weighted average operator of TzCFNs, as follows: TzCFWPA $(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m} w_k a_k$.*

Definition 13: Let $(a_1, a_2, \ldots, a_m)$ be the TzCFNs. The power ordered weighted average operator of TzCFNs is defined as

$$\mathrm{TzCFPOWA}(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m}[w_k a_{\sigma(k)}], \tag{5}$$

where

$$\omega_k = Q\left(\frac{R_k}{TV}\right) - Q\left(\frac{R_{k-1}}{TV}\right), R_k = \sum_{j=1}^{k}[V(a_{\sigma(j)})], TV = \sum_{j=1}^{k} V(a_{\sigma(j)}), V(a_{\sigma(j)}) = 1 + T(a_{\sigma(j)}), \tag{6}$$

where $Q{:}[0, 1] \to [0, 1]$ is a basic unit-interval monotonic function having the properties $Q(0) = 0$, $Q(1) = 1$ and $Q(x) > Q(y)$, if $x > y$. $T(a_{\sigma(j)})$ denotes the support of the j-th largest TzCFN by all the other TzCFNs; hence, $T(a_{\sigma(j)}) = \sum_{l=1, l \neq j}^{m} \mathrm{Sup}(a_{\sigma(j)}, a_{\sigma(l)}) \cdot \mathrm{Sup}(a_{\sigma(j)}, a_{\sigma(l)})$ indicates the support of l-th largest argument for the j-th largest argument. $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a permutation of $(1, 2, \ldots, m)$, such that $a_{\sigma(k-1)} \geq a_{\sigma(k)}$ $k = (1, 2, \ldots, m)$ and the ranking method of TzCFNs.

Theorem 7: *If $Q(x) = x$, then the power ordered weighted average operator of TzCFNs reduces to the power average operator of TzCFNs, i.e.*

$$\mathrm{TzCFPOWA}(a_1, a_2, \ldots, a_m) = \mathrm{TzCFPA}(a_1, a_2, \ldots, a_m).$$

Theorem 8: *If $Sup(a_k, a_j) = c(c \in [0, 1], k \neq j)$ and $Q(x) = x$, then the power ordered weighted average operator of TzCFNs reduces to the arithmetic average operator of TzCFNs as follows: TzCFPOWA $(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m} \frac{a_k}{m}$.*

Theorem 9: *If $Q(x) = 1$ for all $x > 0$, then $\text{TzCFPOWA}(a_1, a_2, \ldots, a_m) = \max\{a_i \mid i = 1, 2, \ldots, m\}$. If $Q(x) = 0$ for all $x < 1$, then TzCFPOWA $(a_1, a_2, \ldots, a_m) = \min\{a_i|i = 1, 2, \ldots, m\}$. Similarly, the power ordered weighted average operator of TzCFNs includes properties such as idempotency, boundary, and commutativity.*

Definition 14: Let $(a_1, a_2, \ldots, a_m)$ be the TzCFNs, then the power hybrid average operator of TzCFNs is defined as

$$\text{TzCFPHA}(a_1, a_2, \ldots, a_m) = \sum_{k=1}^{m}[w_k a_k^{'}], \tag{7}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is the associated vector, satisfying that $0 \leq \omega_k \leq 1$ $(k = 1, 2, \ldots, m)$ and $\sum_{k=1}^{m} \omega_k = 1$. We have $a_k^{'}$ is the k-th largest of the weighted TzCFNs $a_i^{'}(i = 1, 2, \ldots, m)$, $a_i^{'} = mw_i a_i$, $w = (w_1, w_2, \ldots, w_m)^T$ is the weight vector of $a_i(i = 1, 2, \ldots, m)$, satisfying that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{m} w_i = 1$. m is the balancing coefficient.

4.2 Determining Approach of $T(a_k)$ for Power Average Operators of TzCFNs

Before using these power average operators of TzCFNs, the key issue is to determine $T(a_k)$, i.e. the support of a_k by all the other TzCFNs. In the following, a determining approach of $T(a_k)$ for power average operators of TzCFNs is investigated. For TzCFNs $(a_1, a_2, \ldots, a_m)$, the consensus degree matrix is constructed as follows:

$$S = (S_{kj})_{m \times m}, \tag{8}$$

where $S_{kj} = 1 - d(a_k, a_j)$, d is Hamming or Euclidean distance defined as in Definition 7. The average consensus degree of a_k by all the other TzCFNs is defined as follows:

$$AS_k = \frac{1}{n-1} \sum_{j=1, j \neq k}^{m} S_{kj}. \tag{9}$$

To compare easily, we normalize the average consensus degree to obtain the relative consensus degree of a_k by all the other TzCFNs, as follows:

$$RS(a_k) = \frac{AS(a_k)}{\sum_{k=1}^{m} AS(a_k)}. \tag{10}$$

As can be seen from Eqs. (8)–(10), $RS(a_k)$ meets the conditions of support function in Definition 7. Hence, $RS(a_k)$ can be viewed as AS_k in Definitions 11–14.

5 MAGDM Model and Method Using TzCFNs

5.1 MAGDM Problem Using TzCFNs

A MAGDM problem is to find a best compromise solution from all feasible alternatives assessed on multiple attributes. Assume that there is a group consisting of k decision makers $\{P_1, P_2, \ldots, P_k\}$ who have to choose one of (or rank) m alternatives $\{A_1, A_2, \ldots, A_m\}$ based on n attributes $\{a_1, a_2, \ldots, a_n\}$. Denote an alternative set by $A = \{A_1, A_2, \ldots, A_m\}$ and an attribute set by $F = \{a_1, a_2, \ldots, a_n\}$. The weight vector of decision makers is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$, satisfying that $0 \leq \lambda_t \leq 1$ $(t = 1, 2, \ldots, k)$ and $\sum_{t=1}^{k} \lambda_t = 1$. Suppose that the trapezoidal cubic fuzzy rating of an alternative A_i on an attribute a_j given by the decision maker P_t is a TzCFN $a_{ij}^t = [h_{1i}^{(t)}(a_j), h_{2i}^{(t)}(a_j), h_{3i}^{(t)}(a_j), h_{4i}^{(t)}(a_j)]; \langle [w_{ij}^{-(t)}, w_{ij}^{+(t)}], w_{ij}^{(t)} \rangle$, where $[w_{ij}^{-(t)}, w_{ij}^{+(t)}]$ denotes the extent to which alternative A_i belongs to trapezoidal fuzzy number $[h_{1i}^{(t)}(a_j), h_{2i}^{(t)}(a_j), h_{3i}^{(t)}(a_j), h_{4i}^{(t)}(a_j)]$ on the attribute, $w_{ij}^{(t)}$ denotes the extent to which alternative A_i does not belong to the trapezoidal fuzzy number $[h_{1i}^{(t)}(a_j), h_{2i}^{(t)}(a_j), h_{3i}^{(t)}(a_j), h_{4i}^{(t)}(a_j)]$ on the attribute a_j, satisfying that $0 \leq w_{ij}^{-(t)} \leq 1, 0 \leq w_{ij}^{+(t)} \leq 1$ and $0 \leq w_{ij}^{(t)} \leq 1$. Hence, a MAGDM problem using TzCFNs can be concisely expressed in matrix format, as follows: $A^{(t)} = (a_{ij}^{(t)})_{m\times n}$ $(t = 1, 2, \ldots, k)$, which are the trapezoidal cubic fuzzy decision matrices.

5.2 MAGDM Method Based on Power Average Operators of TzCFNs

In general, attributes can be classified into two types: benefit attributes and cost attributes. In other words, the attribute set F can be divided into two subsets: F_1 and F_2, which are the subsets of benefit attributes and cost attributes, respectively. As the n attributes may be measured in different ways, so the matrix

$$A^t = (a_{ij}^{(t)})_{m\times n} \tag{11}$$

needs to be normalized into

$$R^t = (r_{ij}^{(t)})_{m\times n}, \tag{12}$$

where

$$r_{ij}^t = [r_{1i}^{(t)}(a_j), r_{2i}^{(t)}(a_j), r_{3i}^{(t)}(a_j), r_{4i}^{(t)}(a_j)];\ \langle [w_{ij}^{-(t)}, w_{ij}^{+(t)}], w_{ij}^{(t)}\rangle.$$

In this paper, the normalization method is chosen as follows:

$$[r_{1i}^{(t)}(a_j), r_{2i}^{(t)}(a_j), r_{3i}^{(t)}(a_j), r_{4i}^{(t)}(a_j)] = \begin{bmatrix} \dfrac{r_{1i}^{(t)}(a_j)}{r_{4j}^{(t)-}(a_j)}, \dfrac{r_{2i}^{(t)}(a_j)}{r_{3j}^{(t)-}(a_j)} \wedge 1, \\ \dfrac{r_{3i}^{(t)}(a_j)}{r_{2j}^{(t)-}(a_j)} \wedge 1, \dfrac{r_{4i}^{(t)}(a_j)}{r_{1j}^{(t)-}(a_j)} \wedge 1 \end{bmatrix} \quad \text{for } j \in F_1$$

$$[r_{1i}^{(t)}(a_j), r_{2i}^{(t)}(a_j), r_{3i}^{(t)}(a_j), r_{4i}^{(t)}(a_j)] = \begin{bmatrix} \dfrac{r_{1i}^{(t)}(a_j)}{r_{4j}^{(t)+}(a_j)}, \dfrac{r_{2i}^{(t)}(a_j)}{r_{3j}^{(t)+}(a_j)} \wedge 1, \\ \dfrac{r_{3i}^{(t)}(a_j)}{r_{2j}^{(t)+}(a_j)} \wedge 1, \dfrac{r_{4i}^{(t)}(a_j)}{r_{1j}^{(t)+}(a_j)} \wedge 1 \end{bmatrix} \quad \text{for } j \in F_1$$

and

$$[r_{1i}^{(t)}(a_j), r_{2i}^{(t)}(a_j), r_{3i}^{(t)}(a_j), r_{4i}^{(t)}(a_j)] = \begin{bmatrix} \dfrac{r_{1i}^{(t)}(a_j)}{r_{4j}^{(t)}(a_j)}, \dfrac{r_{2i}^{(t)}(a_j)}{r_{3j}^{(t)}(a_j)} \wedge 1, \\ \dfrac{r_{3i}^{(t)}(a_j)}{r_{2j}^{(t)}(a_j)} \wedge 1, \dfrac{r_{4i}^{(t)}(a_j)}{r_{1j}^{(t)}(a_j)} \wedge 1 \end{bmatrix} \quad \text{for } j \in F_2$$

where $r_{kj}^{(t)-} = \max\{r_{1i}^{(t)}(a_j) | i = 1, 2, \ldots, m\}$, $r_{kj}^{(t)+} = \max\{r_{1i}^{(t)}(a_j) | i = 1, 2, \ldots, m\}$, and $r_{kj}^{(t)} = \min\{r_{1i}^{(t)}(a_j) | i = 1, 2, \ldots, m\}$.

The normalization method mentioned above is done to preserve the property that the range of a normalized trapezoidal fuzzy number $[r_{1i}^{(t)}(a_j), r_{2i}^{(t)}(a_j), r_{3i}^{(t)}(a_j), r_{4i}^{(t)}(a_j)]$ belongs to the closed interval [0, 1]. Then, the decision matrix $A^{(t)} = (a_{ij}^{(t)})_{m\times n}$ can be transformed into the normalized trapezoidal cubic fuzzy decision matrix $R^{(t)} = (r_{ij}^{(t)})_{m\times n}$.

Remark 1: The normalization equations for cost attributes given by Wang and Zhang [36] are as follows:

$$r_{ki}(a_j) = \left[\frac{\max_j\{h_{4i}^-(a_j)\} - h_{ki}^-(a_j)}{\max_j\{h_{4i}^-(a_j)\} - \min_j\{h_{1i}^-(a_j)\}}\right],$$

$$r_{ki}(a_j) = \left[\frac{\max_j\{h_{4i}^+(a_j)\} - h_{ki}^+(a_j)}{\max_j\{h_{4i}^+(a_j)\} - \min_j\{h_{1i}^+(a_j)\}}\right],$$

and

$$r_{ki}(a_j) = \left[\frac{\max_j\{h_{4i}(a_j)\} - h_{ki}(a_j)}{\max_j\{h_{4i}(a_j)\} - \min_j\{h_{1i}(a_j)\}}\right].$$

It is easily seen that the above equations cannot ensure that the normalized fuzzy number $[r_{1i}^{(t)}(a_j), r_{2i}^{(t)}(a_j), r_{3i}^{(t)}(a_j), r_{4i}^{(t)}(a_j)]$ is still a TzCFN.

For the i-th line elements of matrix $R^{(t)} = (r_{ij}^{(t)})_{m\times n}$, by Eq. (8) the consensus degree matrix of alternative A_i given by P_t is obtained as follows:

$$S_i^{(t)} = (r_{ij}^{(t)})_{n\times n}, \tag{13}$$

where $S_i^{(t)} = S(r_{ik}^{(t)}, r_{ij}^{(t)})_{n\times n} = 1 - d(r_{ik}^{(t)}, r_{ij}^{(t)})_{n\times n}$; d is Hamming or Euclidean distance defined as in Definition 7.

By Eq. (9), the average consensus degree of attribute a_k by all the other attributes of alternative A_i is computed as

$$AS_k^{(t)} = \frac{1}{n-1}\sum_{j\neq k}^{n} S_{kj}^{(t)}. \tag{14}$$

By Eq. (10), the relative consensus degree of attribute a_k by all the other attributes of alternative A_i is calculated as

$$RS(S_k^{(t)}) = \frac{AS_k^{(t)}}{\sum_{k=1}^{n} AS_k^{(t)}}. \tag{15}$$

Obviously, $RS(S_k^{(t)})$ reflects the support degree of attribute a_k by all the other attributes of alternative A_i given by decision maker p_t. The bigger the value of $RS(S_k^{(t)})$, the higher the support degree of a_k by all the other attributes. As can be seen from Eqs. (13) to (15), $RS(S_k^{(t)})$ meets the conditions of support function in Definition 11. Based on the above analysis, an algorithm and process of the MAGDM problems using TzCFNs may be given as follows:

Step 1: Normalize the decision matrix $A^{(t)} = (a_{ij}^{(t)})_{n\times n}$ according to Eqs. (11) and (12).

Step 2: Calculate the individual overall TzCFNs of all the alternatives. Substituting $T(a_k)$ of Definition 11 by $RS(S_k^{(t)})$, the individual overall TzCFN of alternative A_i given by p_t is derived as follows:

$$a_i^t = \text{TzCFPA}(r_{i1}^t, r_{i2}^t, \ldots, r_{in}^t) \quad (i = 1, 2, \ldots, m;\ t = 1, 2, \ldots, k). \tag{16}$$

Step 3: Using the TzCFPHA operator to integrate a_i^t $(t = 1, 2, \ldots, k)$, the collective overall TzCFN of alternative A_i is obtained as

$$a_i = \text{TzCFPHA}_{\omega,\lambda}(r_i^{(1)}, r_i^{(2)}, \ldots, r_{in}^{(k)}), \tag{17}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the decision makers' weight vector and $\omega = (\omega_1, \omega_2, \ldots, \omega_k)^T$ is the associated vector.

Step 4: The ranking orders of alternatives are generated according to TzCFNs $a_i (i = 1, 2, \ldots, m)$ by the ranking method.

Remark 2: If the weight vector of attributes is known, the algorithm and process of the MAGDM problems using TzCFNs are the same as the above steps except for substituting TzCFPA by TzCFWPAW in Eq. (16), where $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of attributes, satisfying that $0 \leq w_j \leq 1$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$.

Table 1: Decision Matrix Given by Expert p_1.

	a_1	a_2	a_3	a_4
A_1	$\{[0.2, 0.3, 0.4, 0.5], \langle [0.7, 0.14], 0.10 \rangle\}$	$\{[0.5, 0.7, 0.9, 0.11], \langle [0.24, 0.29], 0.25 \rangle\}$	$\{[0.2, 0.3, 0.4, 0.5], \langle [0.7, 0.14], 0.10 \rangle\}$	$\{[0.2, 0.3, 0.4, 0.5], \langle [0.7, 0.10], 0.9 \rangle\}$
A_2	$\{[0.1, 0.3, 0.5, 0.7], \langle [0.10, 0.14], 0.12 \rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle [0.30, 0.39], 0.34 \rangle\}$	$\{[0.20, 0.33, 0.44, 0.54], \langle [0.12, 0.16], 0.13 \rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle [0.30, 0.34], 0.31 \rangle\}$
A_3	$\{[0.22, 0.24, 0.26, 0.28], \langle [0.12, 0.16], 0.14 \rangle\}$	$\{[0.12, 0.14, 0.16, 0.18], \langle [0.9, 0.20], 0.13 \rangle\}$	$\{[0.13, 0.14, 0.15, 0.16], \langle [0.20, 0.28], 0.23 \rangle\}$	$\{[0.2, 0.3, 0.4, 0.5], \langle [0.7, 0.11], 0.8 \rangle\}$
A_4	$\{[0.2, 0.3, 0.4, 0.5], \langle [0.20, 0.28], 0.22 \rangle\}$	$\{[0.2, 0.3, 0.4, 0.5], \langle [0.7, 0.12], 0.10 \rangle\}$	$\{[0.7, 0.8, 0.9, 0.10], \langle [0.1, 0.5], 0.3 \rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle [0.30, 0.37], 0.33 \rangle\}$

Table 2: Decision Matrix Given by Expert p_2.

	a_1	a_2	a_3	a_4
A_1	$\{[0.1, 0.3, 0.5, 0.7], \langle [0.10, 0.12], 0.11 \rangle\}$	$\{[0.12, 0.14, 0.16, 0.18], \langle [0.9, 0.11], 0.10 \rangle\}$	$\{[0.21, 0.31, 0.41, 0.51], \langle [0.71, 0.91], 0.81 \rangle\}$	$\{[0.11, 0.13, 0.14, 0.15], \langle [0.20, 0.22], 0.21 \rangle\}$
A_2	$\{[0.1, 0.3, 0.5, 0.7], \langle [0.10, 0.12], 0.11 \rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle [0.30, 0.32], 0.31 \rangle\}$	$\{[0.12, 0.14, 0.16, 0.18], \langle [0.9, 0.11], 0.10 \rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle [0.30, 0.32], 0.31 \rangle\}$
A_3	$\{[0.22, 0.24, 0.26, 0.28], \langle [0.12, 0.14], 0.13 \rangle\}$	$\{[0.12, 0.14, 0.16, 0.18], \langle [0.9, 0.11], 0.10 \rangle\}$	$\{[0.13, 0.14, 0.15, 0.16], \langle [0.20, 0.22], 0.21 \rangle\}$	$\{[0.11, 0.13, 0.14, 0.15], \langle [0.20, 0.22], 0.21 \rangle\}$
A_4	$\{[0.12, 0.14, 0.16, 0.18], \langle [0.9, 0.11], 0.10 \rangle\}$	$\{[0.1, 0.3, 0.5, 0.7], \langle [0.10, 0.12], 0.11 \rangle\}$	$\{[0.14, 0.15, 0.16, 0.17], \langle [0.12, 0.14], 0.13 \rangle\}$	$\{[0.21, 0.25, 0.27, 0.29], \langle [0.40, 0.42], 0.41 \rangle\}$

6 Numerical Example

6.1 An Enterprise Selection Problem and the Analysis Process

In order to improve the market-competing ability, a motorcycle company wants to select the best enterprise to form the cooperative alliance from four potential enterprises A_1, A_2, A_3, and A_4. The attributes considered in the selection are as follows: the producing ability a_1, the level of technology innovation a_2, the ability of capital currency a_3, and the ability of research a_4. An expert group is formed that consists of three experts (or decision makers) $\{p_1, p_2, p_3\}$, whose weight vector is $k = (0.25, 0.25, 0.25, 0.25)^T$.

Step 1: Using Eq. (11), the decision matrix $A^{(t)} = (a_{ij}^{(t)})_{n\times n}$ is normalized to $R^{(t)} = (r_{ij}^{(t)})_{n\times n}$. The results are listed in Tables 4–6.

Table 3: Decision Matrix Given by Expert p_3.

	a_1	a_2	a_3	a_4
A_1	$\{[0.1, 0.3, 0.5, 0.7], \langle[0.10, 0.12], 0.11\rangle\}$	$\{[0.2, 0.4, 0.6, 0.8], \langle[0.10, 0.12], 0.11\rangle\}$	$\{[0.11, 0.13, 0.14, 0.15], \langle[0.20, 0.22], 0.21\rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle[0.30, 0.32], 0.31\rangle\}$
A_2	$\{[0.22, 0.24, 0.26, 0.28], \langle[0.12, 0.14], 0.13\rangle\}$	$\{[0.2, 0.3, 0.4, 0.5], \langle[0.28, 0.30], 0.29\rangle\}$	$\{[0.12, 0.14, 0.16, 0.18], \langle[0.9, 0.11], 0.10\rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle[0.30, 0.32], 0.31\rangle\}$
A_3	$\{[0.22, 0.24, 0.26, 0.28], \langle[0.12, 0.14], 0.13\rangle\}$	$\{[0.22, 0.24, 0.26, 0.28], \langle[0.12, 0.14], 0.13\rangle\}$	$\{[0.13, 0.14, 0.15, 0.16], \langle[0.20, 0.22], 0.21\rangle\}$	$\{[0.1, 0.5, 0.7, 0.9], \langle[0.30, 0.32], 0.31\rangle\}$
A_4	$\{[0.12, 0.14, 0.16, 0.18], \langle[0.9, 0.11], 0.10\rangle\}$	$\{[0.3, 0.5, 0.7, 0.9], \langle[0.45, 0.47], 0.46\rangle\}$	$\{[0.11, 0.13, 0.14, 0.15], \langle[0.20, 0.22], 0.21\rangle\}$	$\{[0.21, 0.25, 0.27, 0.29], \langle[0.40, 0.42], 0.41\rangle\}$

Table 4: Normalized Decision Matrix Given by Expert p_1.

	a_1	a_2	a_3	a_4
A_1	$\{[0.1428, 0.2142, 0.2857, 0.3571], \langle[0.7446, 0.1489], 0.1036\rangle\}$	$\{[0.2262, 0.3167, 0.4072, 0.0497], \langle[0.3076, 0.3717], 0.3205\rangle\}$	$\{[0.1428, 0.3571, 0.2857, 0.2143], \langle[0.7446, 0.1489], 0.1063\rangle\}$	$\{[0.1428, 0.2142, 0.2666, 0.3571], \langle[0.4117, 0.0588], 0.5294\rangle\}$
A_2	$\{[0.0625, 0.1875, 0.3125, 0.4375], \langle[0.2777, 0.3888], 0.3333\rangle\}$	$\{[0.0454, 0.2272, 0.3181, 0.4090], \langle[0.2912, 0.3786], 0.3300\rangle\}$	$\{[0.1324, 0.2185, 0.2913, 0.3576], \langle[0.2926, 0.3902], 0.3170\rangle\}$	$\{[0.0454, 0.0909, 0.3181, 0.4090], \langle[0.3157, 0.3578], 0.3263\rangle\}$
A_3	$\{[0.22, 0.24, 0.26, 0.28], \langle[0.2857, 0.3809], 0.3333\rangle\}$	$\{[0.2, 0.2333, 0.2666, 0.3], \langle[0.7317, 0.1626], 0.1056\rangle\}$	$\{[0.2241, 0.2413, 0.2586, 0.2758], \langle[0.2816, 0.3943], 0.3239\rangle\}$	$\{[0.1428, 0.2142, 0.2857, 0.3571], \langle[0.4347, 0.0683], 0.4968\rangle\}$
A_4	$\{[0.1428, 0.2142, 0.2857, 0.3571], \langle[0.2857, 0.4000], 0.0314\rangle\}$	$\{[0.1428, 0.2142, 0.2857, 0.3571], \langle[0.7608, 0.1304], 0.1086\rangle\}$	$\{[0.28, 0.32, 0.36, 0.04], \langle[0.1111, 0.5555], 0.3333\rangle\}$	$\{[0.0454, 0.2272, 0.3181, 0.4090], \langle[0.30, 0.37], 0.33\rangle\}$

Step 2: Utilizing the Hamming distance and Eq. (16), the elements in the i-th line of the normalized matrix $R^{(t)} = (r_{ij}^{(t)})_{n\times n}$ can be integrated into the individual overall TzCFNs of the alternative A_i $(i = 1, 2, 3, 4;\ t = 1, 2, 3, 4)$, as follows:

Step 3: By Eq. (6), the associated weighted vector with the CFPHA operator is obtained as $x = (0.25, 0.25, 0.25, 0.25)^T$. Combining the weight vector of experts $k = (0.36, 0.25, 0.39)^T$ and Eq. (17), the collective overall TzCFNs of the alternatives are, respectively derived as follows:

Table 5: Normalized Decision Matrix Given by Expert p_2.

	a_1	a_2	a_3	a_4
A_1	{[0.0625, 0.1875, 0.3125, 0.4375], ⟨[0.3030, 0.3636], 0.3333⟩}	{[0.2, 0.2333, 0.2666, 0.3], ⟨[0.8108, 0.0990], 0.0900⟩}	{[0.1458, 0.2152, 0.2847, 0.3541], ⟨[0.2921, 0.3744], 0.3333⟩}	{[0.2075, 0.2452, 0.2641, 0.2830], ⟨[0.3174, 0.3492], 0.3333⟩}
A_2	{[0.0625, 0.1875, 0.3125, 0.4375], ⟨[0.3030, 0.3636], 0.3333⟩}	{[0.0454, 0.2272, 0.3181, 0.4090], ⟨[0.3225, 0.3440], 0.3333⟩}	{[0.2, 0.2333, 0.2666, 0.3], ⟨[0.8108, 0.0990], 0.0900⟩}	{[0.0454, 0.2272, 0.3181, 0.4090], ⟨[0.3225, 0.3440], 0.3333⟩}
A_3	{[0.22, 0.24, 0.26, 0.28], ⟨[0.3076, 0.3589], 0.3333⟩}	{[0.2, 0.2333, 0.2666, 0.3], ⟨[0.8108, 0.0990], 0.0900⟩}	{[0.2241, 0.2413, 0.2586, 0.2758], ⟨[0.3174, 0.3492], 0.3333⟩}	{[0.2075, 0.2452, 0.2641, 0.2830], ⟨[0.3174, 0.3492], 0.3333⟩}
A_4	{[0.2, 0.2333, 0.2666, 0.3], ⟨[0.8108, 0.0990], 0.0900⟩}	{[0.0625, 0.1875, 0.3125, 0.4375], ⟨[0.3033, 0.3636], 0.3333⟩}	{[0.2258, 0.2419, 0.2580, 0.2741], ⟨[0.3076, 0.3589], 0.3333⟩}	{[0.2058, 0.2450, 0.2647, 0.2843], ⟨[0.3252, 0.4117], 0.3333⟩}

Table 6: Normalized Decision Matrix Given by Expert p_3.

	a_1	a_2	a_3	a_4
A_1	{[0.0625, 0.1875, 0.3125, 0.4375], ⟨[0.3030,0.3636], 0.3333⟩}	{[0.2, 0.4, 0.6, 0.8], ⟨[0.3030,0.3636], 0.3333⟩}	{[0.2053, 0.2452, 0.2641, 0.2830], ⟨[0.3174,0.3492], 0.3333⟩}	{[0.0454, 0.2272, 0.3181, 0.4090], ⟨[0.3225,0.3440], 0.3333⟩}
A_2	{[0.22, 0.24, 0.26, 0.28], ⟨[0.3076,0.3589], 0.3333⟩}	{[0.1428, 0.2142, 0.2857, 0.3571], ⟨[0.3218,0.3448], 0.3333⟩}	{[0.2, 0.2333, 0.2666, 0.3], ⟨[0.8108,0.0990], 0.0900⟩}	{[0.0454, 0.2272, 0.3181, 0.4090], ⟨[0.3225,0.3440], 0.3333⟩}
A_3	{[0.22, 0.24, 0.26, 0.28], ⟨[0.3076,0.3589], 0.3333⟩}	{[0.22, 0.24, 0.26, 0.28], ⟨[0.3076,0.3589], 0.3333⟩}	{[0.2241, 0.2413, 0.2586, 0.2758], ⟨[0.3174,0.3492], 0.3333⟩}	{[0.0454, 0.2272, 0.3181, 0.4090], ⟨[0.3225,0.3440], 0.3333⟩}
A_4	{[0.2, 0.2333, 0.2666, 0.3], ⟨[0.8108,0.0990], 0.0900⟩}	{[0.125, 0.2083, 0.2916, 0.375], ⟨[0.3260,0.3405], 0.3333⟩}	{[0.2053, 0.2452, 0.2641, 0.2830], ⟨[0.3174,0.3492], 0.3333⟩}	{[0.2058, 0.2450, 0.2647, 0.2843], ⟨[0.3252,0.3414], 0.3333⟩}

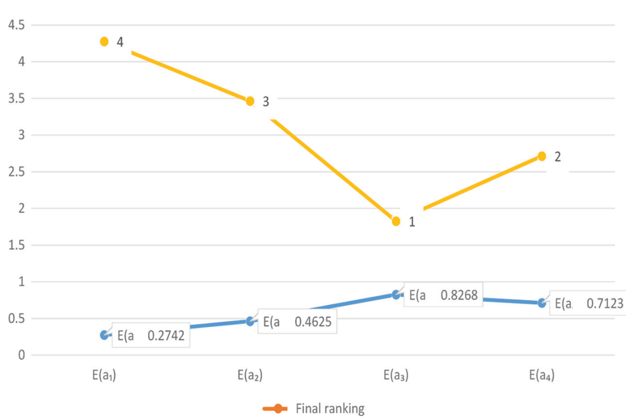
Table 7: Hamming Distance for Trapezoidal Cubic Fuzzy Number.

	a_1	a_2	a_3	a_4
A_1	$\left\langle \begin{bmatrix} 0.3024, \\ 0.1489 \end{bmatrix}, 0.1036 \right\rangle$	$\left\langle \begin{bmatrix} 1.2026, \\ 0.3771 \end{bmatrix}, 0.365 \right\rangle$	$\left\langle \begin{bmatrix} 0.3083, \\ 1.1007 \end{bmatrix}, 1.045 \right\rangle$	$\left\langle \begin{bmatrix} 0.6962, \\ 1.1412 \end{bmatrix}, 0.4386 \right\rangle$
A_2	$\left\langle \begin{bmatrix} 0.2693, \\ 0.3811 \end{bmatrix}, 0.8188 \right\rangle$	$\left\langle \begin{bmatrix} 0.8216, \\ 0.8142 \end{bmatrix}, 0.8243 \right\rangle$	$\left\langle \begin{bmatrix} 1.6808, \\ 0.3902 \end{bmatrix}, 0.3170 \right\rangle$	$\left\langle \begin{bmatrix} 0.3157, \\ 0.3578 \end{bmatrix}, 0.3263 \right\rangle$
A_3	$\left\langle \begin{bmatrix} 0.5319, \\ 0.6205 \end{bmatrix}, 0.5762 \right\rangle$	$\left\langle \begin{bmatrix} 0.7869, \\ 0.5785 \end{bmatrix}, 0.5411 \right\rangle$	$\left\langle \begin{bmatrix} 0.2816, \\ 0.3943 \end{bmatrix}, 0.6439 \right\rangle$	$\left\langle \begin{bmatrix} 0.6732, \\ 1.1317 \end{bmatrix}, 0.4712 \right\rangle$
A_4	$\left\langle \begin{bmatrix} 0.9867, \\ 0.1206 \end{bmatrix}, 0.1096 \right\rangle$	$\left\langle \begin{bmatrix} 0.2882, \\ 1.0466 \end{bmatrix}, 1.0044 \right\rangle$	$\left\langle \begin{bmatrix} 0.9702, \\ 0.6693 \end{bmatrix}, 0.8198 \right\rangle$	$\left\langle \begin{bmatrix} 0.5177, \\ 0.6385 \end{bmatrix}, 0.5695 \right\rangle$

Table 8: Collective Overall Trapezoidal Cubic Fuzzy Number.

	a_1	a_2	a_3	a_4
A_1	$\langle 0.1008 \rangle$	$\langle 0.4008 \rangle$	$\langle 0.3669 \rangle$	$\langle 0.3804 \rangle$
A_2	$\langle 0.2693 \rangle$	$\langle 0.2748 \rangle$	$\langle 0.5602 \rangle$	$\langle 0.1192 \rangle$
A_3	$\langle 0.1269 \rangle$	$\langle 0.2623 \rangle$	$\langle 0.2146 \rangle$	$\langle 0.3772 \rangle$
A_4	$\langle 0.1333 \rangle$	$\langle 0.3488 \rangle$	$\langle 0.3234 \rangle$	$\langle 0.2128 \rangle$

Step 4: Adopting the approach of the rank a_i, the expectation and expectant scores of all alternatives are obtained as follows: $E(a_1) = 0.2742$, $E(a_2) = 0.4625$, $E(a_3) = 0.8268$, $E(a_4) = 0.7123$.



7 Comparison Analyses

To verify the rationality and efficiency of the proposed approach, a comparative study is steered using the methods of IF power average aggregation operator [51], trapezoidal fuzzy number [33], and triangular cubic fuzzy number [11], which are special cases of power average operators of TzCFNs, to the similar expressive example.

7.1 A Comparison Analysis with the Existing MCDM Method Intuitionistic Fuzzy Power Average Aggregation Operator

The IF power average aggregation operator can be considered as a special case of TzCFNs when there is only element in membership and non-membership degree [50]. For comparison, the TzCFNs can be transformed to IF power average aggregation operator by calculating the average value of the membership and non-membership degrees. After transformation, the IF power average aggregation operator is shown in Table 9.

Step 1: The decision matrix $A^{(t)} = (a_{ij}^{(t)})_{n\times n}$ is normalized to $R^{(t)} = (r_{ij}^{(t)})_{n\times n}$ [50]. The results are listed in Table 9.

Step 2: Utilizing the aggregation operator, the elements in the i-th line of the aggregation matrix $R^{(t)} = (r_{ij}^{(t)})_{n\times n}$ can be integrated into the individual overall IFNs of the alternative A_i $(i = 1, 2, 3, 4;\ t = 1, 2, 3, 4)$, as follows: $w = (0.2, 0.2, 0.3, 0.3)$.

Step 3: Find the score value $E(a_1) = 0.2021$, $E(a_2) = -0.0403$, $E(a_3) = 0.0818$, $E(a_4) = -0.0544$.

Step 4: Find the final ranking $E(a_1) > E(a_3) > E(a_4) > E(a_2)$.

Obviously, the ranking being derived from the method proposed by Xu [50] is different from the result of the proposed method. TzCFNs are more flexible than the IF power average aggregation operator because they consider the situations where decision makers would like to use several possible values to express the membership and non-membership degrees.

7.2 A Comparison Analysis with the Existing MCDM Method Trapezoidal Fuzzy Number

Trapezoidal fuzzy number can be considered a special case of TzCFNs when decision makers only consider membership degrees in evaluation [33]. For comparison, the TzCFNs can be transformed to trapezoidal fuzzy number by remaining only the membership degrees and non-membership degrees. After transformation, the trapezoidal fuzzy number information is given in Table 11.

Step 1: Utilizing the aggregation operator, the elements in the i-th line of the normalized matrix $R^{(t)} = (r_{ij}^{(t)})_{n\times n}$ can be integrated into the individual overall TzFNs of the alternative $A_i(i = 1, 2, 3;\ t = 1, 2, 3)$, as follows:

Step 2: Find the score value $E(a_1) = 0.4486$, $E(a_2) = 0.6810$, $E(a_3) = 0.3167$, $E(a_4) = 0.4221$.

Step 3: Find the final ranking $E(a_2) > E(a_1) > E(a_4) > E(a_3)$.

The ranking of all alternatives $Z_2 > Z_1 > Z_4 > Z_3$ and Z_2 is the best selection. Obviously, the ranking being derived from the method proposed by Shaw and Roy [33] is different from the result of the proposed

Table 9: Intuitionistic Fuzzy Power Average Aggregation Operator.

	C_1	C_2	C_3	C_4
B_1	[0.7, 0.14]	[0.24, 0.29]	[0.7, 0.14]	[0.7, 0.10]
B_2	[0.10, 0.14]	[0.30, 0.39]	[0.12, 0.16]	[0.30, 0.34]
B_3	[0.12, 0.16]	[0.9, 0.20]	[0.20, 0.28]	[0.7, 0.11]
B_4	[0.20, 0.28]	[0.7, 0.12]	[0.1, 0.5]	[0.30, 0.37]

Table 10: Aggregating Matrix on Intuitionistic Fuzzy Power Averaging.

B_1	[0.5650, 0.1609]
B_2	[0.1829, 0.2636]
B_3	[0.3554, 0.1918]
B_4	[0.2357, 0.3445]

Table 11: Trapezoidal Fuzzy Decision Matrix.

	C_1	C_2	C_3	C_4
B_1	$\begin{bmatrix}0.1, 0.3,\\ 0.5, 0.7\end{bmatrix}$	$\begin{bmatrix}0.12, 0.14,\\ 0.16, 0.18\end{bmatrix}$	$\begin{bmatrix}0.21, 0.31,\\ 0.41, 0.51\end{bmatrix}$	$\begin{bmatrix}0.11, 0.13,\\ 0.14, 0.15\end{bmatrix}$
B_2	$\begin{bmatrix}0.1, 0.3,\\ 0.5, 0.7\end{bmatrix}$	$\begin{bmatrix}0.1, 0.5,\\ 0.7, 0.9\end{bmatrix}$	$\begin{bmatrix}0.12, 0.14,\\ 0.16, 0.18\end{bmatrix}$	$\begin{bmatrix}0.1, 0.5,\\ 0.7, 0.9\end{bmatrix}$
B_3	$\begin{bmatrix}0.22, 0.24,\\ 0.26, 0.28\end{bmatrix}$	$\begin{bmatrix}0.12, 0.14,\\ 0.16, 0.18\end{bmatrix}$	$\begin{bmatrix}0.13, 0.14,\\ 0.15, 0.16\end{bmatrix}$	$\begin{bmatrix}0.11, 0.13,\\ 0.14, 0.15\end{bmatrix}$
B_4	$\begin{bmatrix}0.12, 0.14,\\ 0.16, 0.18\end{bmatrix}$	$\begin{bmatrix}0.1, 0.3,\\ 0.5, 0.7\end{bmatrix}$	$\begin{bmatrix}0.14, 0.15,\\ 0.16, 0.17\end{bmatrix}$	$\begin{bmatrix}0.21, 0.25,\\ 0.27, 0.29\end{bmatrix}$

Table 12: Trapezoidal Fuzzy Weighted Averaging Operator.

B_1	[0.1333, 0.2024, 0.2560, 0.3056]
B_2	[0.1056, 0.3081, 0.4203, 0.5281]
B_3	[0.1351, 0.1525, 0.1661, 0.1797]
B_4	[0.1433, 0.1980, 0.2351, 0.2678]

method. TzCFNs are more flexible than trapezoidal fuzzy numbers because they consider the situations where decision makers would like to use several possible values to express the membership degrees.

7.3 A Comparison Analysis with the Existing MCDM Triangular Cubic Fuzzy Number

Triangular cubic fuzzy number can be considered as a special case of TzCFNs when there are only three membership and non-membership degrees [11]. For comparison, the triangular cubic fuzzy number can be transformed to triangular cubic fuzzy numbers by calculating the average value of the membership and non-membership degrees. After transformation, the triangular cubic fuzzy number information is given in Table 13.

Step 1: Use the TCFHA operator to aggregate all three decision matrices into a single collective decision matrix with triangular cubic fuzzy ratings. Consider $w = (0.2, 0.3, 0.2, 0.3)$.

Step 2: To find the ranking order of the alternatives, use the score function $E(a_1) = 0.0409$, $E(a_2) = 0.0628$, $E(a_3) = 0.0203$, $E(a_4) = 0.1145$.

Step 3: Find the ranking $E(a_4) > E(a_2) > E(a_1) > E(a_3)$.

Table 13: Triangular Cubic Fuzzy Decision Matrix.

	a_1	a_2	a_3	a_4
A_1	$\begin{Bmatrix}[0.2, 0.3,\\ 0.4], \langle[0.7,\\ 0.14], 0.10\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.5, 0.7,\\ 0.9], \langle[0.24,\\ 0.29], 0.25\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.2, 0.3,\\ 0.4], \langle[0.7,\\ 0.14], 0.10\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.2, 0.3,\\ 0.4], \langle[0.7,\\ 0.10], 0.9\rangle\end{Bmatrix}$
A_2	$\begin{Bmatrix}[0.1, 0.3,\\ 0.5], \langle[0.10,\\ 0.14], 0.12\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.1, 0.5,\\ 0.7], \langle[0.30,\\ 0.39], 0.34\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.20, 0.33,\\ 0.44], \langle[0.12,\\ 0.16], 0.13\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.1, 0.5,\\ 0.7], \langle[0.30,\\ 0.34], 0.31\rangle\end{Bmatrix}$
A_3	$\begin{Bmatrix}[0.22, 0.24,\\ 0.26], \langle[0.12,\\ 0.16], 0.14\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.12, 0.14,\\ 0.16], \langle[0.9,\\ 0.20], 0.13\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.13, 0.14,\\ 0.15], \langle[0.20,\\ 0.28], 0.23\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.2, 0.3,\\ 0.4], \langle[0.7,\\ 0.11], 0.8\rangle\end{Bmatrix}$
A_4	$\begin{Bmatrix}[0.2, 0.3,\\ 0.4], \langle[0.20,\\ 0.28], 0.22\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.2, 0.3,\\ 0.4], \langle[0.7,\\ 0.12], 0.10\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.7, 0.8,\\ 0.9], \langle[0.1,\\ 0.5], 0.3\rangle\end{Bmatrix}$	$\begin{Bmatrix}[0.1, 0.5,\\ 0.7], \langle[0.30,\\ 0.37], 0.33\rangle\end{Bmatrix}$

Table 14: Aggregating Matrix on Triangular Cubic Fuzzy Number.

B_1	$\{[0.22, 0.32, 0.42], \langle[0.5403, 0.1392], 0.2954\rangle\}$
B_2	$\{[0.12, 0.555, 0.702], \langle[0.2472, 0.3096], 0.1461\rangle\}$
B_3	$\{[0.134, 0.164, 0.194], \langle[0.5376, 0.1551], 0.3198\rangle\}$
B_4	$\{[0.28, 0.57, 0.72], \langle[0.4326, 0.3833], 0.1591\rangle\}$

Table 15: Comparison Analysis with Existing Method.

An enterprise selection problem and the analysis process	$E(a_3) > E(a_4) > E(a_2) > E(a_1)$
Intuitionistic fuzzy power average aggregation operator [50]	$E(a_1) > E(a_3) > E(a_4) > E(a_2)$
Trapezoidal fuzzy number [33]	$E(a_2) > E(a_1) > E(a_4) > E(a_3)$
Triangular cubic fuzzy number [11]	$E(a_4) > E(a_2) > E(a_1) > E(a_3)$

The ranking of all alternatives $Z_4 > Z_2 > Z_1 > Z_3$ and Z_4 is the best selection. Obviously, the ranking being derived from the method proposed by Fahmi et al. [11] is different from the result of the proposed method. The main reason is that the triangular cubic fuzzy number only considers the interval-valued triangular fuzzy number and triangular fuzzy number, which may result in the information not being equal.

The advantages of our proposal can be summarized on the basis of the above comparison analyses. TzCFNs are very suitable for illustrating uncertain or fuzzy information in MCDM problems because the membership and non-membership degrees can be two sets of several possible values, which cannot be achieved by the IF power average aggregation operator, trapezoidal fuzzy number, and triangular cubic fuzzy number. On the basis of basic operations, aggregation operators and comparison method of TzCFNs can also be used to process the IF power average aggregation operator, trapezoidal fuzzy number, and triangular cubic fuzzy number after slight adjustments, because TzCFNs can be considered as the generalized form of the IF power average aggregation operator, trapezoidal fuzzy number, and triangular cubic fuzzy number. The defined operations of TzCFNs give us more accuracy than the existing operators.

A comparison analysis with existing methods is shown in Table 15.

8 Conclusions

This paper considers the MAGDM problem, in which the attribute values are in the form of TzCFNs, and a new MAGDM method is offered. Based on the Hausdorff metric, two kinds of distances between TzCFNs are defined: relating Hamming distance and Euclidean distance. The new operation laws for TzCFNs are specified. Hereby, four kinds of power average operators of TzCFNs are defined and their desirable properties are studied. Applying the power average operator and the power hybrid average operator of TzCFNs, the collective overall TzCFNs of alternatives are derived. Then, the ranking order of alternatives is generated according to the collective overall TzCFNs of alternatives. Finally, an illustrative example is given to verify the developed approach and to demonstrate its practicality and effectiveness. Because the power average operators of TzCFNs can sufficiently take into account the information about the relationships among the arguments being aggregated and can reduce the influence of outlier arguments on the decision result by assigning lower weights to those outliers, they can make the decision result more reflective of the total collection of arguments. The proposed decision method in this paper is more objective and reasonable. Meanwhile, the power average operators of TzCFNs greatly enrich the research content of cubic fuzzy MAGDM and provide a new tool of information fusion for solving decision problems under cubic fuzzy environments. The developed method is very suitable for decision making problems in many areas, especially in situations where the problems involve multiple different attributes with different dimensions and some unfair assessment data. It is expected to be applicable to supplier management, water environment assessment, threat evaluation and missile weapon system selection, warship combat plan evaluation, etc. For future research, we will develop some new geometric aggregation operators for TzCFNs, including the power geometric operator of TzCFNs, weighted power geometric operator of TzCFNs, and power ordered weighted geometric operator of TzCFNs, whose weighting vectors depend upon the input arguments and allow values being aggregated to support.

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