J. Intell. Syst. 2020; 29(1): 736–752 DE GRUYTER

Khaista Rahman*, Saleem Abdullah, Asad Ali and Fazli Amin

Pythagorean Fuzzy Einstein Hybrid Averaging Aggregation Operator and its Application to Multiple-Attribute Group Decision Making

https://doi.org/10.1515/jisys-2018-0071
Received February 3, 2018; previously published online July 9, 2018.

Abstract: Pythagorean fuzzy set is one of the successful extensions of the intuitionistic fuzzy set for handling uncertainties in information. Under this environment, in this paper, we introduce the notion of Pythagorean fuzzy Einstein hybrid averaging (PFEHA) aggregation operator along with some of its properties, namely idempotency, boundedness, and monotonicity. PFEHA aggregation operator is the generalization of Pythagorean fuzzy Einstein weighted averaging aggregation operator and Pythagorean fuzzy Einstein ordered weighted averaging aggregation operator. The operator proposed in this paper provides more accurate and precise results as compared to the existing operators. Therefore, this method plays a vital role in real-world problems. Finally, we applied the proposed operator and method to multiple-attribute group decision making.

Keywords: Pythagorean fuzzy set, PFEHA aggregation operator, multiple-attribute group decision-making problem.

1 Introduction

Multi-criteria decision making is one of the processes for finding the optimal alternative from all feasible alternatives according to some criteria or attributes. Traditionally, it has been generally assumed that all data that access the alternative in terms of criteria and their corresponding weights are expressed in the form of crisp numbers. However, most of the decisions in real-life situations are taken in the environment where the goals and constraints are generally imprecise or vague in nature. In order to handle the uncertainties, the intuitionistic fuzzy set [1] theory, one of the successful extensions of the fuzzy set theory [36], which is characterized by the degree of membership and degree of non-membership, has been presented. Xu [25] developed some basic arithmetic aggregation operators, including intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, and intuitionistic fuzzy hybrid averaging operator. Xu and Yager [29] defined some basic geometric aggregation operators, such as intuitionistic fuzzy weighted geometric operator, intuitionistic fuzzy ordered weighted geometric operator, and intuitionistic fuzzy hybrid geometric operator. Wang and Liu [22, 23] introduced the notion of some Einstein aggregation operators, such as intuitionistic fuzzy Einstein weighted geometric operator, intuitionistic fuzzy Einstein ordered weighted geometric operator, intuitionistic fuzzy Einstein weighted averaging operator, and intuitionistic fuzzy Einstein ordered weighted averaging operator, and applied them to group decision making. In Refs. [5, 6, 8, 9, 20, 21, 24, 26, 27, 30, 31, 35], many scholars worked in the field of intuitionistic fuzzy sets and introduced many aggregation operators and applied them to group decision making.

However, there are many cases where the decision maker may provide the degree of membership and non-membership of a particular attribute in such a way that their sum is greater than 1. For example,

***Corresponding author: Khaista Rahman,** Department of Mathematics, Hazara University, Mansehra, KPK, Pakistan, e-mail: khaista355@yahoo.com
Saleem Abdullah: Department of Mathematics, Abdul Wali Khan University, Mardan, KPK, Pakistan
Asad Ali and Fazli Amin: Department of Mathematics, Hazara University, Mansehra, KPK, Pakistan

suppose a man expresses his preferences toward the alternative in such a way that degree of their satisfaction is 0.6 and the degree of rejection is 0.8. Obviously, its sum is greater than 1. Therefore, Yager [33] introduced the concept of another set called Pythagorean fuzzy set. Pythagorean fuzzy set is a more powerful tool to solve uncertain problems. Like intuitionistic fuzzy aggregation operators, Pythagorean fuzzy aggregation operators have also become an interesting and important area for research, after the advent of the Pythagorean fuzzy sets theory. In 2013, Yager and Abbasov [34] introduced the notion of two new Pythagorean fuzzy aggregation operators, such as Pythagorean fuzzy weighted averaging operator and Pythagorean fuzzy ordered weighted averaging operator. In Refs. [10–19], Rahman et al. introduced the concept of many aggregation operators using Pythagorean fuzzy numbers and also applied them to group decision making. In Refs. [2–4], Garg introduced the notion of Einstein averaging aggregation operator and Einstein geometric aggregation operator, and applied them to group decision making. In Ref. [37], Zang and Xu introduced the notion of TOPSIS for multiple-criteria decision making with Pythagorean fuzzy sets. Xue et al. [32], Liang et al. [7], and Xu and Da [28] developed some methods and aggregation operators using Pythagorean fuzzy information.

Thus, keeping the advantages and applications of the above-mentioned aggregation operators, in this paper, we introduce the notion of Pythagorean fuzzy Einstein hybrid averaging (PFEHA) aggregation operator along with its desirable properties, namely idempotency, boundedness, and monotonicity. Actually, Pythagorean fuzzy Einstein weighted averaging (PFEWA) aggregation operator weights only the Pythagorean fuzzy arguments and Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA) aggregation operator weights only the ordered positions of the Pythagorean fuzzy arguments instead of weighting the Pythagorean fuzzy arguments themselves. To overcome these limitations, we introduce the concept of PFEHA aggregation operator, which weights both the given Pythagorean fuzzy value and its ordered position. Thus, the method proposed in this paper is more general, more flexible, and provides more accurate and precise results compared to the existing methods.

The remainder of this paper is structured as follows. In Section 2, we give some basic definitions and results, which will be used in later sections. In Section 3, we introduce the notion of PFEHA aggregation operator and method. In Section 4, we apply the proposed aggregation operator to multiple-attribute group decision-making problems with Pythagorean fuzzy information. In Section 5, we construct a numerical example. In Section 6, we compare the proposed method to other methods. In Section 7, we provide our conclusion.

2 Preliminaries

In the following, we developed Pythagorean fuzzy set, score function, and accuracy function.

Definition 1 ([37]). Let Z be a universal set, then a Pythagorean fuzzy set can be defined as

$$P=\{\langle z, \mu_P(z), \eta_P(z)\rangle \mid z\in Z\}, \tag{1}$$

where $\mu_P(z)$ and $\eta_P(z)$ are mappings from Z to [0, 1], such that $0\le\mu_P(z)\le 1$, $0\le\eta_P(z)\le 1$, and also $0\le\mu_P^2(z)+\eta_P^2(z)\le 1$, for all $z\in Z$, and they denote the membership degree and non-membership degree of element $z\in Z$ to set P, respectively. Let $\pi_P(z)=\sqrt{1-\mu_P^2(z)-\eta_P^2(z)}$, then it is called the Pythagorean fuzzy index of element $z\in Z$ to set P, representing the degree of indeterminacy of z to P. Also, $0\le\pi_P(z)\le 1$, for every $z\in Z$.

Definition 2 ([37]). Let $\alpha=\langle\mu_\alpha, \eta_\alpha\rangle$ be a Pythagorean fuzzy number, then the score function of α can be defined as

$$s(\alpha)=\mu_\alpha^2-\eta_\alpha^2, \tag{2}$$

where $s(\alpha)\in[-1, 1]$.

Definition 3 ([37]). Let $\alpha=\langle\mu_\alpha, \nu_\alpha\rangle$ be a Pythagorean fuzzy number, then the accuracy function of α can be defined as

$$h(\alpha)=\mu_\alpha^2+\eta_\alpha^2, \tag{3}$$

where $h(\alpha)\in[0, 1]$.

Definition 4 ([37]). Let $\alpha_1=\langle\mu_{\alpha_1}, \eta_{\alpha_1}\rangle$ and $\alpha_2=\langle\mu_{\alpha_2}, \eta_{\alpha_2}\rangle$ be the two Pythagorean fuzzy number, then the following conditions hold:

1. If $s(\alpha_1)\prec s(\alpha_2)$, then $\alpha_1\prec\alpha_2$.
2. If $s(\alpha_1)=s(\alpha_2)$, then
1. If $h(\alpha_1)=h(\alpha_2)$, then $\alpha_1=\alpha_2$.
2. If $h(\alpha_1)\prec h(\alpha_2)$, then $\alpha_1\prec\alpha_2$.
3. If $h(\alpha_1)\succ h(\alpha_2)$, then $\alpha_1\succ\alpha_2$.

In the following, we developed some Einstein operational laws for sum and product.

Definition 5 ([2]). Let $\alpha_j=\langle\mu_{\alpha_j}, \eta_{\alpha_j}\rangle (j=1, 2)$ be the three Pythagorean fuzzy values and $\delta\succ0$ be any real number, then

(1) $$\alpha_1\oplus_\varepsilon\alpha_2=\left(\frac{\sqrt{\mu_{\alpha_1}^2+\mu_{\alpha_2}^2}}{\sqrt{1+\mu_{\alpha_1}^2\cdot_\varepsilon\mu_{\alpha_2}^2}}, \frac{\eta_{\alpha_1}\cdot_\varepsilon\eta_{\alpha_2}}{\sqrt{1+(1-\eta_{\alpha_1}^2)\cdot_\varepsilon(1-\eta_{\alpha_2}^2)}}\right).$$

(2) $$\alpha_1\otimes_\varepsilon\alpha_2=\left(\frac{\mu_{\alpha_1}\cdot_\varepsilon\mu_{\alpha_2}}{\sqrt{1+(1-\mu_{\alpha_1}^2)\cdot_\varepsilon(1-\mu_{\alpha_2}^2)}}, \frac{\sqrt{\eta_{\alpha_1}^2+\eta_{\alpha_2}^2}}{\sqrt{1+\eta_{\alpha_1}^2\cdot_\varepsilon\eta_{\alpha_2}^2}}\right).$$

(3) $$\alpha^{\wedge_\varepsilon\delta}=\left(\frac{\sqrt{2(\mu_\alpha^2)^\delta}}{\sqrt{(2-\mu_\alpha^2)^\delta+(\mu_\alpha^2)^\delta}}, \frac{\sqrt{(1+\eta_\alpha^2)^\delta-(1-\eta_\alpha^2)^\delta}}{\sqrt{(1+\eta_\alpha^2)^\delta+(1-\eta_\alpha^2)^\delta}}\right).$$

(4) $$\delta\cdot_\varepsilon\alpha=\left(\frac{\sqrt{(1+\mu_\alpha^2)^\delta-(1-\mu_\alpha^2)^\delta}}{\sqrt{(1+\mu_\alpha^2)^\delta+(1-\mu_\alpha^2)^\delta}}, \frac{\sqrt{2(\eta_\alpha^2)^\delta}}{\sqrt{(2-\eta_\alpha^2)^\delta+(\eta_\alpha^2)^\delta}}\right).$$

In the following, we developed some aggregation operators, such as Pythagorean fuzzy hybrid averaging (PFHA) operator, PFEWA aggregation operator, and PFEOWA aggregation operator.

Definition 6 ([17]). Let $\alpha_j=\langle\mu_{\alpha_j}, \eta_{\alpha_j}\rangle (j=1, 2, 3, \dots, n)$ be a collection of fuzzy Pythagorean values, then PFHA aggregation operator can be defined as

$$\mathrm{PFHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)=\left(\sqrt{1-\prod_{j=1}^n(1-\mu_{\dot\alpha_{\sigma(j)}}^2)^{w_j}}, \prod_{j=1}^n(\eta_{\dot\alpha_{\sigma(j)}})^{w_j}\right), \tag{4}$$

where $\dot\alpha_{\sigma(j)}$ is the j^{th} largest of the weighted Pythagorean fuzzy values $\dot\alpha_j(\dot\alpha_j=n\omega_j\alpha_j)$, $w=(w_1, w_2, w_3, \dots, w_n)^T$ is the weighted vector of the PFHA operator, such that $w_j\in[0, 1]$ and $\sum_{j=1}^n w_j=1$. $\omega=(\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ is the weighted vector of $\alpha_j (j=1, 2, 3, \dots, n)$, such that $\omega_j\in[0, 1]$, $\sum_{j=1}^n\omega_j=1$, and n is the balancing coefficient, which plays a role of balance. If the vector $w=(w_1, w_2, w_3, \dots, w_n)^T$ approaches to $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then the vector $(n\omega_1\alpha_1, n\omega_2\alpha_2, n\omega_3\alpha_3, \dots, n\omega_n\alpha_n)^T$ approaches to $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)^T$.

Definition 7 ([2]). Let $\alpha_j=\langle\mu_{\alpha_j}, \eta_{\alpha_j}\rangle (j=1, 2, 3, \dots, n)$ be a collection of fuzzy Pythagorean values, then the PFEWA aggregation operator can be defined as

$$\mathrm{PFEWA}_w(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \left(\frac{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\alpha_j}^2\right)^{w_j} - \prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\alpha_j}^2\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^2\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(\eta_{\alpha_j}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-\eta_{\alpha_j}^2\right)^{w_j} + \prod_{j=1}^{n}\left(\eta_{\alpha_j}^2\right)^{w_j}}} \right), \tag{5}$$

where $w=(w_1, w_2, w_3, \ldots, w_n)^T$ is the weighted vector of $\alpha_j (j=1, 2, 3, \ldots, n)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 8 ([2]). Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j} \rangle (j=1, 2, 3, \ldots, n)$ be a collection of fuzzy Pythagorean values, then the PFEOWA aggregation operator can be defined as

$$\mathrm{PFEOWA}_w(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \left(\frac{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\alpha_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\alpha_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\alpha_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(\eta_{\alpha_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-\eta_{\alpha_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{n}\left(\eta_{\alpha_{\sigma(j)}}^2\right)^{w_j}}} \right), \tag{6}$$

where $w=(w_1, w_2, w_3, \ldots, w_n)^T$ is the weighted vector of $\alpha_{\sigma(j)} (j=1, 2, 3, \ldots, n)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Also, $(\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\alpha_{\sigma(j)} \le \alpha_{\sigma(j-1)}$ for all j.

3 PFEHA Aggregation Operator

In this section, we introduce the concept of PFEHA aggregation operator along with some of its basic properties, such as idempotency, boundedness, and monotonicity.

Definition 9. The PFEHA aggregation operator can be defined as follows:

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \left(\frac{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}} \right), \tag{7}$$

where $\dot{\alpha}_{\sigma(j)}$ is the j^{th} largest of the weighted Pythagorean fuzzy values $\dot{\alpha}_j (\dot{\alpha}_j = n\omega_j \alpha_j)$, $w=(w_1, w_2, w_3, \ldots, w_n)^T$ is the weighted vector of the PFEHA operator such that $w_j \in [0, 1]$, and $\sum_{j=1}^{n} w_j = 1$. $\omega=(\omega_1, \omega_2, \omega_3, \ldots, \omega_n)^T$ is the weighted vector of $\alpha_j (j=1, 2, 3, \ldots, n)$ such that $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient, which plays a role of balance. If the vector $w=(w_1, w_2, w_3, \ldots, w_n)^T$ approaches to $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the vector $(n\omega_1\alpha_1, n\omega_2\alpha_2, n\omega_3\alpha_3, \ldots, n\omega_n\alpha_n)^T$ approaches to $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)^T$.

Theorem 1. *Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j} \rangle (j=1, 2, \ldots, n)$ be a collection of Pythagorean fuzzy values, then their aggregated value by using the PFEHA aggregation operator is also a Pythagorean fuzzy value, and*

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \left(\frac{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j} - \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j} + \prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^2\right)^{w_j}}} \right). \tag{8}$$

Proof. We can prove this theorem by mathematical induction on n.

For $n=2$

$$w_1\dot{\alpha}_1=\left(\frac{\sqrt{\left(1+\mu^2_{\dot{\alpha}_1}\right)^{w_1}-\left(1-\mu^2_{\dot{\alpha}_1}\right)^{w_1}}}{\sqrt{\left(1+\mu^2_{\dot{\alpha}_1}\right)^{w_1}+\left(1-\mu^2_{\dot{\alpha}_1}\right)^{w_1}}},\ \frac{\sqrt{2\left(\eta^2_{\dot{\alpha}_1}\right)^{w_1}}}{\sqrt{\left(2-\eta^2_{\dot{\alpha}_1}\right)^{w_1}+\left(\eta^2_{\dot{\alpha}_1}\right)^{w_1}}}\right)$$

and

$$w_2\dot{\alpha}_2=\left(\frac{\sqrt{\left(1+\mu^2_{\dot{\alpha}_2}\right)^{w_2}-\left(1-\mu^2_{\dot{\alpha}_2}\right)^{w_2}}}{\sqrt{\left(1+\mu^2_{\dot{\alpha}_2}\right)^{w_2}+\left(1-\mu^2_{\dot{\alpha}_2}\right)^{w_2}}},\ \frac{\sqrt{2\left(\eta^2_{\dot{\alpha}_2}\right)^{w_2}}}{\sqrt{\left(2-\eta^2_{\dot{\alpha}_2}\right)^{w_2}+\left(\eta^2_{\dot{\alpha}_2}\right)^{w_2}}}\right).$$

Then

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1,\alpha_2)=\left(\frac{\sqrt{\prod_{j=1}^{2}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{2}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}},\ \frac{\sqrt{2\prod_{j=1}^{2}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{2}\left(2-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{2}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}\right).$$

Thus, the result is true for $n=2$. Now, we assume that Eq. (8) holds for $n=k$. Thus

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_k)=\left(\frac{\sqrt{\prod_{j=1}^{k}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}},\ \frac{\sqrt{2\prod_{j=1}^{k}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}\right).$$

If Eq. (8) is true for $n=k$, then we show that Eq. (8) is true for $n=k+1$. Thus

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_{k+1})=\left(\frac{\sqrt{\prod_{j=1}^{k}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}},\ \frac{\sqrt{2\prod_{j=1}^{k}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}\right)$$
$$\oplus_{\varepsilon}\left(\frac{\sqrt{\left(1+\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}-\left(1-\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(1+\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}+\left(1-\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}}},\ \frac{\sqrt{2\left(\eta^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}}}{\sqrt{\left(2-\eta^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}+\left(\eta^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}}}\right). \tag{9}$$

Let

$$p_1=\sqrt{\prod_{j=1}^{k}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{k}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}},\ q_1=\sqrt{\prod_{j=1}^{k}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}$$

$$p_2=\sqrt{\left(1+\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}-\left(1-\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}},\ q_2=\sqrt{\left(1+\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}+\left(1-\mu^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}},\ r_1=\sqrt{2\prod_{j=1}^{k}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}$$

$$s_1=\sqrt{\prod_{j=1}^{k}\left(2-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{k}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}},\ s_2=\sqrt{\left(2-\eta^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}+\left(\eta^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}},\ r_2=\sqrt{2\left(\eta^2_{\dot{\alpha}_{k+1}}\right)^{w_{k+1}}}.$$

Now, putting these values in Eq. (9), we have

$$\text{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{k+1}) = \left(\frac{p_1}{q_1}, \frac{r_1}{s_1}\right) \oplus_\varepsilon \left(\frac{p_2}{q_2}, \frac{r_2}{s_2}\right).$$

By using the Einstein operation law, we have

$$\begin{aligned}\text{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{k+1}) &= \left(\frac{p_1}{q_1}, \frac{r_1}{s_1}\right) \oplus_\varepsilon \left(\frac{p_2}{q_2}, \frac{r_2}{s_2}\right) \\ &= \left(\frac{\sqrt{p_1^2 q_2^2 + p_2^2 q_1^2}}{\sqrt{q_1^2 q_2^2 + p_1^2 p_2^2}}, \frac{r_1 r_2}{\sqrt{2 s_1^2 s_2^2 - s_1^2 r_2^2 - r_1^2 s_2^2 + r_1^2 r_2^2}}\right).\end{aligned} \tag{10}$$

Now, putting the values of $p_1^2 q_2^2 + p_2^2 q_1^2$, $q_1^2 q_2^2 + p_1^2 p_2^2$, $r_1 r_2$, $2 s_1^2 s_2^2 - s_1^2 r_2^2 - r_1^2 s_2^2 + r_1^2 r_2^2$ in Eq. (10), then

$$\text{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{k+1}) = \left(\frac{\sqrt{\prod_{j=1}^{k+1}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\mu^2_{\dot{\alpha}_{\delta(j)}}\right)^{w_j} + \prod_{j=1}^{k+1}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{k+1}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{k+1}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}\right).$$

Thus, Eq. (8) is true for $n = k+1$. Thus, Eq. (8) is true for all n. □

Lemma 1 ([24, 27]). *Let* $\alpha_j \succ 0$, $w_j \succ 0$ $(j = 0, 2, \ldots n)$ *and* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\prod_{j=1}^{n} (\alpha_j)^{w_j} \le \sum_{j=1}^{n} w_j \alpha_j, \tag{11}$$

where the equality holds if and only if $\alpha_1 = \alpha_2 = \ldots = \alpha_n$.

Theorem 2. *Let* $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j} \rangle (j = 1, 2, 3, \ldots, n)$ *be a collection of Pythagorean fuzzy values, then*

$$\text{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \le \text{PFHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n). \tag{12}$$

Proof. As

$$\sqrt{\prod_{j=1}^{n}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}} \le \sqrt{\sum_{j=1}^{n} w_j\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right) + \sum_{j=1}^{n} w_j\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)}.$$

Also

$$\sqrt{\sum_{j=1}^{n} w_j\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right) + \sum_{j=1}^{n} w_j\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)} = \sqrt{2},$$

then

$$\sqrt{\prod_{j=1}^{n}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}} \le \sqrt{2},$$

thus

$$\frac{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\delta(j)}}^{2}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\delta(j)}}^{2}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}}=\sqrt{1-\frac{2\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}{\prod_{j=1}^{n}\left(1+\mu_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}+\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}}\leq\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}, \tag{13}$$

where the quality holds if and only if $\mu_{\dot{\alpha}_{\sigma(j)}}$ $(j=1, 2, 3, \ldots, n)$ are equal. Again

$$\sqrt{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}+\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}\leq\sqrt{\sum_{j=1}^{n}w_j\left(2-\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)+\sum_{j=1}^{n}w_j\left(\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)}.$$

Also

$$\sqrt{\sum_{j=1}^{n}w_j\left(2-\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)+\sum_{j=1}^{n}w_j\left(\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)}=\sqrt{2}$$

then

$$\sqrt{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}+\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}\leq\sqrt{2},$$

thus,

$$\frac{\sqrt{2\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(2-\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}+\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}^{2}\right)^{w_j}}}\geq\prod_{j=1}^{n}\left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}, \tag{14}$$

where the quality holds if and only if $\eta_{\dot{\alpha}_{\sigma(j)}}^{2}$ $(j=1, 2, 3, \ldots, n)$ are equal.

Let

$$\mathrm{PFHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)=\alpha \tag{15}$$

and

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)=\alpha^{\varepsilon}. \tag{16}$$

Then, Eqs. (13) and (14) can be transformed into the following forms:

$$\mu_{\dot{\alpha}}\geq\mu_{\dot{\alpha}^{\varepsilon}},\ \eta_{\dot{\alpha}}\leq\eta_{\dot{\alpha}^{\varepsilon}}, \tag{17}$$

thus

$$s(\alpha)\geq s(\alpha^{\varepsilon}). \tag{18}$$

If

$$s(\alpha)\succ s(\alpha^{\varepsilon}), \tag{19}$$

then

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \prec \mathrm{PFHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n). \tag{20}$$

If

$$s(\alpha) = s(\alpha^{\varepsilon}), \tag{21}$$

then

$$h(\alpha) = h(\alpha^{\varepsilon}), \tag{22}$$

thus

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \mathrm{PFHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n). \tag{23}$$

From Eqs. (20) to (23), Eq. (12) always holds. □

Example 1: Let

$$\alpha_1 = (0.4, 0.7), \alpha_2 = (0.5, 0.8),$$
$$\alpha_3 = (0.6, 0.7), \alpha_4 = (0.7, 0.6),$$

and $w = (0.1, 0.2, 0.3, 0.4)^T$, then

$$\dot{\alpha}_1 = (0.259, 0.867), \dot{\alpha}_2 = (0.456, 0.836),$$
$$\dot{\alpha}_3 = (0.643, 0.651), \dot{\alpha}_4 = (0.812, 0.441).$$

By calculating the scores function, we have

$$s(\dot{\alpha}_1) = -0.684, s(\dot{\alpha}_2) = -0.491,$$
$$s(\dot{\alpha}_3) = -0.010, s(\dot{\alpha}_4) = 0.465.$$

Hence,

$$s(\dot{\alpha}_4) \succ s(\dot{\alpha}_3) \succ s(\dot{\alpha}_2) \succ s(\dot{\alpha}_1).$$

Thus

$$\begin{aligned}\mathrm{PFHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= \left(\sqrt{1 - \prod_{j=1}^{4} \left(1 - \mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}, \prod_{j=1}^{4} \left(\eta_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} \right) \\ &= (0.517, 0.717).\end{aligned}$$

Now applying the PFEHA operator, we have

$$\dot{\alpha}_1 = (0.253, 0.882), \dot{\alpha}_2 = (0.448, 0.841),$$
$$\dot{\alpha}_3 = (0.650, 0.641), \dot{\alpha}_4 = (0.833, 0.402).$$

By calculating the scores function, we have

$$s(\dot{\alpha}_1) = -0.711, s(\dot{\alpha}_2) = -0.505,$$
$$s(\dot{\alpha}_3) = 0.012, s(\dot{\alpha}_4) = 0.532.$$

As

$$s(\dot{\alpha}_4) \succ s(\dot{\alpha}_3) \succ s(\dot{\alpha}_2) \succ s(\dot{\alpha}_1),$$

thus

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{\sqrt{\prod_{j=1}^{4}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} - \prod_{j=1}^{4}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(1+\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{4}\left(1-\mu^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}, \frac{\sqrt{2\prod_{j=1}^{4}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{4}\left(2-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j} + \prod_{j=1}^{4}\left(\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}} \right).$$

$$= (0.507,\ 0.742)$$

Theorem 3. *Let $\alpha_j = \langle \mu_{\alpha_j}, \eta_{\alpha_j} \rangle (j = 1, 2, 3, \ldots, n)$ be a collection of Pythagorean fuzzy values, then the following properties hold:*

1. ***Idempotency:*** *If $\dot{\alpha}_{\sigma(j)} = \dot{\alpha}$, then*

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \dot{\alpha}. \tag{24}$$

2. ***Boundedness:***

$$\dot{\alpha}_{\min} \le \mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \le \dot{\alpha}_{\max}, \tag{25}$$

where

$$\dot{\alpha}_{\min} = \left(\min_j \mu_{\dot{\alpha}_{\sigma(j)}},\ \max_j \eta_{\dot{\alpha}_{\sigma(j)}} \right), \tag{26}$$

$$\dot{\alpha}_{\max} = \left(\max_j \mu_{\dot{\alpha}_{\sigma(j)}},\ \min_j \eta_{\dot{\alpha}_{\sigma(j)}} \right). \tag{27}$$

3. ***Monotonicity:*** *Let $\alpha^*_{\sigma(j)} = \langle \mu^*_{\alpha_{\sigma(j)}}, \eta^*_{\alpha_{\sigma(j)}} \rangle (j = 1, 2, \ldots, n)$ be a collection of Pythagorean fuzzy values, and $\mu_{\alpha_{\sigma(j)}} \le \mu^*_{\alpha_{\sigma(j)}}, \eta_{\alpha_{\sigma(j)}} \ge \eta^*_{\alpha_{\sigma(j)}}$, for all j, then*

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \le \mathrm{PFEHA}_{\omega,w}(\alpha^*_1, \alpha^*_2, \alpha^*_3, \ldots, \alpha^*_n). \tag{28}$$

Proof. **Idempotency:** As

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \left(\frac{\sqrt{(1+\mu^2_{\dot{\alpha}})^{\sum_{j=1}^{n} w_j} - (1-\mu^2_{\dot{\alpha}})^{\sum_{j=1}^{n} w_j}}}{\sqrt{(1+\mu^2_{\dot{\alpha}})^{\sum_{j=1}^{n} w_j} + (1-\mu^2_{\dot{\alpha}})^{\sum_{j=1}^{n} w_j}}}, \frac{\sqrt{2(\eta^2_{\dot{\alpha}})^{\sum_{j=1}^{n} w_j}}}{\sqrt{(2-\eta^2_{\dot{\alpha}})^{\sum_{j=1}^{n} w_j} + \eta(\nu^2_{\dot{\alpha}})^{\sum_{j=1}^{n} w_j}}} \right).$$

$$= \left(\frac{\sqrt{(1+\mu^2_{\dot{\alpha}}) - (1-\mu^2_{\dot{\alpha}})}}{\sqrt{(1+\mu^2_{\dot{\alpha}}) + (1-\mu^2_{\dot{\alpha}})}}, \frac{\sqrt{2(\eta^2_{\dot{\alpha}})}}{\sqrt{(2-\eta^2_{\dot{\alpha}}) + \eta(\nu^2_{\dot{\alpha}})}} \right) = \dot{\alpha}$$

Boundedness: Let $f(x) = \sqrt{\frac{2-x^2}{x^2}},\ x \in (0, 1]$, then $f'(x) = \frac{-2}{x^3}\sqrt{\frac{x^2}{2-x^2}} \prec 0$, i.e. $f(x)$ is decreasing function on (0, 1]. As $\mu_{\dot{\alpha}_{\min}} \le \mu_{\dot{\alpha}_{\sigma(j)}} \le \mu_{\dot{\alpha}_{\max}}$, for all j, then $f(\mu_{\dot{\alpha}_{\max}}) \le f(\mu_{\dot{\alpha}_{\sigma(j)}}) \le f(\mu_{\dot{\alpha}_{\min}})$, that is $\sqrt{\frac{2-\mu^2_{\dot{\alpha}_{\max}}}{\mu^2_{\dot{\alpha}_{\max}}}} \le \sqrt{\frac{2-\mu^2_{\dot{\alpha}_{\sigma(j)}}}{\mu^2_{\dot{\alpha}_{\sigma(j)}}}} \le \sqrt{\frac{2-\mu^2_{\dot{\alpha}_{\min}}}{\mu^2_{\dot{\alpha}_{\min}}}}$, then

$$\begin{aligned}
&\Leftrightarrow \sqrt{\prod_{j=1}^{n}\left(\frac{2-\mu^2_{\dot{\alpha}_{\max}}}{\mu^2_{\dot{\alpha}_{\max}}}\right)^{w_j}} \le \sqrt{\prod_{j=1}^{n}\left(\frac{2-\mu^2_{\dot{\alpha}_{\sigma(j)}}}{\mu^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}} \le \sqrt{\prod_{j=1}^{n}\left(\frac{2-\mu^2_{\dot{\alpha}_{\min}}}{\mu^2_{\dot{\alpha}_{\min}}}\right)^{w_j}}\\
&\Leftrightarrow \sqrt{\left(\frac{2-\mu^2_{\dot{\alpha}_{\max}}}{\mu^2_{\dot{\alpha}_{\max}}}\right)^{\sum_{j=1}^{n} w_j}} \le \sqrt{\prod_{j=1}^{n}\left(\frac{2-\mu^2_{\dot{\alpha}_{\sigma(j)}}}{\mu^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}} \le \sqrt{\left(\frac{2-\mu^2_{\dot{\alpha}_{\min}}}{\mu^2_{\dot{\alpha}_{\min}}}\right)^{\sum_{j=1}^{n} w_j}}\\
&\Leftrightarrow \sqrt{\left(\frac{2-\mu^2_{\dot{\alpha}_{\max}}}{\mu^2_{\dot{\alpha}_{\max}}}\right)+1} \le \sqrt{\prod_{j=1}^{n}\left(\frac{2-\mu^2_{\dot{\alpha}_{\sigma(j)}}}{\mu^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}+1} \le \sqrt{\left(\frac{2-\mu^2_{\dot{\alpha}_{\min}}}{\mu^2_{\dot{\alpha}_{\min}}}\right)+1}\\
&\Leftrightarrow \frac{\sqrt{\mu^2_{\dot{\alpha}_{\min}}}}{\sqrt{2}} \le \frac{1}{\sqrt{\prod_{j=1}^{n}\left(\frac{2-\mu^2_{\dot{\alpha}_{\sigma(j)}}}{\mu^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}+1}} \le \frac{\sqrt{\mu^2_{\dot{\alpha}_{\max}}}}{\sqrt{2}}\\
&\Leftrightarrow \mu_{\dot{\alpha}_{\min}} \le \frac{\sqrt{2\prod_{j=1}^{n}(\mu^2_{\dot{\alpha}_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^{n}(2-\mu^2_{\dot{\alpha}_{\sigma(j)}})^{w_j}+\prod_{j=1}^{n}(\mu^2_{\dot{\alpha}_{\sigma(j)}})^{w_j}}} \le \mu_{\dot{\alpha}_{\max}}.
\end{aligned} \tag{29}$$

Again, let $g(y)=\sqrt{\frac{1-y^2}{1+y^2}}$, $y\in[0,\ 1]$, then $g'(y)=\frac{-2y}{(1+y^2)^2}\sqrt{\frac{1+y^2}{1-y^2}}\prec 0$, i.e. $g(y)$ is a decreasing function on $[0, 1]$. As $\eta_{\dot{\alpha}_{\min}}\le\eta_{\dot{\alpha}_{\sigma(j)}}\le\eta_{\dot{\alpha}_{\max}}$ for all j, then $g(\eta_{\dot{\alpha}_{\max}})\le g(\eta_{\dot{\alpha}_{\sigma(j)}})\le g(\eta_{\dot{\alpha}_{\min}})$ for all j, that is $\sqrt{\frac{1-\eta^2_{\dot{\alpha}_{\max}}}{1+\eta^2_{\dot{\alpha}_{\max}}}}\le\sqrt{\frac{1-\eta^2_{\dot{\alpha}_{\sigma(j)}}}{1+\eta^2_{\dot{\alpha}_{\sigma(j)}}}}\le\sqrt{\frac{1-\eta^2_{\dot{\alpha}_{\min}}}{1+\eta^2_{\dot{\alpha}_{\min}}}}$, then

$$\begin{aligned}
&\Leftrightarrow \sqrt{\left(\frac{1-\eta^2_{\dot{\alpha}_{\max}}}{1+\eta^2_{\dot{\alpha}_{\max}}}\right)^{w_j}} \le \sqrt{\left(\frac{1-\eta^2_{\dot{\alpha}_{\sigma(j)}}}{1+\eta^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}} \le \sqrt{\left(\frac{1-\eta^2_{\dot{\alpha}_{\min}}}{1+\eta^2_{\dot{\alpha}_{\min}}}\right)^{w_j}}\\
&\Leftrightarrow \sqrt{\prod_{j=1}^{n}\left(\frac{1-\eta^2_{\dot{\alpha}_{\max}}}{1+\eta^2_{\dot{\alpha}_{\max}}}\right)^{w_j}} \le \sqrt{\prod_{j=1}^{n}\left(\frac{1-\eta^2_{\dot{\alpha}_{\sigma(j)}}}{1+\eta^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}} \le \sqrt{\prod_{j=1}^{n}\left(\frac{1-\eta^2_{\dot{\alpha}_{\min}}}{1+\eta^2_{\dot{\alpha}_{\min}}}\right)^{w_j}}\\
&\Leftrightarrow \sqrt{\left(\frac{1-\eta^2_{\dot{\alpha}_{\max}}}{1+\eta^2_{\dot{\alpha}_{\max}}}\right)^{\sum_{j=1}^{n} w_j}} \le \sqrt{\prod_{j=1}^{n}\left(\frac{1-\eta^2_{\dot{\alpha}_{\sigma(j)}}}{1+\eta^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}} \le \sqrt{\left(\frac{1-\eta^2_{\dot{\alpha}_{\min}}}{1+\eta^2_{\dot{\alpha}_{\min}}}\right)^{\sum_{j=1}^{n} w_j}}\\
&\Leftrightarrow \sqrt{1+\eta^2_{\dot{\alpha}_{\min}}} \le \frac{\sqrt{2}}{\sqrt{\prod_{j=1}^{n}\left(\frac{1-\eta^2_{\dot{\alpha}_{\sigma(j)}}}{1+\eta^2_{\dot{\alpha}_{\sigma(j)}}}\right)^{w_j}+1}} \le \sqrt{1+\eta^2_{\dot{\alpha}_{\max}}}\\
&\Leftrightarrow \eta_{\dot{\alpha}_{\min}} \le \frac{\sqrt{\prod_{j=1}^{n}\left(1+\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}-\prod_{j=1}^{n}\left(1-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}}{\sqrt{\prod_{j=1}^{n}\left(1-\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}+\prod_{j=1}^{n}\left(1+\eta^2_{\dot{\alpha}_{\sigma(j)}}\right)^{w_j}}} \le \eta_{\dot{\alpha}_{\max}}.
\end{aligned} \tag{30}$$

Let

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1,\alpha_2,\alpha_3,\ \ldots,\alpha_n)=\dot{\alpha}=(\mu_{\dot{\alpha}},\eta_{\dot{\alpha}}). \tag{31}$$

Then, Eqs. (29) and (30) can be written as

$$\mu_{\dot{\alpha}_{\min}} \leq \mu_{\dot{\alpha}_{\sigma(j)}} \leq \mu_{\dot{\alpha}_{\min}} \tag{32}$$

and

$$\eta_{\dot{\alpha}_{\min}} \leq \eta_{\dot{\alpha}_{\sigma(j)}} \leq \eta_{\dot{\alpha}_{\max}}, \tag{33}$$

thus

$$s(\dot{\alpha}) \leq s(\dot{\alpha}_{\max}) \tag{34}$$

and

$$s(\dot{\alpha}) \geq s(\dot{\alpha}_{\min}). \tag{35}$$

If

$$s(\dot{\alpha}) \prec s(\dot{\alpha}_{\max}) \tag{36}$$

and

$$s(\dot{\alpha}) \succ s(\dot{\alpha}_{\min}), \tag{37}$$

then

$$\dot{\alpha}_{\min} \prec \mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \prec \dot{\alpha}_{\max.} \tag{38}$$

If

$$s(\dot{\alpha}) = s(\dot{\alpha}_{\max}), \tag{39}$$

then

$$h(\dot{\alpha}) = h(\dot{\alpha}_{\max}). \tag{40}$$

Thus

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \dot{\alpha}_{\max}. \tag{41}$$

If

$$s(\dot{\alpha}) = s(\dot{\alpha}_{\min}), \tag{42}$$

then

$$h(\dot{\alpha}) = h(\dot{\alpha}_{\min}). \tag{43}$$

Thus

$$\mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \dot{\alpha}_{\min}. \tag{44}$$

Thus, from Eqs. (38) to (44), we have

$$\dot{\alpha}_{\min} \leq \mathrm{PFEHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \leq \dot{\alpha}_{\max}.$$

Monotonicity: Proof is similar to 2, so it is omitted here. □

Theorem 4. *The PFEWA operator is a special case of the PFEHA operator.*

Theorem 5. *The PFEOWA operator is a special case of the PFEHA operator.*

4 An Application of the PFEHA Aggregation Operator to Multiple-Attribute Group Decision Making

In this section, we investigate an application of the PFEHA aggregation operators to multiple-attribute group decision making with Pythagorean fuzzy information.

Algorithm: Let $G=\{G_1, G_2, G_3, \dots, G_m\}$ be the set of m alternatives, $A=\{A_1, A_2, A_3, \dots, A_n\}$ be the set of n attributes, and $D=\{D_1, D_2, D_3, \dots, D_k\}$ be the set of k decision makers. Let $\omega=(\omega_1, \omega_2, \omega_3, \dots, \omega_n)^T$ be the weighted vector of the attributes $G_i(i=1, 2, 3, \dots, m)$, such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n}\omega_j=1$. Let $w=(w_1, w_2, w_3, \dots, w_k)^T$ be the weighted vector of the decision makers D^s $(s=1, 2, 3, \dots, k)$, such that $w_s \in [0, 1]$ and $\sum_{s=1}^{k} w_s=1$.

Step 1: Construct the decision-making matrices, $D^s=[\alpha_{ij}^{(s)}]_{m\times n}$, for decision. If the criteria have two types, such as benefit criteria and cost criteria, then decision matrices $D^s=[\alpha_{ij}^{(s)}]_{m\times n}$ can be converted into the decision matrices $R^s=[r_{ij}^{(s)}]_{m\times n}$, where

$$r_{ij}^s=\begin{cases}\alpha_{ij}^s, \text{ for benefit criteria } A_j \\ \bar{a}_{ij}^s, \text{ for cost criteria } A_j,\end{cases}\begin{pmatrix} j=1, 2, \dots, n \\ i=1, 2, \dots, m\end{pmatrix},$$

and $\bar{a}_{ji}^s$ is the complement of α_{ji}^s. If all the criteria have the same type, then there is no need of normalization.

Step 2: Utilize the PFEWA aggregation operators to aggregate all the individual normalized decision matrices $R^s=[r_{ij}^{(s)}]_{m\times n}$ into a single Pythagorean fuzzy decision matrix $R=[r_{ij}]_{m\times n}$, where $r_{ij}=(\mu_{ij}, \eta_{ij})(i=1, 2, \dots, m, j=1, 2, \dots, n)$.

Step 3: Utilize $\dot{\alpha}_{ij}=nw_j\alpha_{ij}$ to derive the overall preference values.

Step 4: Utilize the PFEHA aggregation operators to derive the overall preference values.

Step 5: Calculate the scores of r_j $(i=1, 2, 3, \dots, m)$. If there is no difference between two or more than two scores, then we have to find out the accuracy degrees of the collective overall preference values.

Step 6: Arrange the scores of all alternatives in descending order and select the alternative with the highest score function.

5 Numerical Example

Suppose a company wants to invest its money in the following best option: G_1, car company; G_2, food company; G_3, computer company; G_4, TV company; and G_5, fan company. The company must take a decision according to the following four attributes, whose weighted vector is $\omega=(0.4, 0.3, 0.2, 0.1)^T$. Here, A_1: risk analysis, A_2: growth analysis, A_3: social political impact analysis, and A_4: environmental analysis, where A_1, A_3 are cost-type criteria and A_2, A_4 are benefit-type criteria. There are four experts, D^s $(s=1, 2, 3, 4)$, from a group to act as decision makers, whose weight vector is $w=(0.1, 0.2, 0.3, 0.4)^T$.

Step 1: Construct the decision-making matrices (Tables 1–4).

Step 2: Construct the normalized decision-making matrices (Tables 5–8).

Step 3: Utilize the PFEWA operator, we have Table 9.

Step 4: Utilize $\dot{\alpha}_{ij}=nw_j\alpha_{ij}$, we have

Table 1: Pythagorean Fuzzy Decision Matrix D^1.

	A_1	A_2	A_3	A_4
G_1	(0.8, 0.5)	(0.7, 0.4)	(0.7, 0.4)	(0.7, 0.5)
G_2	(0.8, 0.4)	(0.7, 0.5)	(0.8, 0.5)	(0.8, 0.3)
G_3	(0.5, 0.6)	(0.6, 0.5)	(0.7, 0.5)	(0.8, 0.3)
G_4	(0.6, 0.5)	(0.6, 0.4)	(0.6, 0.4)	(0.8, 0.4)
G_5	(0.6, 0.8)	(0.6, 0.6)	(0.7, 0.3)	(0.6, 0.5)

Table 2: Pythagorean Fuzzy Decision Matrix D^2.

	A_1	A_2	A_3	A_4
G_1	(0.6, 0.5)	(0.8, 0.4)	(0.6, 0.4)	(0.6, 0.5)
G_2	(0.7, 0.3)	(0.8, 0.4)	(0.7, 0.5)	(0.7, 0.4)
G_3	(0.6, 0.6)	(0.6, 0.5)	(0.6, 0.6)	(0.7, 0.4)
G_4	(0.7, 0.5)	(0.6, 0.6)	(0.7, 0.4)	(0.8, 0.5)
G_5	(0.6, 0.4)	(0.7, 0.2)	(0.8, 0.4)	(0.8, 0.4)

Table 3: Pythagorean Fuzzy Decision Matrix D^3.

	A_1	A_2	A_3	A_4
G_1	(0.7, 0.5)	(0.7, 0.4)	(0.6, 0.5)	(0.6, 0.5)
G_2	(0.8, 0.3)	(0.7, 0.3)	(0.8, 0.3)	(0.9, 0.2)
G_3	(0.6, 0.5)	(0.6, 0.6)	(0.7, 0.4)	(0.8, 0.3)
G_4	(0.7, 0.5)	(0.8, 0.5)	(0.9, 0.1)	(0.6, 0.5)
G_5	(0.7, 0.5)	(0.8, 0.2)	(0.8, 0.2)	(0.7, 0.3)

Table 4: Pythagorean Fuzzy Decision Matrix D^4.

	A_1	A_2	A_3	A_4
G_1	(0.8, 0.3)	(0.8, 0.4)	(0.7, 0.4)	(0.7, 0.5)
G_2	(0.8, 0.3)	(0.8, 0.3)	(0.8, 0.3)	(0.8, 0.2)
G_3	(0.6, 0.6)	(0.7, 0.6)	(0.7, 0.4)	(0.8, 0.3)
G_4	(0.7, 0.4)	(0.8, 0.6)	(0.8, 0.2)	(0.7, 0.5)
G_5	(0.6, 0.6)	(0.8, 0.2)	(0.8, 0.2)	(0.8, 0.3)

Table 5: Normalized Decision Matrix R^1.

	A_1	A_2	A_3	A_4
G_1	(0.5, 0.8)	(0.7, 0.4)	(0.4, 0.7)	(0.7, 0.5)
G_2	(0.4, 0.8)	(0.7, 0.5)	(0.5, 0.8)	(0.8, 0.3)
G_3	(0.6, 0.5)	(0.6, 0.5)	(0.5, 0.7)	(0.8, 0.3)
G_4	(0.5, 0.6)	(0.6, 0.4)	(0.4, 0.6)	(0.8, 0.4)
G_5	(0.8, 0.6)	(0.6, 0.6)	(0.3, 0.7)	(0.6, 0.5)

Table 6: Normalized Decision Matrix R^2.

	A_1	A_2	A_3	A_4
G_1	(0.5, 0.6)	(0.8, 0.4)	(0.4, 0.6)	(0.6, 0.5)
G_2	(0.3, 0.7)	(0.8, 0.4)	(0.5, 0.7)	(0.7, 0.4)
G_3	(0.6, 0.6)	(0.6, 0.5)	(0.6, 0.6)	(0.7, 0.4)
G_4	(0.5, 0.7)	(0.6, 0.6)	(0.4, 0.7)	(0.8, 0.5)
G_5	(0.4, 0.6)	(0.7, 0.2)	(0.4, 0.8)	(0.8, 0.4)

Table 7: Normalized Decision Matrix R^3.

	A_1	A_2	A_3	A_4
G_1	(0.5, 0.7)	(0.7, 0.4)	(0.5, 0.6)	(0.6, 0.5)
G_2	(0.3, 0.8)	(0.7, 0.3)	(0.3, 0.8)	(0.9, 0.2)
G_3	(0.5, 0.6)	(0.6, 0.6)	(0.4, 0.7)	(0.8, 0.3)
G_4	(0.5, 0.7)	(0.8, 0.5)	(0.1, 0.9)	(0.6, 0.5)
G_5	(0.5, 0.7)	(0.8, 0.2)	(0.2, 0.8)	(0.7, 0.3)

Table 8: Normalized Decision Matrix R^4.

	A_1	A_2	A_3	A_4
G_1	(0.3, 0.8)	(0.8, 0.4)	(0.4, 0.7)	(0.7, 0.5)
G_2	(0.3, 0.8)	(0.8, 0.3)	(0.3, 0.8)	(0.8, 0.2)
G_3	(0.6, 0.6)	(0.7, 0.6)	(0.4, 0.7)	(0.8, 0.3)
G_4	(0.4, 0.7)	(0.8, 0.6)	(0.2, 0.8)	(0.7, 0.5)
G_5	(0.6, 0.6)	(0.8, 0.2)	(0.2, 0.8)	(0.8, 0.3)

Table 9: Collective Pythagorean Fuzzy Decision Matrix *R*.

	A_1	A_2	A_3	A_4
G_1	(0.432, 0.728)	(0.764, 0.400)	(0.432, 0.649)	(0.653, 0.500)
G_2	(0.311, 0.779)	(0.764, 0.335)	(0.372, 0.779)	(0.823, 0.239)
G_3	(0.572, 0.589)	(0.643, 0.568)	(0.459, 0.679)	(0.782, 0.317)
G_4	(0.463, 0.689)	(0.753, 0.546)	(0.259, 0.789)	(0.684, 0.489)
G_5	(0.568, 0.629)	(0.767, 0.224)	(0.263, 0.789)	(0.757, 0.335)

$$\dot{\alpha}_{11} = (0.542,\ 0.572),\ \dot{\alpha}_{12} = (0.815,\ 0.318),\ \dot{\alpha}_{13} = (0.387,\ 0.718),\ \dot{\alpha}_{14} = (0.424,\ 0.793)$$
$$\dot{\alpha}_{21} = (0.393,\ 0.648),\ \dot{\alpha}_{22} = (0.815,\ 0.318),\ \dot{\alpha}_{23} = (0.333,\ 0.824),\ \dot{\alpha}_{24} = (0.564,\ 0.627)$$
$$\dot{\alpha}_{31} = (0.704,\ 0.390),\ \dot{\alpha}_{32} = (0.695,\ 0.485),\ \dot{\alpha}_{33} = (0.411,\ 0.793),\ \dot{\alpha}_{34} = (0.527,\ 0.687)$$
$$\dot{\alpha}_{41} = (0.578,\ 0.518),\ \dot{\alpha}_{42} = (0.805,\ 0.470),\ \dot{\alpha}_{43} = (0.232,\ 0.830),\ \dot{\alpha}_{44} = (0.453,\ 0.787)$$
$$\dot{\alpha}_{51} = (0.700,\ 0.439),\ \dot{\alpha}_{52} = (0.818,\ 0.156),\ \dot{\alpha}_{53} = (0.249,\ 0.832),\ \dot{\alpha}_{54} = (0.505,\ 0.699).$$

By calculating the score functions, we have Table 10.

Step 5: Utilize the PFEHA aggregation operator, we have

$$r_1 = (0.65,\ 0.49),\ r_2 = (0.65,\ 0.50),\ r_3 = (0.64,\ 0.50),\ r_4 = (0.65,\ 0.57),\ r_5 = (0.70,\ 0.35).$$

Table 10: Pythagorean Fuzzy Hybrid Decision Matrix.

	A_1	A_2	A_3	A_4
G_1	(0.815, 0.318)	(0.542, 0.572)	(0.387, 0.718)	(0.424, 0.793)
G_2	(0.815, 0.318)	(0.564, 0.627)	(0.393, 0.648)	(0.333, 0.824)
G_3	(0.704, 0.390)	(0.695, 0.485)	(0.527, 0.687)	(0.411, 0.793)
G_4	(0.805, 0.470)	(0.578, 0.518)	(0.453, 0.787)	(0.232, 0.830)
G_5	(0.818, 0.156)	(0.700, 0.439)	(0.505, 0.699)	(0.249, 0.832)

Table 11: Comparisons with Previous Operators.

Operators	Score functions	Ranking
PFEWA operator	$s(r_5)\succ s(r_2)\succ s(r_1)\succ s(r_3)\succ s(r_4)$	$5\succ 2\succ 1\succ 3\succ 4$
PFEOWA operator	$s(r_5)\succ s(r_2)\succ s(r_3)\succ s(r_1)\succ s(r_4)$	$5\succ 2\succ 3\succ 1\succ 4$
PFEHA operator	$s(r_5)\succ s(r_1)\succ s(r_2)\succ s(r_3)\succ s(r_4)$	$5\succ 1\succ 2\succ 3\succ 4$

Now we calculate the scores of $s(r_i)$ $(i=1, 2, 3, 4, 5)$, we have

$$s(r_1)=0.18,\ s(r_2)=0.17,\ s(r_3)=0.16,\ s(r_4)=0.09,\ s(r_5)=0.37.$$

Step 6: Arrange the scores in descending order, we have G_5 is the best option (Table 11).

6 Comparison with Other Methods

In order to verify the effectiveness of the proposed method, we can compare the proposed method with other methods. First, we compare the proposed method with the method proposed by Rahman et al. [17]. The aggregation operator proposed by Rahman et al. [17] is based on algebraic operations, and that in this paper is based on Einstein operations. Obviously, the operator or method proposed in this paper is more general, more accurate, and more flexible. The Einstein operators proposed by Garg [2] are only the special cases of the proposed operator in this paper. The methods or operators proposed by Garg [2] are PFEWA aggregation operator, which weights only the Pythagorean fuzzy arguments, and PFEOWA aggregation operator, which weights only the ordered positions of the Pythagorean fuzzy arguments instead of weighting the Pythagorean fuzzy arguments themselves. To overcome these limitations in this paper, we have developed the notion of PFEHA aggregation operator, which weights both the given Pythagorean fuzzy value and its ordered position.

7 Conclusion

The objective of this paper is to present the PFEHA aggregation operator based on Pythagorean fuzzy numbers and to apply it to the multi-attribute group decision-making problems where the attribute values are Pythagorean fuzzy numbers. First, we have developed the PFEHA aggregation operator along with its properties. Furthermore, we have developed a method for multi-criteria group decision making based on this operator, and the operational processes have been illustrated in detail. An illustrative example of selecting the best company to invest money has been considered for demonstrating the approach. The suggested methodology can be used for any type of selection problem involving any number of selection attributes. We ended the paper with an application of the new approach in a group decision-making problem.

In further research, it is necessary and meaningful to give the applications of this operator to the other domains, such as induction, interval numbers, fuzzy numbers, linguistic variables, pattern recognition, fuzzy cluster analysis, uncertain programming, etc.

Bibliography

[1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.* **20** (1986), 87–96.
[2] H. Garg, A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making, *Int. J. Intell. Syst.* **31** (2016), 1–35.
[3] H. Garg, Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process, *Int. J. Intell. Syst.* **32** (2016), 1–34.
[4] H. Garg, Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process, *Comput. Math. Organ. Theory* **23** (2017), 546–571.
[5] X. Gou, Z. Xu and P. Ren, The properties of continuous Pythagorean fuzzy information, *Int. J. Intell. Syst.* **31** (2016), 401–424.
[6] D. Liang and Z. Xu, Projection model for fusing information in Pythagorean fuzzy multi-criteria group decision making based on geometric Bonferroni mean, *Int. J. Intell. Syst.* **9** (2017), 966–987.
[7] D. C. Liang, Y. R. J. Zhang, Z. S. Xu and A. P. Darko, Pythagorean fuzzy Bonferroni mean aggregation operator and its accelerative calculating algorithm with the multithreading, *Int. J. Intell. Syst.* **33** (2018), 615–633.
[8] H. Liao and Z. Xu, Intuitionistic fuzzy hybrid weighted aggregation operators, *Int. J. Intell. Syst.* **29** (2014), 971–993.
[9] Z. M. Ma and Z. Xu, Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multi-criteria decision making problems, *Int. J. Intell. Syst.* **31** (2016), 1198–1219.
[10] K. Rahman, S. Abdullah, F. Husain and M. S. Ali Khan, Approaches to Pythagorean fuzzy geometric aggregation operators, *Int. J. Comput. Sci. Inform. Security* **14** (2016), 174–200.
[11] K. Rahman, S. Abdullah, M. S. Ali Khan and M. Shakeel, Pythagorean fuzzy hybrid geometric aggregation operator and their applications to multiple attribute decision making, *Int. J. Comput. Sci. Inform. Security* **14** (2016), 837–854.
[12] K. Rahman, M. S. Ali Khan, M. Ullah and A. Fahmi, Multiple attribute group decision making for plant location selection with Pythagorean fuzzy weighted geometric aggregation operator, *Nucleus* **1** (2017), 66–74.
[13] K. Rahman, M. S. Ali Khan and M. Ullah, New approaches to Pythagorean fuzzy averaging aggregation operators, *Math. Lett.* **3** (2017), 29–36.
[14] K. Rahman, S. Abdullah, A. Ali and F. Amin, Some induced averaging aggregation operators based on Pythagorean fuzzy numbers, *Math. Lett.* **3** (2017), 40–45.
[15] K. Rahman, S. Abdullah, F. Husain M. S. Ali Khan and M. Shakeel, Pythagorean fuzzy ordered weighted geometric aggregation operator and their application to multiple attribute group decision making, *J. Appl. Environ. Biol. Sci.* **7** (2017), 67–83.
[16] K. Rahman, S. Abdullah, M. S. Ali Khan, M. Ibrar and F. Husain, Some basic operations on Pythagorean fuzzy sets, *J. Appl. Environ. Biol. Sci.* **7** (2017), 111–119.
[17] K. Rahman, M. S. Ali Khan, S. Abdullah, F. Husain and M. Ibrar, Some properties of Pythagorean fuzzy hybrid averaging aggregation operator, *J. Appl. Environ. Biol. Sci.* **7** (2017), 122–133.
[18] K. Rahman, S. Abdullah, R. Ahmed and M. Ullah, Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making, *J. Intell. Fuzzy Syst.* **33** (2017), 635–647.
[19] K. Rahman, A. Ali, M. Shakeel, M. S. Ali Khan and M. Ullah, Pythagorean fuzzy weighted averaging aggregation operator and its application to decision making theory, *Nucleus* **54** (2017), 190–196.
[20] K. Rahman, S. Abdullah, M. Jamil and M. Y. Khan, Some generalized intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute group decision making, *Int. J. Fuzzy Syst.* (2018), 1–9. doi: 10.1007/s40815-018-0452–0.
[21] P. Ren, Z. Xu and X. Gou, Pythagorean fuzzy TODIM approach to multi-criteria decision making, *Appl. Soft Comput.* **42** (2016), 246–259.
[22] W. Wang and X. Liu, Intuitionistic fuzzy geometric aggregation operators based on Einstein operations, *Int. J. Intell. Syst.* **26** (2011), 1049–1075.
[23] W. Wang and X. Liu, Intuitionistic fuzzy information aggregation using Einstein operations, *IEEE Trans. Fuzzy Syst.* **20** (2012), 923–938.
[24] Z. Xu, On consistency of the weighted geometric mean complex judgement matrix in AHP, *Eur. J. Oper. Res.* **126** (2000), 683–687.
[25] Z. S. Xu, Intuitionistic fuzzy aggregation operators, *IEEE Trans. Fuzzy Syst.* **15** (2007), 1179–1187.
[26] Z. Xu, *Intuitionistic Fuzzy Aggregation and Clustering*, Springer, Berlin, 2013.
[27] Z. Xu and Q. L. Da, The ordered weighted geometric averaging operators, *Int. J. Intell. Syst.* **17** (2002), 709–716.
[28] Z. S. Xu and Q. L. Da, An overview of operators for aggregating information, *Int. J. Intell. Syst.* **18** (2003), 953–969.
[29] Z. S. Xu and R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, *Int. J. Gen. Syst.* **35** (2006), 417–433.
[30] Z. Xu and H. Hu, Projection models for intuitionistic fuzzy multiple attribute decision making, *Int. J. Inform. Technol. Decis. Making* **9** (2010), 267–280.
[31] Z. Xu and X. Q. Cai, Recent advances in intuitionistic fuzzy information aggregation, *Fuzzy Optim. Decis. Making* **9** (2010), 359–381.

[32] W. T. Xue, Z. S. Xu, X. L. Zhang and X. L. Tian, Pythagorean fuzzy LINMAP method based on the entropy theory for railway project investment decision making, *Int. J. Intell. Syst.* **33** (2018), 93–125.

[33] R. R. Yager, Pythagorean fuzzy subsets, in: *Proc. Joint IFSA World Congress and NAFIPS Annual Meeting*, Edmonton, Canada, pp. 57–61, 2013.

[34] R. R. Yager and A. M. Abbasov, Pythagorean membership grades, complex numbers and decision making, *Int. J. Intell. Syst.* **28** (2013), 436–452.

[35] D. Yu and H. Liao, Visualization and quantitative research on intuitionistic fuzzy studies, *Int. J. Fuzzy Syst.* **30** (2016), 3653–3663.

[36] L. A. Zadeh, Fuzzy sets, *Inf. Control* **8** (1965), 338–353.

[37] X. Zhang and Z. S. Xu, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, *Int. J. Intell. Syst.* **29** (2014), 1061–1078.