J. Intell. Syst. 2020; 29(1): 154–171 DE GRUYTER

Muhammad Sajjad Ali Khan*, Saleem Abdullah, Asad Ali and Khaista Rahman

Pythagorean Hesitant Fuzzy Information Aggregation and Their Application to Multi-Attribute Group Decision-Making Problems

https://doi.org/10.1515/jisys-2017-0231
Received May 18, 2017; previously published online January 5, 2018.

Abstract: In this paper, we introduce the concept of the Pythagorean hesitant fuzzy set (PHFS), which is the generalization of the intuitionistic hesitant fuzzy set under the restriction that the square sum of its membership degrees is ≤1. In decision making with PHFSs, aggregation operators play a key role because they can be used to synthesize multidimensional evaluation values represented as Pythagorean hesitant fuzzy values into collective values. Under PHFS environments, Pythagorean hesitant fuzzy ordered weighted averaging and Pythagorean fuzzy ordered weighted geometric operators are used to aggregate the Pythagorean hesitant fuzzy values. The main advantage of these operators is that they provide more accurate and valuable results. Furthermore, these operators are applied to decision-making problems in which experts provide their preferences in the Pythagorean hesitant fuzzy environment to show the validity, practicality, and effectiveness of the new approach. Finally, we compare the proposed approach to the existing methods.

Keywords: Pythagorean hesitant fuzzy set, PHFWA operator, PHFWG operator, PHFOWA operator, PHFOWG operator, group decision making.

1 Introduction

The concept of fuzzy set was first proposed by Zadeh in his important paper [31] to handle uncertainty. The aggregation of various inputs into a single output is a major problem and has been discussed by many others [23, 24, 30]. Therefore, in Ref. [4], the concept of fuzzy set used by Bellman and Zadeh in decision making for the solution of uncertainty in information came from human preferences. Dubois [7] compared old and new methods for fuzzy decision analysis. Liu and Liao [17] conducted a bibliometric analysis on fuzzy decision-related research for finding underlying patterns and dynamics in this research direction. The fuzzy set is characterized by membership degrees; therefore, Atanassov defined the intuitionistic fuzzy set (IFS), which is the generalization of fuzzy set and characterized by a membership function and a non-membership function [1, 2]. The notion of IFS is more appropriate for dealing with uncertainty and fuzziness than that of fuzzy set. IFS is very suitable for showing the uncertainty and vagueness of an object, and hence an IFS can be used as a powerful tool to obtain precise data information under different fuzzy environments that receive great attention. In decision-making problems, the concept of IFS is broadly applied [3, 5, 6, 10]. Liao and Xu [13] proposed a series of intuitionistic hybrid operators, namely intuitionistic hybrid weighted average operator, intuitionistic hybrid weighted geometric operator, generalized intuitionistic hybrid weighted average operator, and generalized intuitionistic hybrid weighted geometric operator. In Ref. [15], Liao et al. proposed an enhanced consensus-reaching process for group decision making with intuitionistic fuzzy preference relations (IFPRs). In Ref. [25], Xu and Liao presented a comprehensive survey on decision making with IFPRs with the aim of providing a clear perspective on the originality, consistency, prioritization, and consensus of IFPRs. In Ref. [29], Yu and Liao made a scientometric review on IFS studies to reveal the most cited papers,

***Corresponding author: Muhammad Sajjad Ali Khan,** Department of Mathematics, Hazara University, Mansehra, Pakistan, e-mail: sajjadalimath@yahoo.com
Saleem Abdullah: Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan
Asad Ali and Khaista Rahman: Department of Mathematics, Hazara University, Mansehra, Pakistan

influential authors, and influential journals in this domain, based on the 1318 references retrieved from the Science Citation Index Expanded and Social Science Citation Index databases via Web of Science.

The extended the notion of IFS by Yager in Refs. [26, 27] initiated the notion of the Pythagorean fuzzy set (PFS), under the restriction that the sum of square of membership degree and non-membership degree is ≤1. Many researchers have paid attention to the group decision-making problems by using the concept of Pythagorean fuzzy. In Ref. [28], the relation between Pythagorean membership degrees and complex numbers has been discussed. The authors showed that Pythagorean degrees are a subclass of complex numbers and is said to be $\Pi - i$ numbers. Zhang and Xu in Ref. [33] introduced a method for order preference by similarity to a best solution to solve the multiple criteria decision-making (MCDM) problem with Pythagorean fuzzy information. In Ref. [27], Yager proposed a series of aggregation operators, which are the Pythagorean fuzzy weighted average operator, Pythagorean fuzzy weighted geometric average operator, Pythagorean fuzzy weighted power average operator, and Pythagorean fuzzy weighted power geometric average operator, to aggregate the different Pythagorean fuzzy numbers (PFNs). These proposed operators have been proved with an application to the MCDM problem. Peng and Yang [18] introduced some new operations in PFS, which are division and subtraction, and discussed their corresponding properties. The authors also dealt with the superiority and inferiority ranking method to solve the multi-attribute group decision-making problems with Pythagorean fuzzy information. Liang et al. [11] initiated the concept of Pythagorean fuzzy geometric Bonferroni mean and weighted Pythagorean fuzzy geometric Bonferroni mean operators. In Ref. [8], Garg developed the interval-valued Pythagorean fuzzy weighted average operator and interval-valued Pythagorean fuzzy geometric operator, and introduced the concept of new accuracy function under an interval-valued Pythagorean fuzzy environment.

The concept of fuzzy set was further extended by Torra in Ref. [22], who then introduced the notion of hesitant fuzzy sets (HFSs). HFSs permit the situation of the membership having a set of possible values. Using the concept of HFS, many researchers solved group decision-making problems with aggregation operators in Refs. [16, 22, 23, 30, 32]. Liao and Xu [12] proposed the concepts of hesitant fuzzy hybrid arithmetic averaging (HFHAA) operator, hesitant fuzzy hybrid arithmetic geometric (HFHAG) operator, quasi-HFHAA operator, and quasi-HFHAG operator, and investigated some of their properties. Liao et al. [14] developed a generalized family of hybrid operators under a hesitant fuzzy environment, namely generalized hesitant fuzzy hybrid weighted averaging operator, generalized hesitant fuzzy hybrid weighted geometric operator, generalized quasi-hesitant fuzzy hybrid weighted averaging operator, generalized quasi-hesitant fuzzy hybrid weighted geometric operator, and their induced forms.

In Ref. [21], Qian et al. generalized the notion of HFSs with IFSs and referred to them as generalized HFSs, which, in essence, extended the element of HFSs from a real number to an intuitionistic fuzzy number (IFN). Zhu et al. [34] developed the concept of dual HFSs and also discussed their basic operations and properties. Peng et al. [19] introduced an MCDM approach with hesitant interval-valued IFSs, which are an extension of dual interval-valued HFSs. However, dual HFSs are defined in terms of sets of values, as opposed to precise numbers, for the membership degrees and non-membership degrees of IFSs. In Ref. [20], the authors applied the concept of intuitionistic HFS (IHFS) to group decision-making problems using fuzzy cross-entropy. PFSs, HFSs, and IHFSs have attracted more and more scholars' attention due to their powerfulness in expressing vagueness and uncertainty. IHFS satisfies the condition that the sum of its membership degrees is ≤1. However, there may be a situation where the decision maker may provide the degree of membership and non-membership of a particular attribute in such a way that their sum is >1. To overcome this shortcoming, Khan et al. [9] initiated the concept of Pythagorean HFS (PHFS), which is the generalization of the notion of IHFS. PHFS satisfies the condition that the square sum of its membership degrees and non-membership degree is ≤1. They introduced score and accuracy functions and developed aggregation operators, namely Pythagorean hesitant fuzzy weighted average (PHFWA) operator and Pythagorean hesitant fuzzy weighted geometric (PHFWG) operator. In this paper, we develop aggregation operators, namely Pythagorean hesitant fuzzy ordered weighted average (PHFOWA) operator and Pythagorean hesitant fuzzy ordered weighted geometric (PHFOWG) operator. We discuss some properties, like idempotency, boundedness, and monotonicity, of these operators. In order to do so, the remainder of the paper is organized as follows.

In the next section, we discuss some basic definitions and properties. In Section 3, we develop aggregation operators, such as the PHFOWA and PHFOWG operators. In Section 4, we develop multi-attribute decision making based on the proposed aggregation operators in which experts provide their preferences in the form of Pythagorean hesitant fuzzy numbers (PHFNs).

In Section 5, we give a numerical example to show the validity, practicality, and effectiveness of the proposed approach. In Section 6, we compare the proposed approach to the existing methods. Conclusion is given in Section 7.

2 Preliminaries

In this section, we review some basic definitions and results.

Definition 1 ([26])**:** Let X be a fixed set. Then, a PFS P in X can be defined as follows:

$$P=\{\langle x,\ h_P(x),\ h'_P(x)\rangle \mid x\in X\}, \tag{1}$$

where $h_P(x)$ and $h'_P(x)$ are mappings from X to [0, 1], such that $0\le h_P(x)\le 1$, $0\le h'_P(x)\le 1$, and also $0\le h_P^2(x)+h_P'^2(x)\le 1$, for all $x\in X$; here, $h_P(x)$ and $h'(x)$ denote the membership degree and non-membership degree of element $x\in X$ to set P, respectively. Let $\pi_P(x)=\sqrt{1-h_P^2(x)-h_P'^2(x)}$. Then, it is commonly called the Pythagorean fuzzy index of element $x\in X$ to set P, representing the degree of indeterminacy of x to P. Also, $0\le \pi_P(x)\le 1$, for every $x\in X$. We denote the PFN by $p=\langle \Lambda_{\hat{h}},\ \Gamma_{\hat{h}}\rangle$.

To compare two PFNs in Ref. [33], the authors introduced the concept of score function and accuracy degree. They also discussed some relation between them.

Definition 2 ([33])**:** Let $p_1=\langle \Lambda_{p_1},\Gamma_{p_1}\rangle$ and $p_2=\langle \Lambda_{p_2},\Gamma_{p_2}\rangle$ be two PFNs. Then, $S(p_1)=\Lambda^2_{p_1}-\Gamma^2_{p_1}$ and $S(p_2)=\Lambda^2_{p_2}-\Gamma^2_{p_2}$ are the scores of p_1 and p_2, respectively, and $H(p_1)=\Lambda^2_{p_1}+\Gamma^2_{p_1}$, $H(p_2)=\Lambda^2_{p_2}+\Gamma^2_{p_2}$ are the accuracy degrees of p_1, p_2, respectively. Then, we have

(1) If $S(p_1)<S(p_2)$, then p_1 is smaller than p_2, denoted by $p_1<p_2$.
(2) If $S(p_1)=S(p_2)$, then
  (a) If $H(p_1)=H(p_2)$, then p_1 and p_2 represent the same information, i.e. $\Lambda^2_{p_1}=\Lambda^2_{p_2}$ and $\Gamma^2_{p_1}=\Gamma^2_{p_2}$ denoted by $p_1=p_2$.
  (b) If $H(p_1)<H(p_2)$, then p_1 is smaller than p_2 denoted by $p_1<p_2$.
  (c) If $H(p_1)>H(p_2)$, then p_1 is greater than p_2 denoted by $p_1>p_2$.

Definition 3 ([22])**:** Let X be a fixed set. Then, HFS H in X can be defined as follows:

$$H=\{\langle x,\ h_H(x)\rangle \mid x\in X\}, \tag{2}$$

where $h_H(x)$ denotes the set of some values belonging to [0, 1], i.e. the possible membership degree of the element $x\in X$ to the set H. For convenience, we denote a hesitant fuzzy number (HFN) by $h=h_H(x)$ and the set of all HFNs as *HFN*.

Definition 4 ([22])**:** Let h, h_2, and h_3 be three HFNs. Then, some basic operations on HFNs can be defined as follows:

(1) $h^c=\bigcup_{\delta\in h}\{1-\delta\}$.
(2) $h_1\cup h_2=\bigcup_{\delta_1\in h_1,\delta_2\in h_2}\max\{\delta_1,\ \delta_2\}$.
(3) $h_1\cap h_2=\bigcup_{\delta_1\in h_1,\delta_2\in h_2}\min\{\delta_1,\ \delta_2\}$.

Definition 5 ([20])**:** Let X be a fixed set. Then, IHFS I_H in X can be defined as follows:

$$I_H=\{\langle x,\ \Lambda_{I_H}(x),\ \Gamma_{I_H}(x)\mid x\in X\rangle\}, \tag{3}$$

where $\Lambda_{I_H}(x)$ and $\Gamma_{I_H}(x)$ are mappings from X to [0, 1], denoting a possible degree of membership and non-membership degree of element $x \in X$ in I_H, respectively, and for every element $x \in X$, for all $h_{I_H}(x) \in \Lambda_I(x)$, $\exists h'_{I_H}(x) \in \Gamma_{I_H}(x)$ such that $0 \le h_{I_H}(x) + h'_{I_H}(x) \le 1$, and for all $h'_{I_H}(x) \in \Gamma_{I_H}(x)$, $\exists h_{I_H}(x) \in \Lambda_{I_H}(x)$ such that $0 \le h_{I_H}(x) + h'_{I_H}(x) \le 1$. If X has only one element, $\langle x, \Lambda_{I_H}(x), \Gamma_{I_H}(x)\rangle$ is said to be an intuitionistic HFN (IHFN) and is denoted by $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$. The set of all IHFNs is denoted by *IHFNs*.

2.1 Intuitionistic Hesitant Fuzzy Aggregation Operators

Definition 6 ([20])**:** Let $\hat{h}_i = (\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i})$ $(i = 1, 2, 3, \ldots, n)$ be a collection of all IHFNs, and $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{h}_i$ $(i = 1, 2, 3, \ldots, n)$ with $w_i \ge 0$ $(i = 1, 2, 3, \ldots, n)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the aggregation result using the IHFWA operator is also an IHFN and

$$IHFWA(I_{H1}, I_{H2}, \ldots, I_{Hn}) = \left\langle \begin{array}{l} \bigcup_{h_{I_{\hat{h}_1}} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \left\{ 1 - \prod_{i=1}^{n} (1 - h_{\hat{h}_i})^{w_i} \right\}, \\ \bigcup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_1} \in \Gamma_{\hat{h}_2}, \ldots, h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \left\{ \prod_{i=1}^{n} (h'_{\hat{h}_i})^{w_i} \right\} \end{array} \right\rangle. \tag{4}$$

Definition 7 ([20])**:** Let $\hat{h}_i = (\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i})$ $(i = 1, 2, 3, \ldots, n)$ be a collection of all IHFNs, and $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $\hat{h}_i$ $(i = 1, 2, 3, \ldots, n)$ with $w_i \ge i = 0$ $(i = 1, 2, 3, \ldots, n)$, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the aggregation result using the IHFWG operator is also an IHFN, and

$$IHFWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \left\langle \begin{array}{l} \bigcup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2}, \ldots, h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \left\{ \prod_{i=1}^{n} (h_{\hat{h}_i})^{w_i} \right\}, \\ \bigcup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_2} \in \Gamma_{\hat{h}_2}, \ldots, h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \left\{ 1 - \prod_{i=1}^{n} (1 - h'_{\hat{h}_i})^{w_i} \right\} \end{array} \right\rangle. \tag{5}$$

In Ref. [9], Khan et al. introduced the concept of PHFS, which is the generalization of IHFS. PHFS is defined below.

2.2 Pythagorean Hesitant Fuzzy Sets

In Ref. [9], Khan et al. generalized the concept of IHFS and introduced the concept of PHFSs. PHFS is defined by Definition 8.

Definition 8 ([9])**:** Let X be a fixed set. A PHFS P_H in X is an object with the following notion:

$$P_H = \{\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x) \mid x \in X \rangle\}, \tag{6}$$

where $\Lambda_{P_H}(x)$ and $\Gamma_{P_H}(x)$ are mappings from X to [0, 1], denoting a possible degree of membership and non-membership degree of element $x \in X$ in P_H, respectively, and for each element $x \in X$, $\forall$ $h_{P_H}(x) \in \Lambda_{P_H}(x)$, $\exists h'_{P_H}(x) \in \Gamma_{P_H}(x)$ such that $0 \le h^2_{P_H}(x) + h'^2_{P_H}(x) \le 1$, and $\forall$ $h'_{P_H}(x) \in \Gamma_{P_H}(x)$, $\exists h_{P_H}(x) \in \Lambda_{P_H}(x)$ such that $0 \le h^2_{P_H}(x) + h'^2_{P_H}(x) \le 1$. Moreover, *PHES*($X$) denotes the set of all elements of PHFSs. If X has only one element $\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x)\rangle$, then it is said to be PHFN and is denoted by $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$ for convenience. We denote the set of all PHFNs by *PHFNS*. For all $x \in X$, if $\Lambda_{P_H}(x)$ and $\Gamma_{P_H}(x)$ have only one element. Then, the PHFS become a PFS. If the non-membership degree is {0}, then the PHFS becomes an HFS. For any PHFS $P_H = \{\langle x, \Lambda_{P_H}(x), \Gamma_{P_H}(x) \mid x \in X \rangle\}$ and for all $x \in X$, $\Pi_{P_H}(x) = \bigcup_{h_{P_H} \in \Lambda_{P_H}(x), h'_{P_H}(x) \in \Gamma_{P_H}(x)} \sqrt{1 - h^2_{P_H} - h'^2_{P_H}}$ is said to be the degree of indeterminacy of x to P_H, where $1 - h^2_{P_H} - h'^2_{P_H} \ge 0$.

Definition 9 ([9])**:** Let $\hat{h}=\langle\Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$, $\hat{h}_1=\langle\Lambda_{\hat{h}_1}, \Gamma_{\hat{h}_1}\rangle$, $\hat{h}_2=\langle\Lambda_{\hat{h}_2}, \Gamma_{\hat{h}_2}\rangle$, are three PHFNs, and $\lambda>0$. Then, their operations are defined as follows. The result obtained by these operations is also a PHFN.

(1) $\hat{h}_1\cup\hat{h}_2=\langle\max\{\Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2}\}, \min\{\Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2}\}\rangle$.

(2) $\hat{h}_1\cap\hat{h}_2=\langle\min\{\Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2}\}, \max\{\Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2}\}\rangle$.

(3) $\hat{h}^c=\langle\Gamma_{\hat{h}}, \Lambda_{\hat{h}}\rangle$.

(4) $\hat{h}_1\oplus\hat{h}_2=\left\langle\begin{array}{l}\bigcup_{h_{\hat{h}_1}\in\Lambda_{\hat{h}_1}, h_{\hat{h}_2}\in\Lambda_{\hat{h}_2}}\left\{\sqrt{h_{\hat{h}_1}^2+h_{\hat{h}_2}^2-h_{\hat{h}_1}^2h_{\hat{h}_2}^2}\right\},\\ \bigcup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1}, h'_{\hat{h}_2}\in\Gamma_{\hat{h}_2}}\{h'_{\hat{h}_1}h'_{\hat{h}_2}\}\end{array}\right\rangle$.

(5) $\hat{h}_1\otimes\hat{h}_2=\left\langle\begin{array}{l}\bigcup_{h_{\hat{h}_1}\in\Lambda_{\hat{h}_1}, h_{\hat{h}_2}\in\Lambda_{\hat{h}_2}}\{h_{\hat{h}_1}h_{\hat{h}_2}\},\\ \bigcup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1}, h'_{\hat{h}_2}\in\Gamma_{\hat{h}_2}}\left\{\sqrt{h'^2_{\hat{h}_1}+h'^2_{\hat{h}_2}-h'^2_{\hat{h}_1}h'^2_{\hat{h}_2}}\right\}\end{array}\right\rangle$.

(6) $\lambda\hat{h}=\left\langle\bigcup_{h_{\hat{h}}\in\Lambda_{\hat{h}}}\left\{\sqrt{1-(1-(h_{\hat{h}})^2)^\lambda}\right\}, \bigcup_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}\{(h'_{\hat{h}})^\lambda\}\right\rangle$, $\lambda>0$.

(7) $\hat{h}^\lambda=\left\langle\bigcup_{h_{\hat{h}}\in\Lambda_{\hat{h}}}\{h_{\hat{h}}^\lambda\}, \bigcup_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}\left\{\sqrt{1-(1-(h'_{\hat{h}})^2)^\lambda}\right\}\right\rangle$, $\lambda>0$.

To compare two PHFNs, in the following [9], the score function, accuracy function, and some basic laws on the basis of the score function are defined.

Definition 10 ([9])**:** Let $\hat{h}=\langle\Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$ be a PHFN. Then, we define the score function of $\hat{h}$ as follows:

$$S(\hat{h})=\left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}}h_{\hat{h}}\right)^2-\left(\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}}\sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}h'_{\hat{h}}\right)^2, \tag{7}$$

where $S(\hat{h})\in[-1, 1]$. $l_{h_{\hat{h}}}$ denotes the number of elements in $\Lambda_{\hat{h}}$ and $l_{h'_{\hat{h}}}$ denotes the number of elements in $\Gamma_{\hat{h}}$.

Definition 11 ([9])**:** Let, $\hat{h}=\langle\Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$ be a PHFN. Then, the accuracy degree of $\hat{h}$ is denoted by $\overline{\sigma}(\hat{h})$ and can be defined as follows:

$$\overline{\sigma}(\hat{h})=\left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}}h_{\hat{h}}-S(\hat{h})\right)^2+\left(\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}}\sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}h'_{\hat{h}}-S(\hat{h})\right)^2. \tag{8}$$

Here, we can see that $S(\hat{h})$ is just the mean value in statistics, and $\overline{\sigma}(\hat{h})$ is just the standard variance, which reflects the accuracy degree between all values in the PHFN $\hat{h}$ and their mean value. Let $\hat{h}_1$ and $\hat{h}_2$ be two PHFNs, $S(\hat{h}_1)$ be the score of $\hat{h}_1$, $S(\hat{h}_2)$ be the score of $\hat{h}_2$, and $\overline{\sigma}(\hat{h}_1)$ be the deviation degree of $\hat{h}_1$, $\overline{\sigma}(\hat{h}_2)$ be the deviation degree of $\hat{h}_2$. Then

(1) If $S(\hat{h}_1)<S(\hat{h}_2)$, then $\hat{h}_1<\hat{h}_2$.
(2) If $S(\hat{h}_1)>S(\hat{h}_2)$, then $\hat{h}_1>\hat{h}_2$.
(3) If $S(\hat{h}_1)=S(\hat{h}_2)$, then $\hat{h}_1\sim\hat{h}_2$.
  (i) If $\overline{\sigma}(\hat{h}_1)<\overline{\sigma}(\hat{h}_2)$, then $\hat{h}_1<\hat{h}_2$.
  (ii) If $\overline{\sigma}(\hat{h}_1)>\overline{\sigma}(\hat{h}_2)$, then $\hat{h}_1>\hat{h}_2$.
  (iii) If $\overline{\sigma}(\hat{h}_1)=\overline{\sigma}(\hat{h}_2)$, then $\hat{h}_1\sim\hat{h}_2$.

3 Pythagorean Hesitant Fuzzy Information Aggregation Operators

In this section, we develop some aggregation operators for PHFNs and investigate some of its properties.

3.1 PHFWA/Geometric Operator

Definition 12 ([9])**:** Let $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \rangle$ $(i=1, 2, 3,\ldots,n)$ be a collection of all PHFNs, and $w=(w_1, w_2,\ldots,w_n)^T$ be the weight vector of $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ with $w_i \geq 0$ $(i=1, 2, 3,\ldots,n)$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the *PHFWA* operator is a mapping $PHFWA{:}PHFN^n \to PHFN$ can be defined as

$$PHFWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n) = w_1\hat{h}_1 \oplus w_2\hat{h}_2 \oplus \ldots \oplus w_n\hat{h}_n, \tag{9}$$

and the *PHFWA* operator is said to be a PHFWA operator.

Theorem 1 ([9])**:** *Let $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \rangle$ $(i=1, 2, 3,\ldots,n)$ be a collection of all PHFNs, and $w=(w_1, w_2,\ldots,w_n)^T$ be the weight vector of $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ with $w_i \geq 0$ $(i=1, 2, 3,\ldots,n)$, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, the aggregation result using PHFWA operator is also a PHFN and*

$$PHFWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n) = \left\langle \begin{array}{l} \bigcup_{h_{\hat{h}_1} \in \Lambda_{\hat{h}_1}, h_{\hat{h}_2} \in \Lambda_{\hat{h}_2},\ldots,h_{\hat{h}_n} \in \Lambda_{\hat{h}_n}} \left\{ \sqrt{1 - \prod_{i=1}^{n} (1 - h_{\hat{h}_i}^2)^{w_i}} \right\}, \\ \bigcup_{h'_{\hat{h}_1} \in \Gamma_{\hat{h}_1}, h'_{\hat{h}_1} \in \Gamma_{\hat{h}_2},\ldots,h'_{\hat{h}_n} \in \Gamma_{\hat{h}_n}} \left\{ \prod_{i=1}^{n} (h'_{\hat{h}_i})^{w_i} \right\} \end{array} \right\rangle. \tag{10}$$

Proof. Proof of the theorem follows from Theorem 4.2 in Ref. [9]. □

In the following, we present some properties of the PHFWA operator.

Theorem 2: *Let $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \rangle$ $(i=1, 2, 3,\ldots,n)$ be a collection of all PHFNs, and $w=(w_1, w_2,\ldots,w_n)^T$ be the weight vector of $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ with $w_i \geq 0$ $(i=1, 2, 3,\ldots,n)$ where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then*

(1) *(Idempotency) If all $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \rangle$ $(i=1, 2, 3,\ldots,n)$ are equal, i.e. $\hat{h}_i$ $(i=1, 2, 3,\ldots,n) = \hat{h}$, then*

$$PHFWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n) = \hat{h}. \tag{11}$$

(2) *(Boundedness)*

$$\hat{h}^- \leq PHFWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n) \leq \hat{h}^+, \tag{12}$$

where $\hat{h}^- = \langle h^-, h^{+'} \rangle$, $\hat{h}^+ = \langle h^+, h^{-'} \rangle$, $h^- = \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \min_i \{h_i\}$,

$$h^+ = \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \max_i \{h_i\},\ h^{-'} = \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \min_i \{h'_i\},\ h^{+'} = \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \max_i \{h'_i\}.$$

(3) *(Monotonicity) If $\hat{h}_i > \hat{h}_i^*$,* then

$$PHFWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n) \leq PHFWA(\hat{h}_1^*, \hat{h}_2^*,\ldots,\hat{h}_n^*). \tag{13}$$

Proof. (1) By Theorem 1, we have

$$\begin{aligned} PHFWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n) &= \left\langle \bigcup_{h_{\hat{h}_i} \in \Lambda_{\hat{h}_i}} \left\{ \sqrt{1 - \prod_{i=1}^{n} (1 - h_{\hat{h}_i}^2)^{w_i}} \right\}, \bigcup_{h'_{\hat{h}_i} \in \Gamma_{\hat{h}_i}} \left\{ \prod_{i=1}^{n} (h'_{\hat{h}_i})^{w_i} \right\} \right\rangle \\ &= \left\langle \bigcup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \left\{ \sqrt{1 - \prod_{i=1}^{n} (1 - h_{\hat{h}}^2)^{w_i}} \right\}, \bigcup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \left\{ \prod_{i=1}^{n} (h'_{\hat{h}})^{w_i} \right\} \right\rangle \\ &= \left\langle \bigcup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \left\{ \sqrt{1 - (1 - h_{\hat{h}}^2)^{\sum_{i=1}^{n} w_i}} \right\}, \bigcup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \left\{ (h'_{\hat{h}})^{\sum_{i=1}^{n} w_i} \right\} \right\rangle \\ &= \left\langle \bigcup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \left\{ \sqrt{1 - (1 - h_{\hat{h}}^2)} \right\}, \bigcup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \{h'_{\hat{h}}\} \right\rangle \\ &= \left\langle \bigcup_{h_{\hat{h}} \in \Lambda_{\hat{h}}} \{h_{\hat{h}}\}, \bigcup_{h'_{\hat{h}} \in \Gamma_{\hat{h}}} \{h'_{\hat{h}}\} \right\rangle = \hat{h} \end{aligned}$$

$$\text{(2) As } \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \min_i\{h_i\} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \{h_i\} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \max_i\{h_i\} \tag{14}$$

$$\text{and } \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \min_i\{h'_i\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \{h'_i\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \max_i\{h'_i\}. \tag{15}$$

Thus, from Eq. (14), we have

$$\begin{aligned}
&\bigcup_{h_i \in \Lambda_{\hat{h}_i}} \min_i\{h_i\} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \{h_i\} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \max_i\{h_i\} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\min_i\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\max_i\{(h_i)^2\}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\max_i\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\min_i\{(h_i)^2\}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{(1-\max_i\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{(1-\min_i\{(h_i)^2\})^{w_i}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\prod_{i=1}^{n}(1-\max_i\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\prod_{i=1}^{n}(1-\min_i\{(h_i)^2\})^{w_i}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{(1-\max_i\{(h_i)^2\})^{\sum_{i=1}^{n} w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{(1-\min_i\{(h_i)^2\})^{\sum_{i=1}^{n} w_i}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\max_i\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\min_i\{(h_i)^2\}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{-1+\min_i\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{-\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{-1+\max_i\{(h_i)^2\}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-1+\min_i\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-1+\max_i\{(h_i)^2\}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\min_i\{(h_i)^2\}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{\max_i\{(h_i)^2\}} \\
&\Leftrightarrow \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \min_i\{h_i\} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \sqrt{1-\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}} \le \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \max_i\{h_i\}
\end{aligned}$$

Now, from Eq. (15), we have

$$\begin{aligned}
&\bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \min_i\{h'_i\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \{h'_i\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \max_i\{h'_i\} \\
&\Leftrightarrow \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \min_i\{(h'_i)^{w_i}\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \{(h'_i)^{w_i}\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \max_i\{(h'_i)^{w_i}\} \\
&\Leftrightarrow \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \prod_{i=1}^{n} \min_i\{(h'_i)^{w_i}\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \prod_{i=1}^{n} \{(h'_i)^{w_i}\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \prod_{i=1}^{n} \max_i\{(h'_i)^{w_i}\} \\
&\Leftrightarrow \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \min_i\left\{(h'_i)^{\sum_{i=1}^{n} w_i}\right\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \prod_{i=1}^{n} \{(h'_i)^{w_i}\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \max_i\left\{(h'_i)^{\sum_{i=1}^{n} w_i}\right\} \\
&\Leftrightarrow \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \min_i\{(h'_i)\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \prod_{i=1}^{n} \{(h'_i)^{w_i}\} \le \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \max_i\{(h'_i)\}.
\end{aligned}$$

According to the score function, we have $PHFWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \ge \hat{h}^-$ with equality if and only if $\hat{h}^- = PHFWA(\hat{h})$.

Similarly, $PHFWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \le \hat{h}^+$ is with equality if and only if $PHFWA(\hat{h})$ is the same as $\hat{h}^+$. Hence, $\hat{h}^- \le PHFWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \le \hat{h}^+$

(3) If $\hat{h}_i > \hat{h}_i^*$, then $PHFWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \le PHFWA(\hat{h}_1^*, \hat{h}_2^*, \ldots, \hat{h}_n^*)$. As $\Lambda_{\hat{h}_i^*} \ge \Lambda_{\hat{h}_i}$, $\Gamma_{\hat{h}_i} \ge \Gamma_{\hat{h}_i^*}$. If $\Lambda_{\hat{h}_i} \le \Lambda_{\hat{h}_i^*}$, then

$$\begin{aligned}
&\cup_{h_i\in\Lambda_{\hat{h}_i}}\{h_i\}\le\cup_{h_i^*\in\Lambda_{\hat{h}_i^*}}\{h_i^*\}\Leftrightarrow\cup_{h_i\in\Lambda_{\hat{h}_i}}\{(h_i)^2\}\le\cup_{h_i^*\in\Lambda_{\hat{h}_i^*}}\{(h_i^*)^2\}\\
&\Leftrightarrow\cup_{h_i\in\Lambda_{\hat{h}_i}}\sqrt{\{(h_i)^2\}}\le\cup_{h_i^*\in\Lambda_{\hat{h}_i^*}}\sqrt{\{(h_i^*)^2\}}\\
&\Leftrightarrow\cup_{h_i^*\in\Lambda_{\hat{h}_i^*}}\sqrt{1-\{(h_i^*)^2\}}\le\cup_{h_i\in\Lambda_{\hat{h}_i}}\sqrt{1-\{(h_i)^2\}}\\
&\Leftrightarrow\cup_{h_i^*\in\Lambda_{\hat{h}_i^*}}\sqrt{(1-\{(h_i^*)^2\})^{w_i}}\le\cup_{h_i\in\Lambda_{\hat{h}_i}}\sqrt{(1-\{(h_i)^2\})^{w_i}}\\
&\Leftrightarrow\cup_{h_i^*\in\Lambda_{\hat{h}_i^*}}\sqrt{\prod_{i=1}^{n}(1-\{(h_i^*)^2\})^{w_i}}\le\cup_{h_i\in\Lambda_{\hat{h}_i}}\sqrt{\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}}\\
&\Leftrightarrow\cup_{h_i\in\Lambda_{\hat{h}_i}}\sqrt{1-\prod_{i=1}^{n}(1-\{(h_i)^2\})^{w_i}}\le\cup_{h_i^*\in\Lambda_{\hat{h}_i^*}}\sqrt{\prod_{i=1}^{n}(1-\{(h_i^*)^2\})^{w_i}}.
\end{aligned} \tag{16}$$

Now, if $\Gamma_{\hat{h}_i}\ge\Gamma_{\hat{h}_i^*}$, then $\cup_{h_i'\in\Gamma_{\hat{h}_i}}\{h_i'\}\ge\cup_{h_i^{*\prime}\in\Gamma_{\hat{h}_i^*}}\{h_i^{*\prime}\}$

$$\begin{aligned}
&\Leftrightarrow\cup_{h_i'\in\Gamma_{\hat{h}_i}}\{(h_i')^{w_i}\}\ge\cup_{h_i^{*\prime}\in\Gamma_{\hat{h}_i^*}}\{(h_i^{*\prime})^{w_i}\}\\
&\Leftrightarrow\cup_{h_i'\in\Gamma_{\hat{h}_i}}\left\{\prod_{i=1}^{n}(h_i')^{w_i}\right\}\ge\cup_{h_i^{*\prime}\in\Gamma_{\hat{h}_i^*}}\left\{\prod_{i=1}^{n}(h_i^{*\prime})^{w_i}\right\}.
\end{aligned} \tag{17}$$

Let $\hat{h}=PHFWA(\hat{h}_1,\hat{h}_2,\ldots,\hat{h}_n)$ and $\hat{h}^*=PHFWA(\hat{h}_1^*,\hat{h}_2^*,\ldots,\hat{h}_n^*)$. Then, from Eqs. (16) and (17), we have $S(\hat{h})\le S(\hat{h}^*)$.

If $S(\hat{h})<S(\hat{h}^*)$, then $PHFWA(\hat{h}_1,\hat{h}_2,\ldots,\hat{h}_n)<PHFWA(\hat{h}_1^*,\hat{h}_2^*,\ldots,\hat{h}_n^*)$. If $S(\hat{h})=S(\hat{h}^*)$, then

$$\begin{aligned}
&\left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}}h_{\hat{h}}\right)^2-\left(\frac{1}{l_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}}\sum_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}h_{\hat{h}}'\right)^2=\left(\frac{1}{l_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}h_{\hat{h}^*}^*\right)^2-\left(\frac{1}{l_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}h_{\hat{h}^*}^{*\prime}\right)^2\\
&\Leftrightarrow\left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}}h_{\hat{h}}\right)^2=\left(\frac{1}{l_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}h_{\hat{h}^*}^*\right)^2\text{and}\left(\frac{1}{l_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}}\sum_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}h_{\hat{h}}'\right)^2=\left(\frac{1}{l_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}h_{\hat{h}^*}^{*\prime}\right)^2\\
&\Leftrightarrow\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}}h_{\hat{h}}=\frac{1}{l_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}h_{\hat{h}^*}^*\ \text{and}\ \frac{1}{l_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}}\sum_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}h_{\hat{h}}'=\frac{1}{l_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}h_{\hat{h}^*}^{*\prime}.
\end{aligned}$$

As

$$\begin{aligned}
\bar{\sigma}(\hat{h})&=\left(\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}}\sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}}h_{\hat{h}}-S(\hat{h})\right)^2+\left(\frac{1}{l_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}}\sum_{h_{\hat{h}}'\in\Gamma_{\hat{h}}}h_{\hat{h}}'-S(\hat{h})\right)^2\\
&=\left(\frac{1}{l_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^*\in\Lambda_{\hat{h}^*}}h_{\hat{h}^*}^*-S(\hat{h}^*)\right)^2+\left(\frac{1}{l_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}}\sum_{h_{\hat{h}^*}^{*\prime}\in\Gamma_{\hat{h}^*}}h_{\hat{h}^*}^{*\prime}-S(\hat{h}^*)\right)^2\\
&=\bar{\sigma}(\hat{h}^*),
\end{aligned}$$

therefore, $PHFWA(\hat{h}_1,\hat{h}_2,\ldots,\hat{h}_n)=PHFWA(\hat{h}_1^*,\hat{h}_2^*,\ldots,\hat{h}_n^*)$. □

Definition 13 ([9])**:** Let $\hat{h}_i=\langle\Lambda_{\hat{h}_i},\Gamma_{\hat{h}_i}\rangle$ $(i=1,2,3,\ldots,n)$ be a collection of all PHFNs, and $w=(w_1,w_2,\ldots,w_n)$ be the weight vector of $\hat{h}_i$ $(i=1,2,3,\ldots,n)$ with $w_i\ge 0$ $(i=1,2,3,\ldots,n)$ such that $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$. Then, the PHFWG operator is a mapping $PHFWG{:}PHFN^n\to PHFN$ can be defined as

$$PHFWG(\hat{h}_1,\hat{h}_2,\ldots,\hat{h}_n)=\hat{h}_1^{w_1}\otimes\hat{h}_2^{w_2}\otimes,\ldots,\otimes\hat{h}_n^{w_n}, \tag{18}$$

and *PHFWG* is said to be a PHFWG operator.

Theorem 3 ([9])**:** *Let* $\hat{h}_i=\langle\Lambda_{\hat{h}_i},\Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ *be a collection of all PHFNs, and* $w=(w_1, w_2,\ldots,w_n)^T$ *be the weight vector of* $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ *with* $w_i\geq 0$ $(i=1, 2, 3,\ldots,n)$, *where* $w_i\in[0, 1]$ *and* $\sum_{i=1}^n w_i=1$. *Then, the aggregation result using the PHFWG operator is also a PHFN, and*

$$PHFWG(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)=\left\langle \begin{matrix} \bigcup_{h_{\hat{h}_1}\in\Lambda_{\hat{h}_1},h_{\hat{h}_2}\in\Lambda_{\hat{h}_2},\ldots,h_{\hat{h}_n}\in\Lambda_{\hat{h}_n}}\left\{\prod_{i=1}^{n}(h_{\hat{h}_i})^{w_i}\right\}, \\ \bigcup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1},h'_{\hat{h}_2}\in\Gamma_{\hat{h}_2},\ldots,h'_{\hat{h}_n}\in\Gamma_{\hat{h}_n}}\left\{\sqrt{1-\prod_{i=1}^{n}(1-h'^2_{\hat{h}_i})^{w_i}}\right\} \end{matrix}\right\rangle. \tag{19}$$

Proof. Proof of the theorem follows from Theorem 4.5 in Ref. [9]. □

In the following, we present some properties of the PHFWG operator.

Theorem 4: *Let* $\hat{h}_i=\langle\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ *be a collection of all PHFNs, and* $w=(w_1, w_2,\ldots,w_n)^T$ *be the weight vector of* $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ *with* $w_i\geq 0$ $(i=1, 2, 3,\ldots,n)$ *where* $w_i\in[0, 1]$ *and* $\sum_{i=1}^n w_i=1$. *Then*

(1) *(Idempotency) If all* $\hat{h}_i=\langle\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ *are equal, i.e.* $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)=\hat{h}$, *then*

$$PHFWG(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)=\hat{h}. \tag{20}$$

(2) *(Boundedness)*

$$\hat{h}^-\leq PHFWG(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)\leq\hat{h}^+, \tag{21}$$

where $\hat{h}^-=\langle h^-, h^{+'}\rangle$, $\hat{h}^+=\langle h^+, h^{-'}\rangle$, $h^-=\bigcup_{h_i\in\Lambda_{\hat{h}_i}}\min_i\{h_i\}$,

$$h^+=\bigcup_{h_i\in\Lambda_{\hat{h}_i}}\max_i\{h_i\},\ h^{-'}=\bigcup_{h'_i\in\Gamma_{\hat{h}_i}}\min_i\{h'_i\},\ h^{+'}=\bigcup_{h'_i\in\Gamma_{\hat{h}_i}}\max_i\{h'_i\}.$$

(3) *(Monotonicity) If* $\hat{h}_i>\hat{h}^*_i$, *then*

$$PHFWG(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)\leq PHFWG(\hat{h}^*_1, \hat{h}^*_2,\ldots,\hat{h}^*_n). \tag{22}$$

Proof. Proof of the theorem follows from Theorem 2. □

3.2 PHFOWA/Geometric Operator

In the following, we develop a PHFOWA operator and a PHFOWG operator. We also discuss some properties of the developed operators.

Definition 14: Let $\hat{h}_i=\langle\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ be a collection of all PHFNs, $\hat{h}_{\sigma(i)}$ be the largest in them, and $w=(w_1, w_2,\ldots,w_n)^T$ be the weight vector of $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ with $w_i\geq 0$ $(i=1, 2, 3,\ldots,n)$ such that $w_i\in[0, 1]$ and $\sum_{i=1}^n w_i=1$. Then, the PHFOWA operator is a mapping *PHFOWA*:*PHFN*^{n}→*PHFN* can be defined by

$$PHFOWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)=w_1\hat{h}_{\sigma(1)}\oplus w_2\hat{h}_{\sigma(2)}\oplus,\ldots,\oplus w_n\hat{h}_{\sigma(n)}. \tag{23}$$

Theorem 5: *Let* $\hat{h}_i=\langle\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ *be a collection of all PHFNs,* $\hat{h}_{\sigma(i)}$ *be the largest in them,* $w=(w_1, w_2,\ldots,w_n)$ *be the weight vector of* $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ *with* $w_i\geq 0$ $(i=1, 2, 3,\ldots,n)$ *such that* $w_i\in[0, 1]$ *and* $\sum_{i=1}^n w_i=1$. *Then, the aggregation result using the PHFOWA operator is also a PHFN and*

$$PHFOWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)=\left\langle \begin{matrix} \bigcup_{h_{\hat{h}_{\sigma(1)}}\in\Lambda_{\hat{h}_{\sigma(1)}},h_{\hat{h}_{\sigma(2)}}\in\Lambda_{\hat{h}_{\sigma(2)}},\ldots,h_{\hat{h}_{\sigma(n)}}\in\Lambda_{\hat{h}_{\sigma(n)}}}\left\{\sqrt{1-\prod_{i=1}^{n}(1-h^2_{\hat{\alpha}_{\sigma(i)}})^{w_i}}\right\}, \\ \bigcup_{h'_{\hat{h}_{\sigma(1)}}\in\Gamma_{\hat{h}_{\sigma(1)}},h'_{\hat{h}_{\sigma(2)}}\in\Gamma_{\hat{h}_{\sigma(2)}},\ldots,h'_{\hat{h}_{\sigma(n)}}\in\Gamma_{\hat{h}_{\sigma(n)}}}\left\{\prod_{i=1}^{n}(h'_{\hat{h}_{\sigma(i)}})^{w_i}\right\} \end{matrix}\right\rangle. \tag{24}$$

Proof. Proof of the theorem is the same as Theorem 1. □

Example 1: Suppose there are three experts who are invited to evaluate some decision alternatives. The evaluation of the expert is denoted by PHFNs, $\hat{h}_1=\langle\{0.5, 0.6, 0.9\}, \{0.3, 0.6\}\rangle$, $\hat{h}_2=\langle\{0.3, 0.6, 0.8\}, \{0.2, 0.7, 0.9\}\rangle$, $\hat{h}_3=\langle\{0.3, 0.4, 0.7.0.9\}, \{0.2, 0.8, 0.9\}\rangle$, respectively, with $w=(0.35, 0.4, 0.25)^T$ as the weighted vector of $\hat{h}_i$ $(i=1, 2, 3)$. To calculate the comprehensive evaluation of the three experts on the decision alternative through using the PHFOWA operator, we have

$$PHFOWA(\hat{h}_1, \hat{h}_2, \hat{h}_3)=\left\langle \begin{array}{l} \bigcup_{h_{\hat{h}_{\sigma(1)}}\in\Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}}\in\Lambda_{\hat{h}_{\sigma(2)}}, h_{\hat{h}_{\sigma(3)}}\in\Lambda_{\hat{h}_{\sigma(3)}}} \left\{\sqrt{1-\prod_{i=1}^{3}(1-h^2_{\hat{h}_{\sigma(i)}})^{w_i}}\right\}, \\ \bigcup_{h'_{\hat{h}_{\sigma(1)}}\in\Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}}\in\Gamma_{\hat{h}_{\sigma(2)}}, h'_{\hat{h}_{\sigma(3)}}\in\Gamma_{\hat{h}_{\sigma(3)}}} \left\{\prod_{i=1}^{3}(h'_{\hat{h}_{\sigma(i)}})^{w_i}\right\} \end{array} \right\rangle.$$

First, we calculate the score functions of $\hat{h}_1, \hat{h}_2$ and $\hat{h}_3$. For this, we have

$$S(\hat{h}_1)=0.928,\ \ S(\hat{h}_2)=-0.117,\ \ S(\hat{h}_3)=0.1192.$$

Thus, $S(\hat{h}_1)>S(\hat{h}_3)>S(\hat{h}_2)$. Now

$$PHFOWA(\hat{h}_1, \hat{h}_2, \hat{h}_3)=\left\langle \begin{array}{l} \left\{\begin{array}{l} 0.39, 0.47, 0.57, 0.42, 0.50, 0.59, 0.57, 0.62, 0.68, 0.74, 0.76, 0.80 \\ 0.44, 0.51, 0.60, 0.47, 0.54, 0.62, 0.60, 0.65, 0.70, 0.76, 0.78, 0.81, \\ 0.69, 0.72, 0.76, 0.70, 0.73, 0.77, 0.82, 0.85, 0.86, 0.88, 0.79, 0.76 \end{array}\right\}, \\ \left\{\begin{array}{l} 0.23, 0.32, 0.34, 0.40, 0.55, 0.59, 0.42, 0.58, 0.61, \\ 0.29, 0.43, 0.51, 0.70, 0.75, 0.54, 0.73, 0.78 \end{array}\right\} \end{array} \right\rangle.$$

This is the required degree of satisfaction and degree of dissatisfaction of the three experts on the decision alternative.

Theorem 6: *Let $\hat{h}_i=\langle\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ be a collection of all PHFNs, and $w=(w_1, w_2,\ldots,w_n)^T$ be the weight vector of $\hat{h}_i(i=1, 2, 3,\ldots,n)$ with $w_i\geq 0$ $(i=1, 2, 3,\ldots,n)$, where $w_i\in[0, 1]$ and $\sum_{i=1}^{n}w_i=1$. Then*

(1) *(Idempotency) If all $\hat{h}_i=\langle\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ are equal, i.e. $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)=\hat{h}$, then*

$$PHFOWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)=\hat{h}. \tag{25}$$

(2) *(Boundedness)*

$$\hat{h}^-\leq PHFOWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)\leq\hat{h}^+, \tag{26}$$

where $\hat{h}^-=\langle h^-, h^{+'}\rangle$, $\hat{h}^+=\langle h^+, h^{-'}\rangle$, $h^-=\bigcup_{h_i\in\Lambda_{\hat{h}_i}}\min_i\{h_i\}$,

$$h^+=\bigcup_{h_i\in\Lambda_{\hat{h}_i}}\max_i\{h_i\},\ \ h^{-'}=\bigcup_{h'_i\in\Gamma_{\hat{h}_i}}\min_i\{h'_i\},\ \ h^{+'}=\bigcup_{h'_i\in\Gamma_{\hat{h}_i}}\max_i\{h'_i\}.$$

(3) *(Monotonicity) If $\hat{h}_i>\hat{h}_i^*$, then*

$$PHFOWA(\hat{h}_1, \hat{h}_2,\ldots,\hat{h}_n)\leq PHFOWA(\hat{h}_1^*, \hat{h}_2^*,\ldots,\hat{h}_n^*). \tag{27}$$

Proof. Proof of the theorem follows from Theorem 2. □

Definition 15: Let $\hat{h}_i=\langle\Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle$ $(i=1, 2, 3,\ldots,n)$ be a collection of all PHFNs, $\hat{h}_{\sigma(i)}$ be the largest in them, $w=(w_1, w_2,\ldots,w_n)$ be the weight vector of $\hat{h}_i$ $(i=1, 2, 3,\ldots,n)$ with $w_i\geq 0$ $(i=1, 2, 3,\ldots,n)$ such that $w_i\in[0, 1]$ and $\sum_{i=1}^{n}w_i=1$. Then, the PHFOWG operator is a mapping $PHFOWG$:$PHFN^n\rightarrow PHFN$ can be defined by

$$PHFOWG(\hat{h}_1, \hat{h}_2, \ldots,\hat{h}_n)=\hat{h}^{w_1}_{\sigma(1)}\otimes\hat{h}^{w_2}_{\sigma(2)}\otimes, \ldots, \otimes\hat{h}^{w_n}_{\sigma(n)}. \tag{28}$$

Theorem 7: *Let* $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \rangle$ $(i=1, 2, 3, \ldots, n)$ *be a collection of all PHFNs,* $\hat{h}_{\sigma(i)}$ *be the largest in them,* $w=(w_1, w_2, \ldots, w_n)$ *be the weight vector of* $\hat{h}_i$ $(i=1, 2, 3, \ldots, n)$ *with* $w_i \geq 0$ $(i=1, 2, 3, \ldots, n)$ *such that* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1.$ *Then, the aggregation result using PHFOWG operator is also a PHFNS, and*

$$PHFOWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \left\langle \begin{array}{l} \bigcup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}} \in \Lambda_{\hat{h}_{\sigma(n)}}} \left\{ \prod_{i=1}^{n} (h_{\hat{h}_{\sigma(i)}})^{w_i} \right\}, \\ \bigcup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(n)}} \in \Gamma_{\hat{h}_{\sigma(n)}}} \left\{ \sqrt{1 - \prod_{i=1}^{n} (1 - h'^{2}_{\hat{h}_{\sigma(i)}})^{w_i}} \right\} \end{array} \right\rangle. \tag{29}$$

Proof. Proof of the theorem is the same as Theorem 3. □

Example 2: To compute the comprehensive evaluation of the three experts (Example 2) on the decision alternative through using the Pythagorean hesitant fuzzy order weighted geometric operator, we have

$$PHFOWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \left\langle \begin{array}{l} \bigcup_{h_{\hat{h}_{\sigma(1)}} \in \Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}} \in \Lambda_{\hat{h}_{\sigma(2)}}, h_{\hat{h}_{\sigma(3)}} \in \Lambda_{\hat{h}_{\sigma(3)}}} \left\{ \prod_{i=1}^{3} (h_{\hat{h}_{\sigma(i)}})^{w_i} \right\}, \\ \bigcup_{h'_{\hat{h}_{\sigma(1)}} \in \Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}} \in \Gamma_{\hat{h}_{\sigma(2)}}, h'_{\hat{h}_{\sigma(3)}} \in \Gamma_{\hat{h}_{\sigma(3)}}} \left\{ \sqrt{1 - \prod_{i=1}^{3} (1 - h'^{2}_{\hat{h}_{\sigma(i)}})^{w_i}} \right\} \end{array} \right\rangle.$$

First, we calculate the score functions of $\hat{h}_1$, $\hat{h}_2$ and $\hat{h}_3$. For this, we have

$$S(\hat{h}_1) = 0.928,\ S(\hat{h}_2) = -0.117,\ S(\hat{h}_3) = 0.1192.$$

Thus, $S(\hat{h}_1) > S(\hat{h}_3) > S(\hat{h}_2)$. Now

$$PHFOWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \left\langle \begin{array}{c} \left\{ \begin{array}{c} 0.36,\ 0.43,\ 0.46,\ 0.40,\ 0.48,\ 0.51,\ 0.50,\ 0.60,\ 0.64, \\ 0.56,\ 0.66,\ 0.71,\ 0.38,\ 0.49,\ 0.55,\ 0.54,\ 0.65,\ 0.69, \\ 0.59,\ 0.76,\ 0.44,\ 0.52,\ 0.63,\ 0.62,\ 0.74,\ 0.79,\ 0.68, \\ 0.81,\ 0.81 \end{array} \right\} \\ \left\{ \begin{array}{c} 0.24,\ 0.44,\ 0.61,\ 0.63,\ 0.68,\ 0.76,\ 0.71,\ 0.82,\ 0.41, \\ 0.54,\ 0.67,\ 0.66,\ 0.72,\ 0.75,\ 0.79,\ 0.84 \end{array} \right\} \end{array} \right\rangle.$$

This is the required degree of satisfaction and degree of dissatisfaction of the three experts on the decision alternative.

Theorem 8: *Let* $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \rangle$ $(i=1, 2, 3, \ldots, n)$ *be a collection of all PHFNs, and* $w=(w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* $\hat{h}_i$ $(i=1, 2, 3, \ldots, n)$ *with* $w_i \geq 0$ $(i=1, 2, 3, \ldots, n)$, *where* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1.$ *Then*

(1) *(Idempotency) If all* $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i} \rangle$ $(i=1, 2, 3, \ldots, n)$ *are equal, i.e.* $\hat{h}_i$ $(i=1, 2, 3, \ldots, n) = \hat{h}$, *then*

$$PHFOWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) = \hat{h}. \tag{30}$$

(2) *(Boundedness)*

$$\hat{h}^- \leq PHFOWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \leq \hat{h}^+, \tag{31}$$

where $\hat{h}^- = \langle h^-, h^{+'} \rangle$, $\hat{h}^+ = \langle h^+, h^{-'} \rangle$, $h^- = \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \min_i \{h_i\}$, $h^+ = \bigcup_{h_i \in \Lambda_{\hat{h}_i}} \max_i \{h_i\}$, $h^{-'} = \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \min_i \{h'_i\}$, $h^{+'} = \bigcup_{h'_i \in \Gamma_{\hat{h}_i}} \max_i \{h'_i\}$.

(3) *(Monotonicity) If* $\hat{h}_i > \hat{h}_i^*$, *then*

$$PHFOWG(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \leq PHFOWG(\hat{h}_1^*, \hat{h}_2^*, \ldots, \hat{h}_n^*). \tag{32}$$

Proof. Proof of the theorem follows from Theorem 2. □

4 Group Decision Making Based on Pythagorean Hesitant Fuzzy Information

In this section, we apply the Pythagorean hesitant fuzzy aggregation operators to multi-attribute decision making with anonymity. Suppose that there are n alternatives $X=\{x_1, x_2,\dots,x_n\}$ and m attributes $A=\{A_1, A_2,\dots,A_m\}$ to be evaluated having the weight vector $w=(w_1, w_2,\dots,w_m)^T$ such that $w_j\in[0, 1]$, $j=1, 2,\dots,m$ and $\sum_{j=1}^{m} w_j=1$. To evaluate the performance of the alternative x_i under the attributes A_j, the decision maker is required to provide not only the information that the alternative x_i satisfies the attributes A_j, but also the information that the alternative x_i does not satisfy the attributes A_j. This two-part information can be expressed by Λ_{ij} and Γ_{ij}, which denote the degrees that the alternative x_i satisfy the criterion A_j and does not satisfy the criterion A_j, then the performance of the alternative x_i under the criteria A_j can be expressed by a PHFN $\hat{h}_{ij}=\langle\Lambda_{ij}, \Gamma_{ij}\rangle$ with the condition that for all $h_{ij}\in\Lambda_{ij}$, $\exists\ h'_{ij}\in\Gamma_{ij}$ such that $0\le(h_{ij})^2+(h'_{ij})^2\le 1$, and for all $h_{ij}\in\Gamma_{ij}$, $\exists\ h'_{ij}\in\Lambda_{ij}$ such that $0\le(h_{ij})^2+(h'_{ij})^2\le 1$, $i=1, 2,\dots,n$, $j=1, 2,\dots,m$, and $k=1, 2,\dots,t$. To obtain the ranking of the alternatives, the following steps are given.

Step 1: In this step, we construct the Pythagorean hesitant fuzzy decision matrices $C=(\hat{h}_{ij})_{m\times n}$ for the decision where $\hat{h}_{ij}=\langle\Lambda_{ij}, \Gamma_{ij}\rangle$ $(i=1, 2,\dots,n; j=1, 2,\dots,m)$.

If the attribute has two types, such as cost and benefit attributes, then the Pythagorean hesitant decision matrix can be converted into the normalized Pythagorean hesitant fuzzy decision matrix

$$D_N=(\gamma_{ij})_{m\times n}, \text{ where } \gamma_{ij}=\begin{cases}\hat{h}_{ij} & \text{if the attribute is of benefit type}\\ \hat{h}^c_{ij} & \text{if the attribute is of cost type}\end{cases},$$

where $\hat{h}^c_{ij}=\langle\Gamma_{ij}, \Lambda_{ij}\rangle$ $(i=1, 2,\dots,n; j=1, 2,\dots,m)$. If all the attributes have the same type, then there is no need to normalize the decision matrix.

Step 2: Utilize the developed aggregation operators to obtain the PHFN $\hat{h}_i(i=1, 2,\dots,n)$ for the alternatives X_i, that is the developed operators to derive the collective overall preference values $\hat{h}_i$ $(i=1, 2,\dots,n)$ of the alternative x_i, where $w=(w_1, w_2,\dots,w_n)^T$ is the weighting vector of the attributes.

Step 3: By using Eq. (7), we calculate the scores $S(\hat{h}_i)$ $(i=1, 2,\dots,n)$ and the deviation degree $\bar{\sigma}(\hat{h}_i)$ $(i=1, 2,\dots,n)$ of all the overall values $\hat{h}_i$ $(i=1, 2,\dots,n)$.

Step 4: Rank the alternatives x_i $(i=1, 2,\dots,n)$ and then select the best one.

5 Illustrative Example

In this section, we present a numerical example considering the air-conditioning system selection problem, which will be installed in a university library under Pythagorean fuzzy contexts to demonstrate the applicability and the implementation process of the proposed method. A university is planning to build a library. One of the problems faced by higher authority is to determine what kind of air-conditioning system should be installed in the library. Suppose that there exist five air-conditioning systems (alternatives) X_i $(i=1, 2, 3, 4, 5)$, which might be adapted to the physical structure of the library. Suppose that three attributes (factors), A_1 = economic, A_2 = functional, A_3 = operational, are taken into consideration in the installation problem, where A_1 is the attribute of cost type. The five alternatives are to be evaluated using PHFNs by the decision makers under the above attributes whose weighting vector $w=(0.25, 0.4, 0.35)^T$ are installed, respectively. In the following, we utilize the developed method to get the desire air-conditioning system.

Step 1: In order to avoid manipulating each other, the decision makers are required to provide their preferences in anonymity and the decision matrix $C=(\hat{h}_{ij})_{m\times n}$ is presented in Table 1, where $\hat{h}_{ij}$ $(i=1, 2, 3, 4, 5; j=1, 2, 3)$ are in the form of PHFNs.

Step 2: We utilize the decision information given in matrix, $D_N=(\hat{h}_{ij})_{m\times n}$ and the PHFWA operator to obtain the overall preference values $\hat{h}_i$ of the air-conditioning system X_i $(i=1, 2, 3, 4, 5)$. We have

$$\hat{h}_1=\left\langle \begin{matrix} \{0.4611, 0.5140, 0.5513, 0.4873, 0.5361, 0.5708, 0.5275, 0.5706, \\ 0.6016, 0.5487, 0.5889, 0.6181\}, \{0.5894, 0.7456, 0.61689, 0.7804\} \end{matrix} \right\rangle.$$

$$\hat{h}_2=\langle\{0.4355, 0.5653, 0.4841, 0.5976\}, \{0.5934, 0.6742, 0.7334\}\rangle.$$

$$\hat{h}_3=\left\langle \begin{matrix} \{0.5881, 0.6931, 0.6395, 0.7283\}, \{0.5850, 0.6236, \\ 0.6222, 0.6632, 0.6049, 0.6447, 0.6434, 0.6858\} \end{matrix} \right\rangle.$$

$$\hat{h}_4=\langle\{0.2771, 0.4007, 0.3103, 0.4223\}, \{0.8003, 0.8386, 0.8242, 0.8637\}\rangle.$$

$$\hat{h}_5=\langle\{0.3321, 0.3870\}, \{0.8337, 0.9\}\rangle.$$

Step 3: Calculate the scores $S(\hat{h}_i)$ $(1, 2, 3, 4, 5)$ of the overall PHFNs $\hat{h}_i$ $(1, 2, 3, 4, 5)$:

$$S(\hat{h}_1)=-0.1663,\ S(\hat{h}_2)=-0.17384,\ S(\hat{h}_3)=0.0365,\ S(\hat{h}_4)=-0.5674,\ S(\hat{h}_5)=-0.6574.$$

Step 4: Rank all the alternatives X_i $(i=1, 2, 3, 4, 5)$ in accordance with the scores $S(\hat{h}_i)$ $(1, 2, 3, 4, 5)$ of the overall Pythagorean hesitant fuzzy preference numbers. We have $S(\hat{h}_3)>S(\hat{h}_1)>S(\hat{h}_2)>S(\hat{h}_4)>S(\hat{h}_5)$, which shows that $X_3>X_1>X_2>X_4>X_5$. That is, the most desirable air-conditioning system is X_3.

Next, we apply the PHFWG operator to the same problem and give steps of our proposed algorithm and start from step 2.

Step 2: We utilize the decision information given in matrix $D=(\hat{h}_{ij})_{m\times n}$ and the PHFWG operator to obtain the overall preference values $\hat{h}_i$ of the air-conditioning system X_i $(i=1, 2, 3, 4, 5)$. We have

$$\hat{h}_1=\left\langle \begin{matrix} \{0.3708, 0.4434, 0.4726, 0.4160, 0.4974, 0.5302, 0.3834, 0.4584, \\ 0.4886, 0.4301, 0.5143, 0.5482\}, \{0.6480, 0.8155, 0.6652, 0.8235\} \end{matrix} \right\rangle.$$

$$\hat{h}_2=\langle\{0.3946, 0.4799, 0.4101, 0.4988\}, \{0.7550, 0.7683, 0.7925\}\rangle.$$

$$\hat{h}_3=\left\langle \begin{matrix} \{0.4812, 0.5399, 0.6333, 0.7105\}, \{0.6707, 0.6287, \\ 0.6452, 0.6692, 0.6435, 0.6676, 0.6819, 0.7028\} \end{matrix} \right\rangle.$$

$$\hat{h}_4=\langle\{0.2144, 0.2470, 0.2521, 0.2905\}, \{0.8277, 0.8492, 0.8553, 0.8731\}\rangle.$$

$$\hat{h}_5=\langle\{0.2462, 0.2692\}, \{0.8815, 0.9000\}\rangle.$$

Step 3: Calculate the scores $S(\hat{h}_i)$ $(1, 2, 3, 4, 5)$ of the overall PHFNs $\hat{h}_i$ $(1, 2, 3, 4, 5)$:

$$S(\hat{h}_1)=-0.3306,\ S(\hat{h}_2)=-0.3971,\ S(\hat{h}_3)=-0.0910,\ S(\hat{h}_4)=-0.6618,\ S(\hat{h}_5)=-0.7270.$$

Table 1: Pythagorean Hesitant Fuzzy Decision Matrix *C*.

	A_1	A_2	A_3
X_1	⟨{0.5, 0.6}, {0.7, 0.8}⟩	⟨{0.3, 0.4}, {0.5, 0.9}⟩	⟨{0.3, 0.5, 0.6}, {0.8}⟩
X_2	⟨{0.3, 0.5, 0.7}, {0.6, 0.7}⟩	⟨{0.3}, {0.9}⟩	⟨{0.4, 0.7}, {0.6}⟩
X_3	⟨{0.7, 0.8}, {0.2, 0.6}⟩	⟨{0.6, 0.8}, {0.6, 0.7}⟩	⟨{0.7}, {0.5, 0.6}⟩
X_4	⟨{0.8, 0.9}, {0.1}⟩	⟨{0.2, 0.3}, {0.9}⟩	⟨{0.4, 0.6}, {0.7, 0.8}⟩
X_5	⟨{0.8, 0.9}, {0.4}⟩	⟨{0.3, 0.4}, {0.9}⟩	⟨{0.1}, {0.9}⟩

Step 4: Rank all the alternatives X_i $(i=1, 2, 3, 4, 5)$ in accordance with the scores $S(\hat{h}_i)$ $(1, 2, 3, 4, 5)$ of the overall Pythagorean hesitant fuzzy preference numbers. We have $S(\hat{h}_3) > S(\hat{h}_1) > S(\hat{h}_2) > S(\hat{h}_4) > S(\hat{h}_5)$, which shows that $X_3 > X_1 > X_2 > X_4 > X_5$. That is, the most desirable air-conditioning system is X_3.

Moreover, we apply the PHFOWA operator to the problem and the order decision matrix is as follows.

Step 2: We utilize the decision information given in matrix $C=(\hat{h}_{ij})_{m\times n}$ and the PHFOWA operator to obtain the overall preference values $\hat{h}_i$ of the air-conditioning system X_i $(i=1, 2, 3, 4, 5)$. We have

$$\hat{h}_1 = \left\langle \begin{array}{c} \{0.4611, 0.5140, 0.5513, 0.4873, 0.5361, 0.5708, 0.5275, 0.5706, \\ 0.6016, 0.5487, 0.5888, 0.6181\}, \{0.5894, 0.7456, 0.6169, 0.7804\} \end{array} \right\rangle.$$
$$\hat{h}_2 = \langle \{0.4393, 0.5822, 0.4873, 0.6127\}, \{0.5815, 0.6607, 0.7187\} \rangle.$$
$$\hat{h}_3 = \left\langle \begin{array}{c} \{0.5506, 0.6287, 0.6681, 0.7209\}, \{0.6051, 0.6340, \\ 0.6435, 0.6743, 0.6333, 0.6636, 0.6735, 0.7058\} \end{array} \right\rangle.$$
$$\hat{h}_4 = \langle \{0.2450, 0.2783, 0.3490, 0.3716\}, \{0.8063, 0.8452, 0.8337, 0.8739\} \rangle.$$
$$\hat{h}_5 = \langle \{0.3321, 0.3870\}, \{0.8739, 0.9\} \rangle.$$

Step 3: Calculate the scores $S(\hat{h}_i)$ $(1, 2, 3, 4, 5)$ of the overall PHFNs $\hat{h}_i$ $(1, 2, 3, 4, 5)$

$$S(\hat{h}_1) = -0.1663,\ S(\hat{h}_2) = -0.1459,\ S(\hat{h}_3) = 0.2221,\ S(\hat{h}_4) = -0.6085,\ S(\hat{h}_5) = -0.6574.$$

Step 4: Rank all the alternatives X_i $(i=1, 2, 3, 4, 5)$ in accordance with the scores $S(\hat{h}_i)$ $(1, 2, 3, 4, 5)$ of the overall Pythagorean hesitant fuzzy preference numbers. We have $S(\hat{h}_3) > S(\hat{h}_1) > S(\hat{h}_2) > S(\hat{h}_4) > S(\hat{h}_5)$, which shows that $X_3 > X_1 > X_2 > X_4 > X_5$. That is, the most desirable air-conditioning system is X_3.

Finally, we apply the PHFOWG operator to the problem and the order decision matrix is in Table 2.

Step 2: We utilize the decision information given in matrix $C=(\hat{h}_{ij})_{m\times n}$ and the PHFOWG operator to obtain the overall preference values $\hat{h}_i$ of the air-conditioning system X_i $(i=1, 2, 3, 4, 5)$. We have

$$\hat{h}_1 = \left\langle \begin{array}{c} \{0.3708, 0.4434, 0.4726, 0.4160, 0.4974, 0.5302, 0.3834, 0.4584, \\ 0.4886, 0.4301, 0.5143, 0.5482\}, \{0.6480, 0.8155, 0.6652, 0.8235\} \end{array} \right\rangle.$$
$$\hat{h}_2 = \langle \{0.4003, 0.5007, 0.41560, 0.5204\}, \{0.7370, 0.7515, 0.7776\} \rangle.$$
$$\hat{h}_3 = \left\langle \begin{array}{c} \{0.4245, 0.6236, 0.4763, 0.6996\}, \{0.6205, 0.6750, \\ 0.6621, 0.7091, 0.6395, 0.6905, 0.6784, 0.7226\} \end{array} \right\rangle.$$
$$\hat{h}_4 = \langle \{0.1866, 0.2195, 0.2065, 0.2429\}, \{0.8342, 0.8699, 0.8492, 0.8815\} \rangle.$$
$$\hat{h}_5 = \langle \{0.2462, 0.2692\}, \{0.8815, 0.9000\} \rangle.$$

Step 3: Calculate the scores $S(\hat{h}_i)$ $(1, 2, 3, 4, 5)$ of the overall PHFNs $\hat{h}_i$ $(1, 2, 3, 4, 5)$

$$S(\hat{h}_1) = -0.3306,\ S(\hat{h}_2) = -0.3597,\ S(\hat{h}_3) = -0.1461,\ S(\hat{h}_4) = -0.6817,\ S(\hat{h}_5) = -0.7270.$$

Table 2: Normalized Pythagorean Hesitant Fuzzy Decision Matrix D_N.

	A_1	A_2	A_3
X_1	⟨{0.7, 0.8}, {0.5, 0.6}⟩	⟨{0.3, 0.4}, {0.5, 0.9}⟩	⟨{0.3, 0.5, 0.6}, {0.8}⟩
X_2	⟨{0.6, 0.7}, {0.3, 0.5, 0.7}⟩	⟨{0.3}, {0.9}⟩	⟨{0.4, 0.7}, {0.6}⟩
X_3	⟨{0.2, 0.6}, {0.7, 0.8}⟩	⟨{0.6, 0.8}, {0.6, 0.7}⟩	⟨{0.7}, {0.5, 0.6}⟩
X_4	⟨{0.1}, {0.8, 0.9}⟩	⟨{0.2, 0.3}, {0.9}⟩	⟨{0.4, 0.6}, {0.7, 0.8}⟩
X_5	⟨{0.4}, {0.8, 0.9}⟩	⟨{0.3, 0.4}, {0.9}⟩	⟨{0.1}, {0.9}⟩

Step 4: Rank all the alternatives X_i $(i=1, 2, 3, 4, 5)$ in accordance with the scores $S(\hat{h}_i)$ (1, 2, 3, 4, 5) of the overall Pythagorean hesitant fuzzy preference numbers. We have $S(\hat{h}_3)>S(\hat{h}_1)>S(\hat{h}_2)>S(\hat{h}_4)>S(\hat{h}_5)$, which shows that $X_3>X_1>X_2>X_4>X_5$. That is, the most desirable air-conditioning system is X_3.

6 Comparison Analysis

In order to verify the validity and effectiveness of the proposed approach, a comparative study is conducted using the methods of HFNs by Torra [22] and IHFSs by Peng et al. [20], which are special cases of PHFNs, to the same illustrative example (Table 3).

For comparison with the HFNs, the PHFNs can be transformed to HFNs by retaining only the membership degrees. The hesitant fuzzy information is represented in Table 4.

The ordered hesitant fuzzy decision matrix is given in Table 5.

Using the HFOWA operator [23], the score values are as follows:

$$S(\hat{h}_1)=0.5326,\quad S(\hat{h}_2)=0.5145,\quad S(\hat{h}_3)=0.6268,\quad S(\hat{h}_4)=0.2907,\quad S(\hat{h}_5)=0.2865.$$

The ranking of the alternatives is $X_3>X_1>X_2>X_4>X_5$.

Using the HFOWG operator [23], the score values are as follows:

$$S(\hat{h}_1)=0.4746,\quad S(\hat{h}_2)=0.4594,\quad S(\hat{h}_3)=0.5564,\quad S(\hat{h}_4)=0.2213,\quad S(\hat{h}_5)=0.2329.$$

The ranking of the alternatives is $X_3>X_1>X_2>X_5>X_4$.

Table 3: Ordered Pythagorean Hesitant Fuzzy Decision Matrix.

	A_1	A_2	A_3
X_1	⟨{0.7, 0.8}, {0.5, 0.6}⟩	⟨{0.3, 0.4}, {0.5, 0.9}⟩	⟨{0.3, 0.5, 0.6}, {0.8}⟩
X_2	⟨{0.6, 0.7}, {0.3, 0.5, 0.7}⟩	⟨{0.4, 0.7}, {0.6}⟩	⟨{0.3}, {0.9}⟩
X_3	⟨{0.7}, {0.5, 0.6}⟩	⟨{0.6, 0.8}, {0.6, 0.7}⟩	⟨{0.2, 0.6}, {0.7, 0.8}⟩
X_4	⟨{0.4, 0.6}, {0.7, 0.8}⟩	⟨{0.2, 0.3}, {0.9}⟩	⟨{0.1}, {0.8, 0.9}⟩
X_5	⟨{0.4, 0.5}, {0.9}⟩	⟨{0.1}, {0.9}⟩	⟨{0.4}, {0.8, 0.9}⟩

Table 4: Hesitant Fuzzy Decision Matrix.

	A_1	A_2	A_3
X_1	{0.7, 0.8}	{0.3, 0.4}	{0.3, 0.5, 0.6}
X_2	{0.6, 0.7}	{0.3}	{0.4, 0.7}
X_3	{0.2, 0.6}	{0.6, 0.8}	{0.7}
X_4	{0.1}	{0.2, 0.3}	{0.4, 0.6}
X_5	{0.4}	{0.3, 0.4}	{0.1}

Table 5: Ordered Hesitant Fuzzy Decision Matrix.

	A_1	A_2	A_3
X_1	{0.7, 0.8}	{0.3, 0.5, 0.6}	{0.3, 0.4}
X_2	{0.6, 0.7}	{0.4, 0.7}	{0.3}
X_3	{0.6, 0.8}	{0.7}	{0.2, 0.6}
X_4	{0.4, 0.6}	{0.2, 0.3}	{0.1}
X_5	{0.4}	{0.3, 0.4}	{0.1}

Table 6: Intuitionistic Hesitant Fuzzy Decision Matrix.

	A_1	A_2	A_3
X_1	⟨{0.3, 0.4}, {0.5, 0.6}⟩	⟨{0.3, 0.4}, {0.5, 0.6}⟩	⟨{0.3, 0.5, 0.6}, {0.4}⟩
X_2	⟨{0.3, 0.4}, {0.3, 0.5, 0.6}⟩	⟨{0.3}, {0.7}⟩	⟨{0.4, 0.5}, {0.5}⟩
X_3	⟨{0.2, 0.6}, {0.3, 0.4}⟩	⟨{0.6, 0.8}, {0.1, 0.2}⟩	⟨{0.6}, {0.4, 0.5}⟩
X_4	⟨{0.1}, {0.8, 0.9}⟩	⟨{0.2, 0.3}, {0.9}⟩	⟨{0.4, 0.5}, {0.5, 0.6}⟩
X_5	⟨{0.4}, {0.5, 0.6}⟩	⟨{0.3, 0.4}, {0.6}⟩	⟨{0.1}, {0.9}⟩

Table 7: Ordered Intuitionistic Hesitant Fuzzy Decision Matrix.

	A_1	A_2	A_3
X_1	⟨{0.5, 0.6}, {0.3, 0.4}⟩	⟨{0.3, 0.5, 0.6}, {0.4}⟩	⟨{0.3, 0.4}, {0.5, 0.6}⟩
X_2	⟨{0.4, 0.5}, {0.5}⟩	⟨{0.3, 0.4}, {0.3, 0.5, 0.6}⟩	⟨{0.3}, {0.7}⟩
X_3	⟨{0.6, 0.8}, {0.1, 0.2}⟩	⟨{0.6}, {0.4, 0.5}⟩	⟨{0.2, 0.6}, {0.3, 0.4}⟩
X_4	⟨{0.4, 0.5}, {0.5, 0.6}⟩	⟨{0.2, 0.3}, {0.9}⟩	⟨{0.1}, {0.8, 0.9}⟩
X_5	⟨{0.4}, {0.5, 0.6}⟩	⟨{0.3, 0.4}, {0.6}⟩	⟨{0.1}, {0.9}⟩

Obviously, the ranking of the alternatives is the same as derived from the proposed method. However, PHFSs are more suitable than HFSs. The main reason is that HFNs only consider the membership degrees of an element and ignore the non-membership degrees, which may result in information distortion and loss.

For comparison with the IHFN, the PHFN can be transformed to IHFN by restricting the square sum of its membership and non-membership degree to ≤1; the sum of membership degrees and non-membership degree is ≤1; and the intuitionistic hesitant fuzzy information is represented in Table 6. The ordered intuitionistic hesitant fuzzy decision matrix is given in Table 7.

Using the IHFOWA operator [20], the score values are as follows:

$$S(\hat{h}_1)=0.0257,\ S(\hat{h}_2)=-0.1809,\ S(\hat{h}_3)=0.2727,\ S(\hat{h}_4)=-0.5179,\ S(\hat{h}_5)=-0.3896.$$

The ranking of the alternatives is $X_3>X_1>X_2>X_5>X_4$.

Using the IHFOWG operator [20], the score values are as follows:

$$S(\hat{h}_1)=-0.0134,\ S(\hat{h}_2)=-0.2219,\ S(\hat{h}_3)=0.1719,\ S(\hat{h}_4)=-0.6257,\ S(\hat{h}_5)=-0.5139.$$

The ranking of the alternatives is, $X_3>X_1>X_2>X_4>X_5$.

Obviously, the ranking of the alternatives is the same as derived from the proposed method. However, PHFSs are more suitable than IHFSs. The main reason for this is that IHFNs consider the membership degrees and non-membership degrees, which satisfy the condition that the sum of its membership and non-membership degree is ≤1, while in the proposed approach the square sum of membership degree and non-membership degree is ≤1.

7 Conclusion

PHFS is a very powerful tool for dealing with uncertainty and fuzziness. It was proposed by Khan et al. [9]. Therefore, in this paper, we developed a multi-attribute decision making approach under Pythagorean hesitant fuzzy information. We discussed some properties of the developed operators, namely PHFWA operator and PHFWG operator [9]. Furthermore, we generalized these operators and developed the PHFOWA and PHFOWG operators to solve the decision- making problems with anonymity. By the illustrative example, we have roughly shown the change trends of the results derived by the developed aggregation operators. Finally, a comparison method has been discussed between the proposed approach and existing methods.

In the future, we will introduce the concept of generalized Pythagorean hesitant fuzzy aggregation operators, Pythagorean hesitant fuzzy hybrid aggregation operators, and their generalizations, TOPSIS and TODIM methods, under the Pythagorean hesitant fuzzy environment.

Bibliography

[1] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.* **20** (1986), 87–96.
[2] K. T. Atanassov, *Intuitionistic fuzzy sets: theory and applications*, Physica-Verlag, Heidelberg, 1999.
[3] I. Beg and T. Rashid, Multi-criteria trapezoidal valued intuitionistic fuzzy decision making with Choquet integral based TOPSIS, *OPSEARCH* **51** (2014), 98–9129.
[4] R. E. Bellman and L. A. Zadeh, Decision-making in a fuzzy environment, *Manage. Sci.* **17** (1970), 141–164.
[5] F. E. Boran, S. Gen, M. Kurt and D. Akay, A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method, *Expert Syst. Appl.* **36** (2009), 11363–11368.
[6] S. K. De, R. Biswas and A. R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis, *Fuzzy Sets Syst.* **117** (2001), 209–213.
[7] D. Dubois, The role of fuzzy sets in decision sciences: old techniques and new directions, *Fuzzy Sets Syst.* **184** (2011), 3–28.
[8] H. Garg, A novel accuracy function under interval-valued Pythagorean fuzzy environment for solving multi-criteria decision making problem, *J. Intell. Fuzzy Syst.* **31** (2016), 529–540.
[9] M. S. A. Khan, S. Abdullah, A. Ali, N. Siddiqui and F. Amin, Pythagorean hesitant fuzzy sets and their application to group decision making with incomplete weight information, *J. Intell. Fuzzy Syst.* **33** (2017), 3971–3985.
[10] D. F. Li, Multi-attribute decision making models and methods using intuitionistic fuzzy sets, *J. Comput. Syst. Sci.* **70** (2005), 73–85.
[11] D. Liang, Z. Xu and A. P. Darko, Projection model for fusing the information of Pythagorean fuzzy multi-criteria group decision making based on geometric Bonferroni mean, *Int. J. Intell. Syst.* **32** (2017), 966–987.
[12] H. C. Liao and Z. Xu, Some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision making environment, *J. Intell. Fuzzy Syst.* **26** (2014), 1601–1617.
[13] H. Liao and Z. Xu, Intuitionistic fuzzy hybrid weighted aggregation operators, *Int. J. Intell. Syst.* **29** (2014), 971–993.
[14] H. Liao and Z. Xu, Extended hesitant fuzzy hybrid weighted aggregation operators and their application in decision making, *Soft Comput.* **19** (2015), 2551–2564.
[15] H. Liao, Z. Xu, X. J. Zeng and D. L. Xu, An enhanced consensus reaching process in group decision making with intuitionistic fuzzy preference relations, *Inform. Sci.* **329** (2016), 274–286.
[16] J. Liu and M. Sun, Generalized power average operator of hesitant fuzzy numbers and its application in multiple attribute decision making, *J. Comput. Inform. Syst.* **9** (2013), 3051–3058.
[17] W. Liu and H. C. Liao, A bibliometric analysis of fuzzy decision research during 1970–2015, *Int. J. Fuzzy Syst.* **19** (2017), 1–14.
[18] X. Peng and Y. Yang, Some results for Pythagorean fuzzy sets, *Int. J. Intell. Syst.* **30** (2015), 1133–1160.
[19] J. J. Peng, J. Q. Wang, J. Wang and X. H. Chen, Multi-criteria decision-making approach with hesitant interval valued intuitionistic fuzzy set, *Sci. World J.*, **2014** Article ID 868515 (2014), 22.
[20] J. J. Peng, J. Q. Wang, X. H. Wu, H. Y. Zhang and X. H. Chen, The fuzzy cross-entropy for intuitionistic hesitant fuzzy sets and their application in multi-criteria decision-making, *Int. J. Syst. Sci.* **46** (2015), 2335–2350.
[21] G. Qian, G. H. Wang and X. Q. Feng, Generalized hesitant fuzzy sets and their application in decision support system, *Knowl. Based Syst.* **37** (2013), 357–365.
[22] V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.* **25** (2010), 529–539.
[23] M. Xia and Z. Xu, Hesitant fuzzy information aggregation in decision making, *Int. J. Approx. Reason.* **52** (2011), 395–407.
[24] Z. Xu and X. Zhang, Hesitant fuzzy multi-attribute decision making based on TOPSIS with incomplete weight information, *Knowl. Based Syst.* **52** (2013), 53–64.
[25] Z. S. Xu and H. C. Liao, A survey of approaches to decision making with intuitionistic fuzzy preference relations, *Knowl. Based Syst.* **80** (2015), 131–142.
[26] R. R. Yager, Pythagorean fuzzy subsets, in: *Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting*, pp. 57–61, Edmonton, Canada, 2013.
[27] R. R. Yager, Pythagorean membership grades in multi-criteria decision making, *IEEE Trans. Fuzzy Syst.* **22** (2014), 958–965.
[28] R. R. Yager and A. M. Abbasov, Pythagorean membership grades, complex numbers and decision making, *Int. J Intell. Syst.* **28** (2013), 436–452.
[29] D. Yu and H. C. Liao, Visualization and quantitative research on intuitionistic fuzzy studies, *J. Intell. Fuzzy Syst.* **30** (2016), 3653–3663.
[30] D. Yu, Y. Wu and W. Zhou, Multi-criteria decision making based on Choquet integral under hesitant fuzzy environment, *J. Comput. Inform. Syst.* **7** (2011), 4506–4513.

[31] L. A. Zadeh, Fuzzy sets, *Inf. Control* **8** (1965), 338–353.
[32] Z. Zhang, Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making, *Inform. Sci.* **234** (2013), 150–181.
[33] X. L. Zhang and Z. S. Xu, Extension of TOPSIS to multi-criteria decision making with Pythagorean fuzzy sets, *Int. J. Intell. Syst.* **29** (2014), 1061–1078.
[34] B. Zhu, Z. Xu and M. Xia, Dual hesitant fuzzy sets, *J. Appl. Math.* **2012** (2012), 13, Article ID 879629.