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Some Interval-Valued Pythagorean Fuzzy Einstein Weighted Averaging Aggregation Operators and Their Application to Group Decision Making

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Abstract: In this paper, we introduce the notion of Einstein aggregation operators, such as the interval-valued Pythagorean fuzzy Einstein weighted averaging aggregation operator and the interval-valued Pythagorean fuzzy Einstein ordered weighted averaging aggregation operator. We also discuss some desirable properties, such as idempotency, boundedness, commutativity, and monotonicity. The main advantage of using the proposed operators is that these operators give a more complete view of the problem to the decision makers. These operators provide more accurate and precise results as compared the existing method. Finally, we apply these operators to deal with multiple-attribute group decision making under interval-valued Pythagorean fuzzy information. For this, we construct an algorithm for multiple-attribute group decision making. Lastly, we also construct a numerical example for multiple-attribute group decision making.

Keywords: Interval-valued Pythagorean fuzzy Einstein weighted averaging operator, interval-valued Pythagorean fuzzy Einstein ordered weighted averaging operator, group decision making.

1 Introduction

Atanassov [1] introduced the concept of intuitionistic fuzzy sets (IFSs) characterized by a membership function and a non-membership function. It is more suitable for dealing with fuzziness and uncertainty than the ordinary fuzzy set developed by Zadeh [33] characterized by membership function. In 1986, many scholars [2–6, 22] have done works in the field of IFS and its applications. Particularly, information aggregation is a very crucial research area in IFS theory that has been receiving more and more focus. Xu [23] developed some basic arithmetic aggregation operators, including intuitionistic fuzzy weighted averaging (IFWA) aggregation operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) aggregation operator, and intuitionistic fuzzy hybrid averaging (IFHA) aggregation operator, and applied them to group decision making. Xu and Yager [26] defined some basic geometric aggregation operators, such as intuitionistic fuzzy weighted geometric (IFWG) aggregation operator, intuitionistic fuzzy ordered weighted geometric (IFOWG) aggregation operator, and intuitionistic fuzzy hybrid geometric (IFHG) aggregation operator. In Refs. [24, 25], Chen and Xu familiarized a series of a new types of aggregation operators, such as interval-valued IFWA (IIFWA) aggregation operator, interval-valued IFOWA (IIFOWA) aggregation operator, interval-valued IFHA (IIFHA) aggregation operator, interval-valued IFWG (IIFWG) aggregation operator, interval-valued IFOWG (IIFOWG) aggregation operator, and interval-valued IFHG (IIFHG) aggregation operator. In Refs. [20, 21], Wang and Liu introduced the concept of intuitionistic fuzzy Einstein weighted geometric (IFEWG) aggregation operator, intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) aggregation operator, intuitionistic fuzzy Einstein weighted averaging (IFEWA) aggregation operator, and intuitionistic fuzzy Einstein ordered weighted averaging (IFEOWA) aggregation operator, and applied them to group decision making. In Refs. [29–32], Yu also worked in the field of IFS theory and introduced many aggregation operators and applied them to group decision making. However, there are many cases where the decision maker may provide the degree

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of membership and non-membership of a particular attribute in such a way that their sum is greater than one. Therefore, Yager [27] introduced the concept of another set called Pythagorean fuzzy set. The Pythagorean fuzzy set is a more powerful tool for solving uncertain problems. Like intuitionistic fuzzy aggregation operators, Pythagorean fuzzy aggregation operators have also become an interesting and important area for research, after the advent of the Pythagorean fuzzy set theory. In Ref. [28], Yager and Abbasov introduced the notion of two new Pythagorean fuzzy aggregation operators, such as Pythagorean fuzzy weighted averaging (PFWA) aggregation operator and Pythagorean fuzzy ordered weighted averaging (PFOWA) operator. In Refs. [12–14, 16, 17], Rahman et al. introduced the concept of Pythagorean fuzzy hybrid averaging (PFHA) aggregation operator, Pythagorean fuzzy weighted geometric (PFWG) aggregation operator, Pythagorean fuzzy ordered weighted geometric (PFOWG) operator, Pythagorean fuzzy hybrid geometric (PFHG) aggregation operator, and Pythagorean fuzzy Einstein weighted geometric (PFEWG) operator, and applied them to group decision making. In Refs. [7, 8], Garg introduced the notion of Pythagorean fuzzy Einstein weighted averaging (PFEWA) aggregation operator, Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA) operator, generalized PFEWA (GPFEWA) aggregation operator, generalized PFEOWA (GPFEOWA) aggregation operator, PFWG aggregation operator, PFOWG aggregation operator, PFEWG aggregation operator, Pythagorean fuzzy Einstein ordered weighted geometric (PFEOWG) aggregation operator, GPFEOWA aggregation operator, and generalized PFEOWG (GPFEOWG) aggregation operator, and applied them to group decision making. In Ref. [11], Peng and Yang introduced the notion of interval-valued PFWA (IVPFWA) aggregation operator and interval-valued PFWG (IVPFWG) aggregation operator. In Refs. [15, 18], Rahman et al. introduced the concept of interval-valued PFOWA (IVPFOWA) aggregation operator, interval-valued PFHA (IVPFHWA) aggregation operator, interval-valued PFOWG (IVPFOWG) aggregation operator, and interval-valued Pythagorean fuzzy hybrid weighted geometric operator, and applied them to group decision making. In Refs. [9, 10, 19, 34], many scholars worked in Pythagorean fuzzy set theory and aggregation operators.

Thus, keeping the advantages of these operators, in this paper, we introduce the notion of interval-valued PFEWA (IVPFEWA) aggregation operator and interval-valued PFEOWA (IVPFEOWA) aggregation operator. By comparison with the existing method, it is decided that the method developed in this paper is a good complement to the existing method.

The remainder of this paper is structured as follows. In Section 2, we give some basic definitions and results, which will be used in later sections. In Section 3, we introduce some Einstein operations for interval-valued Pythagorean fuzzy values. In Section 4, we introduce the notion of IVPFEWA and IVPFEOWA aggregation operators. In Section 5, we apply these operators to deal with multiple-attribute group decision-making (MAGDM) problems with Pythagorean fuzzy information. In Section 6, we develop a numerical example. In Section 7, we present our conclusion.

2 Preliminaries

Definition 1 ([11]): Let K be a fixed set, then an IVPFS can be defined as

$$I=\{\langle k,\ u_I(k),\ v_I(k)\rangle \,|\, k\in K\},\tag{1}$$

where

$$u_I(k)=[u_I^a(k),\ u_I^b(k)]\subset[0,\ 1],\tag{2}$$

$$v_I(k)=[v_I^a(k),\ v_I^b(k)]\subset[0,\ 1].\tag{3}$$

As

$$u_I^a(k)=\inf(u_I(k)),\tag{4}$$

$$u_I^b(k)=\sup(u_I(k)),\tag{5}$$

$$v_I^a(k)=\inf(v_I(k)), \tag{6}$$

$$v_I^b(k)=\sup(v_I(k)), \tag{7}$$

and

$$0\le(u_I^b(k))^2+(v_I^b(k))^2\le 1. \tag{8}$$

If

$$\pi_I(k)=[\pi_I^a(k),\ \pi_I^b(k)], \tag{9}$$

then it is called the interval-valued Pythagorean fuzzy index of k to I, where

$$\pi_I^a(k)=\sqrt{1-(u_I^b(k))^2-(v_I^b(k))^2}, \tag{10}$$

$$\pi_I^b(k)=\sqrt{1-(u_I^a(k))^2-(v_I^a(k))^2}. \tag{11}$$

Definition 2 ([11]). Let $\lambda=([u_\lambda, v_\lambda], [x_\lambda, y_\lambda])$ be an IVPFN, then the score function and accuracy function of λ can be defined as

$$S(\lambda)=\frac{1}{2}[(u_\lambda)^2+(v_\lambda)^2-(x_\lambda)^2-(y_\lambda)^2], \tag{12}$$

and

$$H(\lambda)=\frac{1}{2}[(u_\lambda)^2+(v_\lambda)^2+(x_\lambda)^2+(y_\lambda)^2]. \tag{13}$$

If λ_1 and λ_2 are two IVPFNs, then

1. If $S(\lambda_1)<S(\lambda_2)$, then $\lambda_1<\lambda_2$.
2. If $S(\lambda_1)=S(\lambda_2)$, then we have the following three conditions:
   (i) If $H(\lambda_1)=H(\lambda_2)$, then $\lambda_1=\lambda_2$.
   (ii) If $H(\lambda_1)<H(\lambda_2)$, then $\lambda_1<\lambda_2$.
   (iii) If $H(\lambda_1)>H(\lambda_2)$, then $\lambda_1>\lambda_2$.

Definition 3 ([11]): Let $\lambda_j=([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])\ (j=1, 2, \ldots, n)$ be a collection of IVPFVs, and let IVPFWA: $\Theta^n\to\Theta$, if

$$\text{IVPFWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)=\left(\begin{array}{c}\left[\sqrt{1-\prod_{j=1}^{n}(1-(u_{\lambda_j})^2)^{w_j}},\ \sqrt{1-\prod_{j=1}^{n}(1-(v_{\lambda_j})^2)^{w_j}}\right],\\ \left[\prod_{j=1}^{n}(x_{\lambda_j})^{w_j},\ \prod_{j=1}^{n}(y_{\lambda_j})^{w_j}\right]\end{array}\right), \tag{14}$$

where $w=(w_1, w_2, \ldots, w_n)^T$ is the weighted vector of λ_j with $w_j\in[0, 1]$ and $\sum_{j=1}^{n} w_j=1$. Then, IVPFWA is called interval-valued Pythagorean fuzzy weighted averaging operator.

Definition 4 ([18]): Let $\lambda_j=([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])\ (j=1, 2, \ldots, n)$ be a collection of IVPFVs, and let IVPFOWA: $\Theta^n\to\Theta$, if

$$\text{IVPFOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)=\left(\begin{array}{c}\left[\sqrt{1-\prod_{j=1}^{n}(1-(u_{\lambda_{\sigma(j)}})^2)^{w_j}},\ \sqrt{1-\prod_{j=1}^{n}(1-(v_{\lambda_{\sigma(j)}})^2)^{w_j}}\right],\\ \left[\prod_{j=1}^{n}(x_{\lambda_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{n}(y_{\lambda_{\sigma(j)}})^{w_j}\right]\end{array}\right), \tag{15}$$

where $w=(w_1, w_2, \ldots, w_n)^T$ is the weighted vector of λ_j with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and $\lambda_{\sigma(j)}$ is the j^{th} largest value of λ_j. Then, IVPFOWA is called interval-valued Pythagorean fuzzy ordered weighted averaging operator.

Definition 5 ([18]): An IVPFHA operator of dimension n is a mapping IVPFHA: $\Theta^n \to \Theta$, which has an associated vector $w=(w_1, w_2, \ldots, w_n)^T$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore

$$\mathrm{IVPFHA}_{w,w}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = \left(\begin{array}{c} \left[\sqrt{1-\prod_{j=1}^{n}(1-(u_{\dot{\lambda}_{\sigma(j)}})^2)^{w_j}}, \sqrt{1-\prod_{j=1}^{n}(1-(v_{\dot{\lambda}_{\sigma(j)}})^2)^{w_j}} \right], \\ \left[\prod_{j=1}^{n}(x_{\dot{\lambda}_{\sigma(j)}})^{w_j}, \prod_{j=1}^{n}(y_{\dot{\lambda}_{\sigma(j)}})^{w_j} \right] \end{array} \right), \tag{16}$$

where $\dot{\lambda}_{\sigma(j)}$ is the j^{th} largest of the weighted PFVs, $\dot{\lambda}_{\sigma(j)}$ $(\dot{\lambda}_{\sigma(j)} = nw_j\lambda_j)$, and $w=(w_1, w_2, \ldots, w_n)^T$ is the weighted vector of λ_j $(j=1, 2, \ldots, n)$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ and n is the balancing coefficient, which plays a role of balancing. If the vector $(w_1, w_2, \ldots, w_n)^T$ approaches $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the vector $(nw_1\lambda_1, nw_2\lambda_2, \ldots, nw_n\lambda_n)^T$ approaches $(\lambda_1, \lambda_2, \ldots, \lambda_n)^T$.

3 Some Einstein Operations of Interval-Valued Pythagorean Fuzzy Sets

Definition 6: Let $\lambda=([u, v], [x, y])$, $\lambda_1=([u_1, v_1], [x_1, y_1])$, and $\lambda_2=([u_2, v_2], [x_2, y_2])$ be three IVPFNs and $\delta > 0$, then some Einstein operations for $\lambda, \lambda_1, \lambda_2$ can be defined as follows:

$$\lambda_1 \oplus_\varepsilon \lambda_2 = \left(\begin{array}{c} \left[\frac{\sqrt{u_1^2+u_2^2}}{\sqrt{1+u_1^2u_2^2}}, \frac{\sqrt{v_1^2+v_2^2}}{\sqrt{1+v_1^2v_2^2}} \right], \\ \left[\frac{x_1x_2}{\sqrt{1+(1-x_1^2)(1-x_2^2)}}, \frac{y_1y_2}{\sqrt{1+(1-y_1^2)(1-y_2^2)}} \right] \end{array} \right). \tag{17}$$

$$\lambda_1 \otimes_\varepsilon \lambda_2 = \left(\begin{array}{c} \left[\frac{u_1u_2}{\sqrt{1+(1-u_1^2)(1-u_2^2)}}, \frac{v_1v_2}{\sqrt{1+(1-v_1^2)(1-v_2^2)}} \right], \\ \left[\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{1+x_1^2x_2^2}}, \frac{\sqrt{y_1^2+y_2^2}}{\sqrt{1+y_1^2y_2^2}} \right] \end{array} \right). \tag{18}$$

$$\delta\lambda = \left(\begin{array}{c} \left[\frac{\sqrt{(1+u^2)^\delta-(1-u^2)^\delta}}{\sqrt{(1+u^2)^\delta+(1-u^2)^\delta}}, \frac{\sqrt{(1+v^2)^\delta-(1-v^2)^\delta}}{\sqrt{(1+v^2)^\delta+(1-v^2)^\delta}} \right], \\ \left[\frac{\sqrt{2(x^2)^\delta}}{\sqrt{(2-x^2)^\delta+(x^2)^\delta}}, \frac{\sqrt{2(y^2)^\delta}}{\sqrt{(2-y^2)^\delta+(y^2)^\delta}} \right] \end{array} \right). \tag{19}$$

$$\lambda^\delta = \left(\begin{array}{c} \left[\frac{\sqrt{2(u^2)^\delta}}{\sqrt{(2-u^2)^\delta+(u^2)^\delta}}, \frac{\sqrt{2(v^2)^\delta}}{\sqrt{(2-v^2)^\delta+(v^2)^\delta}} \right], \\ \left[\frac{\sqrt{(1+x^2)^\delta-(1-x^2)^\delta}}{\sqrt{(1+x^2)^\delta+(1-x^2)^\delta}}, \frac{\sqrt{(1+y^2)^\delta-(1-y^2)^\delta}}{\sqrt{(1+y^2)^\delta+(1-y^2)^\delta}} \right] \end{array} \right). \tag{20}$$

4 Some Interval-Valued Pythagorean Fuzzy Einstein Averaging Aggregation Operators

In this section, we introduce two interval-valued Einstein aggregation operators, namely the IVPFEWA and IVPFEOWA operators. We also discuss some desirable properties of these proposed operators, such as idempotency, boundedness, commutativity, and monotonicity. These operators provide more accurate and precise results as compared to the existing method.

4.1 IVPFEWA Aggregation Operator

Definition 7. Let $\lambda_j=([u_j, v_j], [x_j, y_j])$ $(j=1, 2, 3, \dots, n)$ be the collection of IVPFVs, then an IVPFEWA operator of dimension n is a mapping $\text{IVPFEWA}_w: \Theta^n \to \Theta$, and

$$\text{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)=\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^n(1+u_{\lambda_j}^2)^{w_j}-\prod_{j=1}^n(1-u_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(1+u_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(1-u_{\lambda_j}^2)^{w_j}}}, \dfrac{\sqrt{\prod_{j=1}^n(1+v_{\lambda_j}^2)^{w_j}-\prod_{j=1}^n(1-v_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(1+v_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(1-v_{\lambda_j}^2)^{w_j}}}\right],\\ \left[\dfrac{\sqrt{2\prod_{j=1}^n(x_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(2-x_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(x_{\lambda_j}^2)^{w_j}}}, \dfrac{\sqrt{2\prod_{j=1}^n(y_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(2-y_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(y_{\lambda_j}^2)^{w_j}}}\right]\end{array}\right), \tag{21}$$

where $w=(w_1, w_2, w_3, \dots, w_n)^T$ is the weighted vector of λ_j such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j=1$.

Theorem 1. *Let* $\lambda_j=([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])$ $(j=1, 2, \dots, n)$ *be the collection of IVPFVs, then their aggregated value by using the IVPFEWA operator is also an IVPFV, and*

$$\text{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)=\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^n(1+u_{\lambda_j}^2)^{w_j}-\prod_{j=1}^n(1-u_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(1+u_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(1-u_{\lambda_j}^2)^{w_j}}}, \dfrac{\sqrt{\prod_{j=1}^n(1+v_{\lambda_j}^2)^{w_j}-\prod_{j=1}^n(1-v_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(1+v_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(1-v_{\lambda_j}^2)^{w_j}}}\right],\\ \left[\dfrac{\sqrt{2\prod_{j=1}^n(x_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(2-x_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(x_{\lambda_j}^2)^{w_j}}}, \dfrac{\sqrt{2\prod_{j=1}^n(y_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^n(2-y_{\lambda_j}^2)^{w_j}+\prod_{j=1}^n(y_{\lambda_j}^2)^{w_j}}}\right]\end{array}\right), \tag{22}$$

where $w=(w_1, w_2, w_3, \dots, w_n)^T$ *is the weighted vector of* λ_j $(j=1, 2, 3, \dots, n)$ *such that* $w_j \in [0, 1]$ $(j=1, 2, 3, \dots, n)$ *and* $\sum_{j=1}^n w_j=1$.

Proof. We can prove this theorem by mathematical induction. First we show that Eq. (22) holds for $n=1$.

Taking the left-hand side,

$$\text{IVPFEWA}_w(\lambda)=\left(\begin{array}{c}\left[\dfrac{\sqrt{(1+u_\lambda^2)^w-(1-u_\lambda^2)^w}}{\sqrt{(1+u_\lambda^2)^w+(1-u_\lambda^2)^w}}, \dfrac{\sqrt{(1+v_\lambda^2)^w-(1-v_\lambda^2)^w}}{\sqrt{(1+v_\lambda^2)^w+(1-v_\lambda^2)^w}}\right],\\ \left[\dfrac{\sqrt{2(x_\lambda^2)^w}}{\sqrt{(2-x_\lambda^2)^w+(x_\lambda^2)^w}}, \dfrac{\sqrt{2(y_\lambda^2)^w}}{\sqrt{(2-y_\lambda^2)^w+(y_\lambda^2)^w}}\right]\end{array}\right). \tag{23}$$

Taking the right-hand side,

$$\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^{n}(1+u_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{n}(1-u_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{n}(1+u_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{n}(1-u_{\lambda_j}^2)^{w_j}}},\ \dfrac{\sqrt{\prod_{j=1}^{n}(1+v_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{n}(1-v_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{n}(1+v_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{n}(1-v_{\lambda_j}^2)^{w_j}}}\right],\\ \left[\dfrac{\sqrt{2\prod_{j=1}^{n}(x_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{n}(2-x_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{n}(x_{\lambda_j}^2)^{w_j}}},\ \dfrac{\sqrt{2\prod_{j=1}^{n}(y_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{n}(2-y_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{n}(y_{\lambda_j}^2)^{w_j}}}\right]\end{array}\right)$$

$$=\left(\begin{array}{c}\left[\dfrac{\sqrt{(1+u_\lambda^2)^w-(1-u_\lambda^2)^w}}{\sqrt{(1+u_\lambda^2)^w+(1-u_\lambda^2)^w}},\ \dfrac{\sqrt{(1+v_\lambda^2)^w-(1-v_\lambda^2)^w}}{\sqrt{(1+v_\lambda^2)^w+(1-v_\lambda^2)^w}}\right],\\ \left[\dfrac{\sqrt{2(x_\lambda^2)^w}}{\sqrt{(2-x_\lambda^2)^w+(x_\lambda^2)^w}},\ \dfrac{\sqrt{2(y_\lambda^2)^w}}{\sqrt{(2-y_\lambda^2)^w+(y_\lambda^2)^w}}\right]\end{array}\right). \tag{24}$$

From Eqs. (23) and (24), we have Eq. (22) holds for $n=1$. Now we show that Eq. (22) holds for $n=k$. That is

$$\mathrm{IVPFEWA}_w(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_k)=\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^{k}(1+u_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{k}(1-u_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(1+u_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{n}(1-u_{\lambda_j}^2)^{w_j}}},\ \dfrac{\sqrt{\prod_{j=1}^{k}(1+v_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{k}(1-v_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(1+v_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(1-v_{\lambda_j}^2)^{w_j}}}\right],\\ \left[\dfrac{\sqrt{2\prod_{j=1}^{k}(x_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(2-x_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(x_{\lambda_j}^2)^{w_j}}},\ \dfrac{\sqrt{2\prod_{j=1}^{k}(y_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(2-y_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(y_{\lambda_j}^2)^{w_j}}}\right]\end{array}\right). \tag{25}$$

If Eq. (25) holds for $n=k$, then we show that Eq. (25) holds for $n=k+1$.

$$\mathrm{IVPFEWA}_w(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_{k+1})=\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^{k}(1+u_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{k}(1-u_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(1+u_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{n}(1-u_{\lambda_j}^2)^{w_j}}},\ \dfrac{\sqrt{\prod_{j=1}^{k}(1+v_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{k}(1-v_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(1+v_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(1-v_{\lambda_j}^2)^{w_j}}}\right],\\ \left[\dfrac{\sqrt{2\prod_{j=1}^{k}(x_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(2-x_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(x_{\lambda_j}^2)^{w_j}}},\ \dfrac{\sqrt{2\prod_{j=1}^{k}(y_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k}(2-y_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(y_{\lambda_j}^2)^{w_j}}}\right]\end{array}\right)$$

$$\oplus_\varepsilon\left(\begin{array}{c}\left[\dfrac{\sqrt{(1+u_{\lambda_{k+1}}^2)^{w_{k+1}}-(1-u_{\lambda_{k+1}}^2)^{w_{k+1}}}}{\sqrt{(1+u_{\lambda_{k+1}}^2)^{w_{k+1}}+(1-u_{\lambda_{k+1}}^2)^{w_{k+1}}}},\ \dfrac{\sqrt{(1+v_{\lambda_{k+1}}^2)^{w_{k+1}}-(1-v_{\lambda_{k+1}}^2)^{w_{k+1}}}}{\sqrt{(1+v_{\lambda_{k+1}}^2)^{w_{k+1}}+(1-v_{\lambda_{k+1}}^2)^{w_{k+1}}}}\right],\\ \left[\dfrac{\sqrt{2(x_{\lambda_{k+1}}^2)^{w_{k+1}}}}{\sqrt{(2-x_{\lambda_{k+1}}^2)^{w_{k+1}}+(x_{\lambda_{k+1}}^2)^{w_{k+1}}}},\ \dfrac{\sqrt{2(y_{\lambda_{k+1}}^2)^{w_{k+1}}}}{\sqrt{(2-y_{\lambda_{k+1}}^2)^{w_{k+1}}+(y_{\lambda_{k+1}}^2)^{w_{k+1}}}}\right]\end{array}\right). \tag{26}$$

Let

$$t_1=\sqrt{\prod_{j=1}^{k}(1+u_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{k}(1-u_{\lambda_j}^2)^{w_j}}.$$

$$t_2=\sqrt{\prod_{j=1}^{k}(1+u_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{n}(1-u_{\lambda_j}^2)^{w_j}}.$$

$$p_1=\sqrt{\prod_{j=1}^{k}(1+v_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{k}(1-v_{\lambda_j}^2)^{w_j}}.$$

$$p_2=\sqrt{\prod_{j=1}^{k}(1+v_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(1-v_{\lambda_j}^2)^{w_j}}.$$

$$w_1=\sqrt{(1+u_{\lambda_{k+1}}^2)^{w_{k+1}}-(1-u_{\lambda_{k+1}}^2)^{w_{k+1}}}.$$

$$w_2=\sqrt{(1+u_{\lambda_{k+1}}^2)^{w_{k+1}}+(1-u_{\lambda_{k+1}}^2)^{w_{k+1}}}.$$

$$a_1=\sqrt{(1+v_{\lambda_{k+1}}^2)^{w_{k+1}}-(1-v_{\lambda_{k+1}}^2)^{w_{k+1}}}.$$

$$a_2=\sqrt{(1+v_{\lambda_{k+1}}^2)^{w_{k+1}}+(1-v_{\lambda_{k+1}}^2)^{w_{k+1}}}.$$

$$r_2=\sqrt{\prod_{j=1}^{k}(2-x_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(x_{\lambda_j}^2)^{w_j}}.$$

$$r_1=\sqrt{2\prod_{j=1}^{k}(x_{\lambda_j}^2)^{w_j}},\ s_1=\sqrt{2\prod_{j=1}^{k}(y_{\lambda_j}^2)^{w_j}}.$$

$$s_2=\sqrt{\prod_{j=1}^{k}(2-y_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{k}(y_{\lambda_j}^2)^{w_j}}.$$

$$b_2=\sqrt{(2-x_{\lambda_{k+1}}^2)^{w_{k+1}}+(x_{\lambda_{k+1}}^2)^{w_{k+1}}}.$$

$$c_1=\sqrt{2(y_{\lambda_{k+1}}^2)^{w_{k+1}}},\ b_1=\sqrt{2(x_{\lambda_{k+1}}^2)^{w_{k+1}}}.$$

$$c_2=\sqrt{(2-y_{\lambda_{k+1}}^2)^{w_{k+1}}+(y_{\lambda_{k+1}}^2)^{w_{k+1}}}.$$

Now putting these values in Eq. (26), we have

$$\mathrm{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{k+1}) = \left(\left[\frac{t_1}{t_2}, \frac{p_1}{p_2}\right], \left[\frac{r_1}{r_2}, \frac{s_1}{s_2}\right]\right) \oplus_\varepsilon \left(\left[\frac{w_1}{w_2}, \frac{a_1}{a_2}\right], \left[\frac{b_1}{b_2}, \frac{c_1}{c_2}\right]\right)$$

$$= \left(\begin{array}{c} \left[\dfrac{\sqrt{(t_1w_2)^2+(t_2w_1)^2}}{\sqrt{(t_2w_2)^2+(t_1w_1)^2}}, \dfrac{\sqrt{(p_1a_2)^2+(a_1p_2)^2}}{\sqrt{(p_2a_2)^2+(p_1a_1)^2}}\right], \\ \left[\dfrac{r_1b_1}{\sqrt{2r_2^2b_2^2+r_1^2b_1^2-r_2^2b_1^2-r_1^2b_2^2}}, \dfrac{s_1c_1}{\sqrt{2s_2^2c_2^2+s_1^2c_1^2-s_2^2c_1^2-s_1^2c_2^2}}\right] \end{array}\right). \tag{27}$$

Again putting the values of $(t_1w_2)^2+(t_2w_1)^2$, $(t_2w_2)^2+(t_1w_1)^2$, $(p_1a_2)^2+(a_1p_2)^2$, $(p_2a_2)^2+(p_1a_1)^2$, r_1b_1, $2r_2^2b_2^2+r_1^2b_1^2-r_2^2b_1^2-r_1^2b_2^2$, s_1c_1, $2s_2^2c_2^2+s_1^2c_1^2-s_2^2c_1^2-s_1^2c_2^2$, in Eq. (27), we have

$$\mathrm{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{k+1}) = \left(\begin{array}{c} \left[\dfrac{\sqrt{\prod_{j=1}^{k+1}(1+u_{\lambda_j}^2)^{w_j} - \prod_{j=1}^{k+1}(1-u_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}(1+u_{\lambda_j}^2)^{w_j} + \prod_{j=1}^{k+1}(1-u_{\lambda_j}^2)^{w_j}}}, \dfrac{\sqrt{\prod_{j=1}^{k+1}(1+v_{\lambda_j}^2)^{w_j} - \prod_{j=1}^{k+1}(1-v_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}(1+v_{\lambda_j}^2)^{w_j} + \prod_{j=1}^{k+1}(1-v_{\lambda_j}^2)^{w_j}}}\right], \\ \left[\dfrac{\sqrt{2\prod_{j=1}^{k+1}(x_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}(2-x_{\lambda_j}^2)^{w_j} + \prod_{j=1}^{k+1}(x_{\lambda_j}^2)^{w_j}}}, \dfrac{\sqrt{2\prod_{j=1}^{k+1}(y_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{k+1}(2-y_{\lambda_j}^2)^{w_j} + \prod_{j=1}^{k+1}(y_{\lambda_j}^2)^{w_j}}}\right] \end{array}\right).$$

Hence, Eq. (22) holds for $n=k+1$. Thus, Eq. (22) holds for all n. □

Lemma 1 ([16]). *Let* $\lambda_j>0$, $w_j>0$ $(j=1, 2, \ldots, n)$ *and* $\sum_{j=1}^n w_j=1$, *then*

$$\prod_{j=1}^n(\lambda_j)^{w_j} \leqslant \sum_{j=1}^n w_j\lambda_j, \tag{28}$$

where the equality holds if and only if $\lambda_1=\lambda_2=\ldots=\lambda_n$.

Theorem 2. *Let* $\lambda_j=([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])$ $(j=1, 2, \ldots, n)$ *be the collection of IVPFVs, where the weighted vector of* λ_j *is* $w=(w_1, w_2, \ldots, w_n)^T$ *such that* $w_j\in[0, 1]$ *and* $\sum_{j=1}^n w_j=1$, *then*

$$\mathrm{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \le \mathrm{IVPFWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n). \tag{29}$$

Proof. Straightforward. □

Example 1. Let

$$\lambda_1=([0.3, 0.4], [0.5, 0.7]), \lambda_2=([0.2, 0.6], [0.3, 0.6]),$$
$$\lambda_3=([0.3, 0.6], [0.3, 0.5]), \lambda_4=([0.4, 0.7], [0.2, 0.6]),$$

and let $w=(0.1, 0.2, 0.3, 0.4)^T$ be the weighted vector of λ_j $(j=1, 2, 3, 4)$, then we have

$$\text{IVPFEWA}_w(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\left(\begin{array}{c}\left[\dfrac{\sqrt{\prod_{j=1}^{4}(1+u_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{4}(1-u_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{4}(1+u_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{4}(1-u_{\lambda_j}^2)^{w_j}}},\dfrac{\sqrt{\prod_{j=1}^{4}(1+v_{\lambda_j}^2)^{w_j}-\prod_{j=1}^{4}(1-v_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{4}(1+v_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{4}(1-v_{\lambda_j}^2)^{w_j}}}\right],\\ \left[\dfrac{\sqrt{2\prod_{j=1}^{4}(x_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{4}(2-x_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{4}(x_{\lambda_j}^2)^{w_j}}},\dfrac{\sqrt{2\prod_{j=1}^{4}(y_{\lambda_j}^2)^{w_j}}}{\sqrt{\prod_{j=1}^{4}(2-y_{\lambda_j}^2)^{w_j}+\prod_{j=1}^{4}(y_{\lambda_j}^2)^{w_j}}}\right]\end{array}\right)$$

$$=([0.3289,\ 0.6293],\ [0.2693,\ 0.5790]).$$

Now

$$\text{IVPFWA}_w(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\left(\begin{array}{c}\left[\sqrt{1-\prod_{j=1}^{4}(1-(u_{\lambda_j})^2)^{w_j}},\ \sqrt{1-\prod_{j=1}^{4}(1-(v_{\lambda_j})^2)^{w_j}}\right],\\ \left[\prod_{j=1}^{4}(x_{\lambda_j})^{w_j},\ \prod_{j=1}^{4}(y_{\lambda_j})^{w_j}\right]\end{array}\right)$$

$$=([0.3306,\ 0.6321],\ [0.2684,\ 0.5768]).$$

Theorem 3. *Commutativity: Let λ_j and λ'_j $(j=1,2,\ldots,n)$ be two collection of IVPFVs, where $(\lambda'_1,\lambda'_2,\ldots,\lambda'_n)$ is any permutation of $(\lambda_1,\lambda_2,\ldots,\lambda_n)$, then*

$$\text{IVPFEWA}_w(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n)=\text{IVPFEWA}_w(\lambda'_1,\lambda'_2,\lambda'_3,\ldots,\lambda'_n). \tag{30}$$

Proof. As we know that

$$\text{IVPFEWA}_w(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n)=w_1\lambda_1\oplus_\varepsilon w_2\lambda_2\oplus_\varepsilon\ \ldots\ \oplus_\varepsilon w_n\lambda_n, \tag{31}$$

and

$$\text{IVPFEWA}_w(\lambda'_1,\lambda'_2,\lambda'_3,\ldots,\lambda'_n)=w_1\lambda'_1\oplus_\varepsilon w_2\lambda'_2\oplus_\varepsilon\ \ldots\ \oplus_\varepsilon w_n\lambda'_n, \tag{32}$$

as $(\lambda'_1,\lambda'_2,\lambda'_3,\ldots,\lambda'_n)$ is any permutation of $(\lambda_1,\lambda_1,\lambda_3,\ldots,\lambda_n)$. Thus, Eq. (33) always holds. □

Theorem 4. *Idempotency: If $\lambda_j=\lambda$ for all j $(j=1,2,3,\ldots,n)$, where $\lambda=([u,v],[x,y])$, then*

$$\text{IVPFEWA}_w(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n)=\lambda. \tag{33}$$

Proof. As $\lambda_j=\lambda$ for all j, then we have

$$\text{IVPFEWA}_w(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n)=w_1\lambda\oplus_\varepsilon w_2\lambda\oplus_\varepsilon w_3\lambda\oplus_\varepsilon\ \ldots\ \oplus_\varepsilon w_n\lambda=(w_1\oplus_\varepsilon w_2\oplus_\varepsilon w_3\oplus_\varepsilon\ \ldots\ \oplus_\varepsilon w_n)\lambda=\lambda.$$

This completes the proof. □

Theorem 5. *Boundedness: Let $\lambda_j=([u_{\lambda_j},v_{\lambda_j}],[x_{\lambda_j},y_{\lambda_j}])$ $(j=1,2,\ldots,n)$ be a collection of IVPFVs and let $w=(w_1, w_2,\ldots,w_n)^T$ be the weighted vector of λ_j, such that $w_j\in[0,1]$, $\sum_{j=1}^{n}w_j=1$, then*

$$\lambda_{\min}\le\text{IVPFEWA}(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n)\le\lambda_{\max}, \tag{34}$$

for all w_j and also

$$\lambda_{\max}=\max_j(\lambda_j), \tag{35}$$

$$\lambda_{\min} = \min_j(\lambda_j). \tag{36}$$

Proof. Let

$$\text{IVPFEWA} = \lambda = ([u, v], [x, y]). \tag{37}$$

Now by the score function, we have

$$([u_{\min}, v_{\min}], [x_{\max}, y_{\max}]) \le ([u, v], [x, y]), \tag{38}$$

$$([u_{\max}, v_{\max}], [x_{\min}, y_{\min}]) \ge ([u, v], [x, y]), \tag{39}$$

as from Eqs. (38) and (39), we have

$$\lambda_{\min} \le \text{IVPFEWA}\ (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \le \lambda_{\max}.$$

Thus, Eq. (34) always holds. □

Theorem 6. *Monotonicity: If $\lambda_j \le \lambda_j'$ for all j, where $j = 1, 2, 3, \ldots, n$, then*

$$\text{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \le \text{IVPFEWA}_w(\lambda_1', \lambda_2', \lambda_3', \ldots, \lambda_n'). \tag{40}$$

Proof. As we know that

$$\text{IVPFEWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = w_1\lambda_1 \oplus_\varepsilon w_2\lambda_2 \oplus_\varepsilon \ldots \oplus_\varepsilon w_n\lambda_n. \tag{41}$$

and

$$\text{IVPFEWA}_w(\lambda_1', \lambda_2', \lambda_3', \ldots, \lambda_n') = w_1\lambda_1' \oplus_\varepsilon w_2\lambda_2' \oplus_\varepsilon \ldots \oplus_\varepsilon w_n\lambda_n', \tag{42}$$

as $\lambda_j \le \lambda_j'$ for all *j*. Thus, Eq. (40) always holds. □

4.2 IVPFEOWA Aggregation Operator

Definition 8: Let λ_j ($j = 1, 2, 3, \ldots, n$) be a collection of IVPFVs, then an IVPFEOWA operator of dimension *n* is a mapping $\text{IVPFEOWA}_w: \Theta^n \to \Theta$, and

$$\text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = \left(\begin{array}{c} \left[\dfrac{\sqrt{\prod_{j=1}^n (1+u^2_{\lambda_{\sigma(j)}})^{w_j} - \prod_{j=1}^n (1-u^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^n (1+u^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (1-u^2_{\lambda_{\sigma(j)}})^{w_j}}}, \dfrac{\sqrt{\prod_{j=1}^n (1+v^2_{\lambda_{\sigma(j)}})^{w_j} - \prod_{j=1}^n (1-v^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^n (1+v^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (1-v^2_{\lambda_{\sigma(j)}})^{w_j}}} \right], \\ \left[\dfrac{\sqrt{2\prod_{j=1}^n (x^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^n (2-x^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (x^2_{\lambda_{\sigma(j)}})^{w_j}}}, \dfrac{\sqrt{2\prod_{j=1}^n (y^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^n (2-y^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^n (y^2_{\lambda_{\sigma(j)}})^{w_j}}} \right] \end{array} \right), \tag{43}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\sigma(j) \leqslant \sigma(j-1)$ for all *j*, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weighted vector of $\lambda_{\sigma(j)}$ ($j = 1, 2, \ldots, n$) such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Theorem 7. *Let $\lambda_j = ([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])$ ($j = 1, 2, \ldots, n$) be the collection of IVPFVs, then their aggregated value by using the IVPFEOWA operator is also an IVPFV, and*

$$\text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = \left(\begin{array}{c} \left[\dfrac{\sqrt{\prod_{j=1}^{n}(1+u^2_{\lambda_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n}(1-u^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^{n}(1+u^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n}(1-u^2_{\lambda_{\sigma(j)}})^{w_j}}}, \dfrac{\sqrt{\prod_{j=1}^{n}(1+v^2_{\lambda_{\sigma(j)}})^{w_j} - \prod_{j=1}^{n}(1-v^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^{n}(1+v^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n}(1-v^2_{\lambda_{\sigma(j)}})^{w_j}}} \right], \\ \left[\dfrac{\sqrt{2\prod_{j=1}^{n}(x^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^{n}(2-x^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n}(x^2_{\lambda_{\sigma(j)}})^{w_j}}}, \dfrac{\sqrt{2\prod_{j=1}^{n}(y^2_{\lambda_{\sigma(j)}})^{w_j}}}{\sqrt{\prod_{j=1}^{n}(2-y^2_{\lambda_{\sigma(j)}})^{w_j} + \prod_{j=1}^{n}(y^2_{\lambda_{\sigma(j)}})^{w_j}}} \right] \end{array} \right), \quad (44)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\sigma(j) \leqslant \sigma(j-1)$ *for all j, and* $w = (w_1, w_2, \ldots, w_n)^T$ *is the weighted vector of* $\lambda_{\sigma(j)}$ $(j = 1, 2, \ldots, n)$ *such that* $w_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ *and* $\sum_{j=1}^{n} w_j = 1$.

Proof. Proof is similar to Theorem 1. □

Theorem 8. *Let* $\lambda_j = ([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])$ $(j = 1, 2, \ldots, n)$ *be a collection of IVPFVs, where the weighted vector of* λ_j *is* $w = (w_1, w_2, w_3, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *then*

$$\text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \leq \text{IVPFOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n). \quad (45)$$

Proof. Straightforward. □

Theorem 9. *Commutativity: If* λ'_j $(j = 1, \ldots, n)$ *is any permutation of* λ_j $(j = 1, \ldots, n)$, *then*

$$\text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = \text{IVPFEOWA}_w(\lambda'_1, \lambda'_2, \lambda'_3, \ldots, \lambda'_n). \quad (46)$$

Proof. Proof is similar to Theorem 3. □

Theorem 10. *Idempotency: If* $\lambda_j = \lambda$ *for all j* $(j = 1, 2, 3, \ldots, n)$, *where* $\lambda = ([u, v], [x, y])$, *then*

$$\text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) = \lambda. \quad (47)$$

Proof. Proof is similar to Theorem 4. □

Theorem 11. *Boundedness: Let* $\lambda_j = ([u_{\lambda_j}, v_{\lambda_j}], [x_{\lambda_j}, y_{\lambda_j}])$ $(j = 1, 2, \ldots, n)$ *be a collection of IVPFVs and let* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weighted vector of* $\lambda_{\sigma(j)}$, *such that* $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, *then*

$$\lambda_{\min} \leq \text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \leq \lambda_{\max}, \quad (48)$$

for all w_j and

$$\lambda_{\max} = \max_j(\lambda_j), \quad (49)$$

$$\lambda_{\min} = \min_j(\lambda_j). \quad (50)$$

Proof. Proof is similar to Theorem 5. □

Theorem 12. *Monotonicity: If* $\lambda_j \leq \lambda'_j$ *for all j, where* $j = 1, 2, \ldots, n$, *then*

$$\text{IVPFEOWA}_w(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \leq \text{IVPFEOWA}_w(\lambda'_1, \lambda'_2, \lambda'_3, \ldots, \lambda'_n). \quad (51)$$

Proof. Proof is similar to Theorem 6. □

5 An Approach to the MAGDM Problem Based on Interval-Valued Pythagorean Fuzzy Information

Algorithm 1. *Let $X=\{X_1, X_2, X_3, \dots, X_m\}$ be a finite set of m alternatives, and $C=\{C_1, C_2, C_3, \dots, C_n\}$ be a finite set of attributes. Suppose the grade of the alternatives X_i $(i=1, 2, \dots, m)$ on attributes C_j $(j=1, 2, \dots, n)$ given by decision makers are IVPFNs. Let $D=\{D_1, D_2, D_3, \dots, D_k\}$ be the set of k decision makers, and let $w=(w_1, w_2, \dots, w_n)^T$ be the weighted vector of the attributes C_j $(j=1, 2, \dots, n)$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and let $\omega=(\omega_1, \omega_2, \dots, \omega_k)^T$ be the weighted vector of the decision makers D^s $(s=1, 2, \dots, k)$, such that $\omega_s \in [0, 1]$ and $\sum_{s=1}^{k} \omega_s = 1$. Let $D=(a_{ji})=\langle[u_{ji}, v_{ji}], [x_{ji}, y_{ji}]\rangle$ $(i=1, 2, \dots, m, j=1, 2, 3, \dots, n)$, where $[u_{ji}, v_{ji}]$ indicates the interval degree that the alternative X_i satisfies the attribute C_j and $[x_{ji}, y_{ji}]$ indicates the interval degree that the alternative X_i does not satisfy the attribute C_j. Also, $[u_{ji}, v_{ji}] \in [0, 1]$, $[x_{ji}, y_{ji}] \in [0, 1]$ with condition $0 \le (v_{ji})^2 + (y_{ji})^2 \le 1$ $(i=1, 2, \dots, m, j=1, 2, \dots, n)$. This method has the following steps.*

- Step 1: In this step, we construct the interval-valued Pythagorean fuzzy decision-making matrices, $D^s=[a_{ji}^{(s)}]_{n\times m}$ $(s=1, 2, \dots, k)$ for the decision.
- Step 2: If the criteria have two types, such as benefit criteria and cost criteria, then the interval-valued Pythagorean fuzzy decision matrices, $D^s=[a_{ji}^{s}]_{n\times m}$ can be converted into the normalized interval-valued Pythagorean fuzzy decision matrices, $R^s=[r_{ji}^{(s)}]_{n\times m}$, where

$$r_{ji}^{(s)} = \begin{cases} a_{ji}^{(s)}, \text{ for benefit criteria } C_j, \\ \bar{a}_{ji}^{(s)}, \text{ for cost criteria } C_j, \end{cases} \begin{pmatrix} j=1, 2, \dots, n, \\ i=1, 2, \dots, m \end{pmatrix}$$

  and $\bar{a}_{ji}^{(s)}$ is the complement of α_{ji}^{s}. If all the criteria have the same type, then there is no need for normalization.
- Step 3: In this step, we apply the IVPFEWA operator to aggregate all the individual normalized interval-valued Pythagorean fuzzy decision matrices, $R^s=[r_{ji}^{(s)}]_{n\times m}$ $(s=1, \dots, k)$, into a single interval-valued Pythagorean fuzzy decision matrix, $R=[r_{ji}]_{n\times m}$.
- Step 4: In this step, we apply the IVPFEWA operator to aggregate all preference values.
- Step 5: In this step, we calculate the score functions. If there is no difference between two or more than two scores, then we must find out the accuracy degrees of the collective overall preference values.
- Step 6: Arrange the scores of all alternatives in descending order and select that alternative having the highest score function.

6 Illustrative Example

Suppose a company wants to invest money in the following best options: X_1, car company; X_2, food company; and X_3, computer company. There are three experts D^s $(s=1, 2, 3)$ from a group to act as decision makers, whose weight vector is $\omega=(0.2, 0.3, 0.5)^T$. There are many factors that must be considered when selecting the most suitable company; however, here, we have to consider only the following four criteria, whose weighted vector is $w=(0.1, 0.2, 0.3, 0.4)^T$:

1. C_1: risk analysis,
2. C_2: growth analysis,
3. C_3: social political impact analysis,
4. C_4: environmental analysis,

where C_1 and C_3, are cost-type criteria and C_2 and C_4 are benefit-type criteria, i.e. the attributes have two types of criteria. Thus, we must change the cost-type criteria into the benefit-type criteria.

- Step 1: The decision makers give their decision in Tables 1–11.
- Step 2: In this step, we normalize the decision matrices.

Table 1: Interval-Valued Pythagorean Fuzzy Decision Matrix of D^1.

	X_1	X_2	X_3
C_1	([0.5, 0.8], [0.3, 0.4])	([0.6, 0.7], [0.3, 0.6])	([0.3, 0.7], [0.3, 0.5])
C_2	([0.3, 0.5], [0.6, 0.7])	([0.3, 0.7], [0.2, 0.6])	([0.3, 0.6], [0.4, 0.7])
C_3	([0.5, 0.7], [0.3, 0.7])	([0.5, 0.6], [0.3, 0.7])	([0.2, 0.6], [0.3, 0.7])
C_4	([0.3, 0.6], [0.6, 0.7])	([0.6, 0.5], [0.2, 0.7])	([0.3, 0.4], [0.5, 0.6])

Table 2: Interval-Valued Pythagorean Fuzzy Decision Matrix of D^2.

	X_1	X_2	X_3
C_1	([0.5, 0.6], [0.3, 0.5])	([0.5, 0.7], [0.3, 0.6])	([0.2, 0.8], [0.3, 0.4])
C_2	([0.3, 0.4], [0.6, 0.8])	([0.3, 0.8], [0.2, 0.6])	([0.3, 0.6], [0.3, 0.7])
C_3	([0.4, 0.5], [0.3, 0.8])	([0.5, 0.7], [0.3, 0.6])	([0.2, 0.6], [0.3, 0.8])
C_4	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.4], [0.2, 0.8])	([0.3, 0.5], [0.5, 0.7])

Table 3: Interval-Valued Pythagorean Fuzzy Decision Matrix of D^3.

	X_1	X_2	X_3
C_1	([0.3, 0.8], [0.5, 0.6])	([0.3, 0.5], [0.5, 0.7])	([0.2, 0.4], [0.5, 0.7])
C_2	([0.5, 0.7], [0.3, 0.4])	([0.4, 0.6], [0.5, 0.8])	([0.5, 0.7], [0.2, 0.5])
C_3	([0.3, 0.6], [0.4, 0.6])	([0.3, 0.5], [0.5, 0.6])	([0.2, 0.8], [0.4, 0.6])
C_4	([0.5, 0.7], [0.3, 0.4])	([0.5, 0.7], [0.2, 0.4])	([0.5, 0.6], [0.3, 0.5])

Table 4: Normalized Pythagorean Fuzzy Decision Matrix R^1.

	X_1	X_2	X_3
C_1	([0.3, 0.4], [0.5, 0.8])	([0.3, 0.6], [0.6, 0.7])	([0.3, 0.5], [0.3, 0.7])
C_2	([0.3, 0.5], [0.6, 0.7])	([0.3, 0.7], [0.2, 0.6])	([0.3, 0.6], [0.4, 0.7])
C_3	([0.3, 0.7], [0.5, 0.7])	([0.3, 0.7], [0.5, 0.6])	([0.3, 0.7], [0.2, 0.6])
C_4	([0.3, 0.6], [0.6, 0.7])	([0.6, 0.5], [0.2, 0.7])	([0.3, 0.4], [0.5, 0.6])

Table 5: Normalized Pythagorean Fuzzy Decision Matrix R^2.

	X_1	X_2	X_3
C_1	([0.3, 0.5], [0.5, 0.6])	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.4], [0.2, 0.8])
C_2	([0.3, 0.4], [0.6, 0.8])	([0.3, 0.8], [0.2, 0.6])	([0.3, 0.6], [0.3, 0.7])
C_3	([0.3, 0.8], [0.4, 0.5])	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.8], [0.2, 0.6])
C_4	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.4], [0.2, 0.8])	([0.3, 0.5], [0.5, 0.7])

Table 6: Normalized Pythagorean Fuzzy Decision Matrix R^3.

	X_1	X_2	X_3
C_1	([0.5, 0.6], [0.3, 0.8])	([0.5, 0.7], [0.3, 0.5])	([0.5, 0.7], [0.2, 0.4])
C_2	([0.5, 0.7], [0.3, 0.4])	([0.4, 0.6], [0.5, 0.8])	([0.5, 0.7], [0.2, 0.5])
C_3	([0.4, 0.6], [0.3, 0.6])	([0.5, 0.6], [0.3, 0.5])	([0.4, 0.6], [0.2, 0.8])
C_4	([0.5, 0.7], [0.3, 0.4])	([0.5, 0.7], [0.2, 0.4])	([0.5, 0.6], [0.3, 0.5])

Table 7: Collective Interval-Valued Pythagorean Fuzzy Decision Matrix R.

	X_1	X_2	X_3
C_1	([0.413, 0.537], [0.389, 0.738])	([0.413, 0.653], [0.405, 0.595])	([0.413, 0.593], [0.216, 0.562])
C_2	([0.413, 0.593], [0.429, 0.563])	([0.352, 0.692], [0.320, 0.697])	([0.413, 0.653], [0.259, 0.595])
C_3	([0.352, 0.692], [0.363, 0.587])	([0.413, 0.622], [0.389, 0.576])	([0.352, 0.693], [0.200, 0.697])
C_4	([0.413, 0.653], [0.405, 0.536])	([0.475, 0.593], [0.200, 0.563])	([0.413, 0.538], [0.389, 0.576])

Table 8: Pythagorean Fuzzy Ordered Decision Matrix R^1.

	X_1	X_2	X_3
C_1	([0.3, 0.7], [0.5, 0.7])	([0.3, 0.7], [0.2, 0.6])	([0.3, 0.7], [0.2, 0.6])
C_2	([0.3, 0.6], [0.6, 0.7])	([0.6, 0.5], [0.2, 0.7])	([0.3, 0.6], [0.4, 0.7])
C_3	([0.3, 0.5], [0.6, 0.7])	([0.3, 0.7], [0.5, 0.6])	([0.3, 0.5], [0.3, 0.7])
C_4	([0.3, 0.4], [0.5, 0.8])	([0.3, 0.6], [0.6, 0.7])	([0.3, 0.4], [0.5, 0.6])

Table 9: Pythagorean Fuzzy Ordered Decision Matrix R^2.

	X_1	X_2	X_3
C_1	([0.3, 0.8], [0.4, 0.5])	([0.3, 0.8], [0.2, 0.6])	([0.3, 0.8], [0.2, 0.6])
C_2	([0.3, 0.5], [0.5, 0.6])	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.6], [0.3, 0.7])
C_3	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.6], [0.5, 0.7])	([0.3, 0.5], [0.5, 0.7])
C_4	([0.3, 0.4], [0.6, 0.8])	([0.3, 0.4], [0.2, 0.8])	([0.3, 0.4], [0.2, 0.8])

Table 10: Pythagorean Fuzzy Ordered Decision Matrix R^3.

	X_1	X_2	X_3
C_1	([0.5, 0.7], [0.3, 0.4])	([0.5, 0.7], [0.2, 0.4])	([0.5, 0.7], [0.2, 0.4])
C_2	([0.5, 0.7], [0.3, 0.4])	([0.5, 0.7], [0.3, 0.5])	([0.5, 0.7], [0.2, 0.5])
C_3	([0.4, 0.6], [0.3, 0.6])	([0.5, 0.6], [0.3, 0.5])	([0.5, 0.6], [0.3, 0.5])
C_4	([0.5, 0.6], [0.3, 0.8])	([0.4, 0.6], [0.5, 0.8])	([0.4, 0.6], [0.2, 0.8])

Table 11: Collective Pythagorean Fuzzy Ordered Decision Matrix R.

	X_1	X_2	X_3
C_1	([0.413, 0.734], [0.363, 0.481])	([0.413, 0.734], [0.200, 0.495])	([0.413, 0.734], [0.200, 0.492])
C_2	([0.413, 0.630], [0.404, 0.509])	([0.476, 0.638], [0.323, 0.595])	([0.413, 0.653], [0.259, 0.595])
C_3	([0.352, 0.582], [0.404, 0.649])	([0.413, 0.622], [0.389, 0.576])	([0.413, 0.553], [0.351, 0.595])
C_4	([0.413, 0.512], [0.412, 0.800])	([0.352, 0.550], [0.399, 0.779])	([0.352, 0.512], [0.241, 0.758])

- Step 3: In this step, we apply the IVPFEWA operator to aggregate all the individual normalized interval-valued Pythagorean fuzzy decision matrices, $R^s = [r_{ji}^{(s)}]_{n\times m}$, into a single interval-valued Pythagorean fuzzy decision matrix, $R = [r_{ji}]_{n\times m}$.
- Step 4: In this step, we apply the IVPFEWA aggregation operator to aggregate all preference values:

$$r_1 = ([0.395,\ 0.644],\ [0.394,\ 0.575]).$$

$$r_2 = ([0.427,\ 0.617],\ [0.289,\ 0.596]).$$

$$r_3 = ([0.495,\ 0.619],\ [0.277,\ 0.613]).$$

- Step 5: In this step, we calculate the score functions:

$$S(r_1) = 0.042, \quad S(r_2) = 0.069, \quad S(r_3) = 0.083.$$

- Step 6: Arrange the scores of all alternatives in descending order and select that alternative having the highest score function. Hence, $X_3 > X_2 > X_1$. Thus, the best alternative is X_3.

For the IVPFEOWA aggregation operator:
- Step 1: In this step, we construct the interval-valued Pythagorean fuzzy ordered decision matrices.
- Step 2: In this step, we apply the IVPFEOWA operator to aggregate all the individual interval-valued Pythagorean fuzzy ordered decision matrices, $R^s = [r_{ji}^{(s)}]_{n\times m}$, into a single interval-valued Pythagorean fuzzy decision matrix, $R = [r_{ji}]_{n\times m}$.
- Step 3: In this step, we apply the IVPFEOWA aggregation operator to aggregate all preference values:

$$r_1 = ([0.395,\ 0.579],\ [0.402,\ 0.659]).$$

$$r_2 = ([0.403,\ 0.612],\ [0.354,\ 0.649]).$$

$$r_3 = ([0.389,\ 0.576],\ [0.268,\ 0.646]).$$

- Step 4: In this step, we calculate the score functions:

$$S(r_1) = -0.052,\ S(r_1) = -0.004,\ S(r_1) = -0.003.$$

- Step 5: Arrange the scores of all alternatives in descending order and select that alternative having the highest score function. Hence, $X_3 > X_2 > X_1$. Thus, the best alternative is X_3.

7 Conclusion

In this paper, we have developed the notions of the IVPFEWA operator, IVPFEOWA operator, and interval-valued Pythagorean fuzzy Einstein hybrid weighted averaging operator. We have also discussed some of their desirable properties, such as idempotency, boundedness, commutativity, and monotonicity. Finally, we have applied these operators to deal with the MAGDM problem under interval-valued Pythagorean fuzzy information. For this, we constructed an algorithm for the MAGDM problem. Lastly, we also developed a numerical example for MAGDM.

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