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Graded Soft Expert Set as a Generalization of Hesitant Fuzzy Set

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Abstract: Hesitant fuzzy sets play a vital role in decision analysis. Although they have been proved to be a landmark in evaluating information, there are certain deficiencies in their structure. Also, in decision analysis with the aid of hesitant fuzzy sets, the relative importance of the decision makers according to their area of expertise is ignored completely, which may be misleading in some situations. These sorts of issues have been resolved in this work by using graded soft expert (GSE) sets. The proposed structure is a modified form of soft expert sets. Some basic operations have been introduced, and certain laws satisfied by them have carefully been investigated. With the aid of GSE sets, a decision-making algorithm (accompanied with an example) has been developed in which experts have been given due weightage according to their area of expertise.

Keywords: Hesitant fuzzy set, soft set, soft expert set, graded soft expert set.

1 Introduction

The classical set theory, also called the crisp set theory, serves as one of the fundamental concepts in mathematics. However, only a limited number of traditional methods of modeling and computing can be dealt with the help of the crisp set theory. In practice, most of the problems in fields such as economics, engineering, environmental sciences, medical science, and social sciences involve information sets that are vague rather than precise. Due to the vagueness and uncertainties in these domains, traditional methods cannot be applied here.

To this end, several theories have been developed. One of them is the soft set theory, which was initiated by Molodtsov [18]. It attracted the attention of many researchers, as the theory was well equipped with parameters. Maji et al. [12, 13] discussed decision-making problems through soft sets and fuzzy soft sets. Maji et al. [14] defined the operations of union and intersection on soft sets. Ali et al. [3] improved those operations that were based on the selection of parameters in particular. Ali et al. [4] examined soft sets algebraically using these new operations. Sezgin and Atagun [24] proved certain De Morgan's laws for soft set theory and extended the theoretical aspect of operations on soft sets. Chen et al. [6] and Ali [2] studied parametrization reduction of soft sets and discussed its application in decision analysis. Feng et al. [8] extended soft sets to soft rough sets, and Shabir et al. [25] improved the structure by introducing modified soft rough sets. Further extensions can be seen in Refs. [1, 7, 15, 16].

In the context of decision-making analysis, Alkhazaleh and Salleh [5] introduced the concept of soft expert sets. This structure can be considered as a generalization of soft sets in which experts and their opinions have been added to make decision analysis easier to handle.

To analyze decision-making problems, the hesitant fuzzy set theory has been proven to be rather worthwhile. It was presented by Torra [28] and Torra and Narukawa [29] as a generalization of the fuzzy set theory. The motivation behind this theory was the degree of hesitancy while making a decision. They introduced some

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basic operations and also discussed briefly its role in decision-making analysis. Yang et al. [37] extended the hesitant fuzzy set to the hesitant fuzzy rough set and also discussed operational laws in hesitant fuzzy sets. Xia and Xu [33], Meng et al. [17] and Tan et al. [26] developed a series of aggregation operators for hesitant fuzzy information and discussed their application in decision-making problems. Keeping in view the probabilities of possible values in the hesitant fuzzy elements and the linguistic terms representing opinions and preferences, hesitant fuzzy sets have been improved in Refs. [27, 30, 36, 38–40, 43]. Xu and Xia [35] proposed a variety of distance and similarity measures on hesitant fuzzy sets. Liang and Liu [11] introduced hesitant fuzzy sets into decision theoretic rough sets and explored their decision mechanism. Zhang and Wu [41] investigated the deviation of the priority weights from hesitant multiplicative preference relations in group decision-making environments. Although this theory proved to be valuable in the context of decision analysis, it has some deficiencies. The absence of a standard inclusion measure in hesitant fuzzy sets is a main hurdle in applying them to real-life decision-making problems. Wang and Xu [31] introduced some admissible orders. However, their relative importance subject to a given multiple-criteria decision-making (MCDM) problem has not been discussed, which makes it difficult to choose the best suitable admissible order. It is also stated that admissible orders can be generated using functions. However, in practical situations, it may be very difficult, which limits their applicability. Some more approaches to handle MCDM problems using neutrosophic and intuitionistic fuzzy sets can be seen in Refs. [9, 19–21, 32].

Also, while applying hesitant fuzzy sets in decision-making problems, there is no room to specify alternatives and decision makers separately, which makes it difficult to interpret the results. To overcome this deficiency, we have defined graded soft expert (GSE) sets that specify the collections of decision makers and alternatives separately in such a way that suitable weights can be assigned to each decision maker. This will help in reducing complexity in many decision-making algorithms. The operations over GSE sets have been defined, keeping in view the opinions of experts as well. Moreover, in hesitant fuzzy sets, containment of two hesitant fuzzy sets in one another does not lead to their equality. However, in GSE sets, if two GSE sets are contained in each other, they are equal. The hesitant fuzzy sets are suitable if the experts hesitate among several possible values and can be used both for individual and group decision making. These properties have been retained in GSE sets. GSE sets provide a way to handle the information efficiently so that there is no loss of information and weightage can be given according to the given problem.

No standard inclusion measure has yet been developed. In its application in decision analysis, experts' individual weightage has totally been ignored. To overcome these problems, we introduce in this paper GSE sets that can be treated as a generalization of hesitant fuzzy sets. This structure is a modified form of soft expert sets; however, its structural and operational approach is totally different. We mainly focused to fill the gaps in the hesitant fuzzy set theory.

To facilitate our discussion, basic concepts related to soft sets, soft expert sets, and hesitant fuzzy sets have been presented in Section 2. In Section 3, GSE sets have been introduced. Some basic operations have been defined and related laws have been proved. Section 4 has been devoted to the study of decision-making problems with the aid of GSE sets. At the end, Section 5 contains some concluding remarks.

2 Preliminaries

In this section, we give some basic concepts related to soft sets, soft expert sets and hesitant fuzzy sets. These will be required in the later sections.

2.1 Soft Sets

Let U be a non-empty set representing a universe set and $P(U)$ denote the power set of U. Let E be the set of parameters and A, B be non-empty subsets of E.

Definition 1 ([18]): A pair (F, A) is called a soft set over U, where F is a mapping given by $F:A \to P(U)$.

Thus, a soft set can be considered a parameterized family of subsets of the universe U. For $e \in A$, $F(e)$ gives the set of e-approximate elements of the soft set (F, A).

Definition 2 ([14]): For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B), denoted by $(F, A) \tilde{\subseteq} (G, B)$, if

1. $A \subseteq B$ and
2. $F(e) \subseteq G(e)$ for all $e \in A$.

In this case, (G, B) is said to be a soft super set of (F, A).

Definition 3 ([14]): Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 4 ([3]): Let U be an initial universe set, E be the set of parameters, and $A \subseteq E$.

(a) (F, A) is called a relative null soft set (with respect to the parameter set A), denoted by $\varnothing_A$, if $F(a) = \varnothing$ for all $a \in A$.

(b) (G, A) is called a relative whole soft set (with respect to the parameter set A), denoted by $A_{\mathcal{U}}$, if $G(e) = U$ for all $e \in A$.

Remark 1: If a relative null soft set is taken over E, it is called null soft set over U and is denoted by $\varnothing_E$. In a similar way, a relative whole soft set with respect to the set of parameters E is called the absolute soft set over U and is denoted by E_U.

Empty soft set over U, denoted by $\varnothing_\varnothing$, is a unique soft set over U with an empty parameter set.

The operations of union and intersection on soft sets have been defined as below.

Definition 5 ([3]): (1) The extended union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$:

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

We write $(F, A) \cup_\varepsilon (G, B) = (H, C)$.

(2) Let (F, A) and (G, B) be two soft sets over the same universe U, such that $A \cap B \neq \varnothing$. The restricted union of (F, A) and (G, B) is denoted by $(F, A) \cup_{\mathcal{R}} (G, B)$ and is defined as $(F, A) \cup_{\mathcal{R}} (G, B) = (H, C)$, where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cup G(e)$. If $A \cap B = \varnothing$, then $(F, A) \cup_{\mathcal{R}} (G, B) = \varnothing_\varnothing$.

Definition 6 ([3]): (1) The extended intersection of two soft sets (F, A) and (G, B) over a common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

We write $(F, A) \cap_\varepsilon (G, B) = (H, C)$.

(2) Let (F, A) and (G, B) be two soft sets over the same universe U such that $A \cap B \neq \varnothing$. The restricted intersection of (F, A) and (G, B) is denoted by $(F, A) \cap_{\mathcal{R}} (G, B)$ and is defined as $(F, A) \cap_{\mathcal{R}} (G, B) = (H, A \cap B)$, where $H(e) = F(e) \cap G(e)$ for all $e \in A \cap B$. If $A \cap B = \varnothing$, then $(F, A) \cap_{\mathcal{R}} (G, B) = \varnothing_\varnothing$.

2.2 Soft Expert Sets

Soft sets are enriched with parameters but they lack experts' individual opinions and the grades associated with each element of the universe. To strengthen the concept of soft sets, soft expert sets have been introduced. Here, we give some basic concepts related to soft expert sets. All the definitions related to soft expert sets have been taken from Ref. [5].

Let U be a universe set, E be a set of parameters, X be a set of experts, and O be a set of opinions. Let A be a non-empty subset of Z, where $Z=E\times X\times O$. With these notations, Alkhazaleh and Salleh [5] defined the soft expert set as stated below.

Definition 7: A pair (F, A) is called a soft expert set over U, where F is a mapping given by $F{:}A\to P(U)$.

Thus, a soft expert set can be considered as a soft set in which the parameter set is replaced with a Cartesian product of the set of parameters, set of experts, and set of opinions.

Definition 8: For two soft expert sets (F, A) and (G, B) over U, (F, A) is called a soft expert subset of (G, B) if
1. $A\subseteq B$ and
2. $F(a)\subseteq G(a)$ for all $a\in A$.

In that case, (G, B) will be called the soft expert superset of (F, A).

Definition 9: Two soft expert sets (F, A) and (G, B) over U are said to be equal if (F, A) is a soft expert subset of (G, B) and (G, B) is a soft expert subset of (F, A).

Definition 10: Let $E=\{e_1, e_2,\ldots,e_n\}$ be a set of parameters. The NOT set of E denoted by $'IE$ is defined by $'IE=\{'Ie_1, 'Ie_2,\ldots,'Ie_n\}$, where $'Ie_i$ represents "not e_i" for all i.

Definition 11: The complement of a soft expert set (F, A) is denoted and defined as $(F, A)^c=(F^c, 'IA)$, where F^c: $'IA\to P(U)$ is a mapping given by

$$F^c(a)=U-F('IA)\ \text{ for all } a\in 'IA.$$

Definition 12: The union of two soft expert sets (F, A) and (G, B) over U, denoted by $(F, A)\tilde{U}(G, B)$, is a soft expert set (H, C), where $C=A\cup B$ and for all $a\in C$:

$$H(a)=\begin{cases} F(a) & \text{if } a\in A-B\\ G(a) & \text{if } a\in B-A\\ F(a)\cap G(a) & \text{if } a\in A\cap B.\end{cases}$$

Definition 13: The intersection of two soft expert sets (F, A) and (G, B) over U, denoted by $(F, A)\tilde{\cap}(G, B)$, is a soft expert set (H, C), where $C=A\cup B$, and for all $a\in C$:

$$H(a)=\begin{cases} F(a) & \text{if } a\in A-B\\ G(a) & \text{if } a\in B-A\\ F(a)\cap G(a) & \text{if } a\in A\cap B.\end{cases}$$

2.3 Hesitant Fuzzy Sets

Definition 14 ([28, 33]): Let X be a fixed set. A hesitant fuzzy set on X is in terms of a function that when applied to X returns a subset of [0, 1].

Thus, if h is a hesitant fuzzy set on X, then $h(x)$ $(x \in X)$, being a subset of [0, 1], gives the possible degrees of membership. For any $x \in X$, $h(x)$ is called hesitant fuzzy element.

Remark 2: Torra [28] defined lower and upper bounds for a hesitant fuzzy element as below:
1. Lower bound: $h^-(x) = \min\{\gamma : \gamma \in h(x)\}$.
2. Upper bound: $h^+(x) = \max\{\gamma : \gamma \in h(x)\}$.

Basic operations on hesitant fuzzy sets are given below.

Definition 15 ([28, 37]): For hesitant fuzzy sets h, h_1, and h_2 on X, the following operations have been defined:
1. Containment: h_1 is contained in h_2, denoted by $h_1 \preceq h_2$, if and only if $h_1^-(x) \le h_2^-(x)$ and $h_1^+(x) \le h_2^+(x)$ for all $x \in X$;
2. Union: union of h_1 and h_2, denoted by $h_1 \Cup h_2$, is defined for any $x \in X$ as $(h_1 \Cup h_2)(x) = \{h \in h_1(x) \cup h_2(x) : h \ge \max\{h_1^-(x), h_2^-(x)\}$;
3. Intersection: intersection of h_1 and h_2, denoted by $h_1 \Cap h_2$, is defined for any $x \in X$ as $(h_1 \Cap h_2)(x) = \{h \in h_1(x) \cup h_2(x) : h \le \min\{h_1^+(x), h_2^+(x)\}$;
4. Complement: complement of h is denoted by h^c and is defined for any $x \in X$ as $h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}$.

Operational laws investigated by Yang et al. [37] are stated in the next theorem.

Theorem 1: *For hesitant fuzzy sets h_1, h_2, and h_3 on X, the following properties hold:*
1. *Idempotent:* $h_1 \Cup h_1 = h_1$, $h_1 \Cap h_1 = h_1$;
2. *Commutative:* $h_1 \Cup h_2 = h_2 \Cup h_1$, $h_1 \Cap h_2 = h_2 \Cap h_1$;
3. *Associative:* $h_1 \Cup (h_2 \Cup h_3) = (h_1 \Cup h_2) \Cup h_3$, $h_1 \Cap (h_2 \Cap h_3) = (h_1 \Cap h_2) \Cap h_3$;
4. *Distributive:* $h_1 \Cup (h_2 \Cap h_3) = (h_1 \Cup h_2) \Cap (h_1 \Cup h_3)$, $h_1 \Cap (h_2 \Cup h_3) = (h_1 \Cap h_2) \Cup (h_1 \Cap h_3)$;
5. *De Morgan's laws:* $(h_1 \Cup h_2)^c = h_1^c \Cap h_2^c$, $(h_1 \Cap h_2)^c = h_1^c \Cup h_2^c$;
6. *Double negation*: $(h^c)^c = h$.

3 GSE Set versus Hesitant Fuzzy Set and Soft Expert Set

In this section, the soft expert set defined by Alkhazalah and Salleh [5] has been redefined and revised, which may be called the GSE set. In order to strengthen the structure, its basic operations have been redefined in a more fruitful manner. Several laws and related results have also been investigated, some of which do not hold in hesitant fuzzy sets.

Hesitant fuzzy sets are basically introduced to handle decision-making problems in which there are several alternatives and decision makers. However, in the definition of hesitant fuzzy sets, alternatives and decision makers have not been specified. This may lead to the wrong use and interpretation of the set. Also, if we take x_1, x_2, and x_3 as three alternatives and hesitant fuzzy set h represents a particular criteria, then for each i $(i = 1, 2, \ldots, n)$, $h(x_i)$ represents opinions of various decision makers in which there is no space to highlight an individual decision maker's opinions separately. For that purpose, different techniques and algorithms were introduced, which make the decision-making problems somehow difficult to handle. One of them is to assign weights to the opinions. However, again, as opinions of the decision makers have been collected in a set without specifying their individual decisions, it is not possible to give more weightage to a particular decision maker. It may be possible by introducing a complex algorithm.

To avoid such type of situations, the GSE set can prove its worth. In the GSE set, each alternative (or attribute) and decision maker have been specified separately. Formally, it is stated as below.

Definition 16: Let U be a finite universe set containing n alternatives, E a set of criteria, and X a set of experts (or decision makers). Let O be a set of opinions with a given preference relation $\precsim$ among the opinions.

A GSE set (F, A, Y) is characterized by a mapping $F{:}A\times Y\to P(U\times O)$ defined for every $e\in A$ and $p\in Y$ by $F(e, p)=\{(u_i, o_i){:}\ i\in I\}$, where $I=\{1, 2, 3,\ldots, n\}$, $A\subseteq E$, $Y\subseteq X$, and $P(U\times O)$ denotes the power set of $U\times O$. Here, the set of opinions O contains graded values of the given parameters, i.e. the values $o_1, o_2,\ldots,o_n$ can be graded as $o_1 \precsim o_2 \precsim \ldots, \precsim o_n$, which means that o_n is the most preferred value while o_1 is the least preferred one, and so forth.

The above definition states that for a given criteria e, the decision maker p gives the opinion o_i for each alternative u_i $(i=1, 2, \ldots, n)$. As an example of the preference relation in the above definition, consider the set of opinions $O=\{$excellent, very good, good, poor, very poor$\}$. It is obvious that "excellent" is preferred over "very good," which is preferred over "good," which is preferred over "poor," and the least preferred one is "very poor." For simplicity, we can fuzzify these values according to their grading and preference; that is, the opinions can be assigned values from the interval [0, 1] based on their preference. For example, in the above-mentioned set O of opinions, "excellent" is the most preferred opinion, so it can be assigned value 1 from the interval [0, 1] while "very poor" is the least preferred opinion, so it can be assigned the value 0. The rest of the opinions will be assigned values between 0 and 1.

Thus, GSE can be regarded as a generalization of the hesitant fuzzy set in the sense that in addition to assigning hesitancy degree to the objects/alternatives, it considers experts' individual opinions separately for each given parameter. From the application point of view, GSE sets are particularly important to handle decision-making problems in which one needs to assign weightage to the experts' opinion separately according to their expertise. In the rest of the paper, the set of opinions O will be taken as a subset of [0, 1].

Example 1: Let $U=\{u_1, u_2, u_3, u_4, u_5\}$ be a set of wheat types (alternatives), $E=\{e_1=$moisture content, $e_2=$protein content, $e_3=$milling quality, $e_4=$baking quality$\}$ be a set of criteria, $X=\{a, b, c\}$ be a set of experts, and $O=\{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ be the set of possible grades for the given parameters.

Suppose that a farmer has distributed a questionnaire to the team of experts to judge the quality of wheat types on the basis of given criteria. The decision of experts in the form of GSE set $F{:}A\times X\to P(U\times O)$ is given below:

– $F(e_1, a)=\{(u_1, 0.5), (u_2, 0.1), (u_3, 0.7), (u_4, 0.9), (u_5, 0.2)\}$, $F(e_1, b)=\{(u_1, 0.5), (u_2, 0.2), (u_3, 0.7), (u_4, 0.3), (u_5, 0.4)\}$, $F(e_1, c)=\{(u_1, 0.4), (u_2, 0.3), (u_3, 0.3), (u_4, 0.6), (u_5, 0.7)\}$, $F(e_2, a)=\{(u_1, 0.9), (u_2, 0.3), (u_3, 0.2), (u_4, 0.3), (u_5, 0.6)\}$, $F(e_2, b)=\{(u_1, 0.8), (u_2, 0.9), (u_3, 0.4), (u_4, 0.1), (u_5, 0.4)\}$, $F(e_2, c)=\{(u_1, 0.7), (u_2, 0.0), (u_3, 0.3), (u_4, 0.3), (u_5, 0.6)\}$, $F(e_3, a)=\{(u_1, 0.5), (u_2, 0.5), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_3, b)=\{(u_1, 0.4), (u_2, 0.4), (u_3, 0.8), (u_4, 0.2), (u_5, 0.3)\}$, $F(e_3, c)=\{(u_1, 0.4), (u_2, 0.4), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_4, a)=\{(u_1, 0.6), (u_2, 0.7), (u_3, 0.5), (u_4, 0.9), (u_5, 0.7)\}$, $F(e_4, b)=\{(u_1, 0.5), (u_2, 0.8), (u_3, 0.4), (u_4, 0.6), (u_5, 0.3)\}$, $F(e_4, c)=\{(u_1, 0.3), (u_2, 0.9), (u_3, 0.5), (u_4, 0.0), (u_5, 0.6)\}$.

In the soft set theory, the basic concept is parametrization of objects in a given universe set. The various operations thus defined on soft sets depend upon the e-approximate elements of a given set for all attributes e. As a soft expert set does not only depend upon the various parameters involved but also on the opinion of experts, which is basically the main purpose of introducing soft expert sets, the operations on soft expert sets should consider these opinions as well. In the rest of the section, we define operations on GSE sets taking into consideration the respective opinions as well.

In particular, we can see that the operation of complement on the soft expert set defined in Ref. [5] takes into consideration the objects of universe and their respective attributes only ignoring their respective opinions. As in example 3.9 of Ref. [5], we can see that the complement of $F(e_1, p, 1)=\{u_3\}$ is given as $F^c(Ie_1, p, 1)=\{u_1, u_2, u_4\}$, which means that according to the expert p, only the object u_3 has attribute e_1 and its complement states that according to the same opinion of expert p the objects u_1, u_2, and u_4 do not have attribute e_1. This idea can work if we are taking only two opinions (agree 1, disagree 0). If we consider more than two opinions (as in GSE sets), the idea may not work. In the same above case, if we take $F(e_1, p, 0.3)=\{u_3\}$ and $F^c(Ie_1, p, 0.3)=\{u_1, u_2, u_4\}$, then the objects not having attribute e_1 in the same degree 0.3 as the objects having that attribute do not sound accurate. Thus, for more than two opinions, we define the complement of the GSE set, as follows.

Definition 17: The complement of a GSE set (F, A, Y), denoted by $(F, A, Y)^c$, is defined as $(F, A, Y)^c=(F^c, A^c, Y)$, where $F^c:A^c\times Y\to P(U\times O^c)$ is a mapping given as

$$F^c(e^c, p)=\{(u_i, o_i^c):i\in I\}$$

whenever

$$F(e, p)=\{(u_i, o_i):i\in I\} \text{ and } o_i^c=1-o_i.$$

Definition 18: The union of two GSE sets (F, A, Y) and (G, B, Z) over U, denoted by $(F, A, Y)\cup(G, B, Z)$, is a GSE set (H, C, Y), where $C=A\cup B$, $X=Y\cup Z$ and for all $e\in C$ and $p\in X$:

$$H(e, p)=\begin{cases}\{(u_i, \max\{o_i, o_i'\}):i\in I\} & \text{if}(e, p)\in(A\cap B, Y\cap Z)\\ \{(u_i, o_i):i\in I\} & \text{if}(e, p)\in(A, Y)\backslash(B, Z),\\ \{(u_i, o_i'):i\in I\} & \text{if}(e, p)\in(B, Z)\backslash(A, Y)\end{cases}$$

whenever $F(e, p)=\{(u_i, o_i):i\in I\}$ and $G(e, p)=\{(u_i, o_i'):i\in I\}$.

Definition 19: The intersection of two GSE sets (F, A, Y) and (G, B, Z) over U, denoted by $(F, A, Y)\cap(G, B, Z)$, is a GSE set (H, C, X), where $C=A\cap B$, $X=Y\cap Z$ and for all $e\in C$ and $p\in X$:

$$H(e, p)=\{(u_i, \min\{o_i, o_i'\}):i\in I\}$$

whenever $F(e, p)=\{(u_i, o_i):i\in I\}$ and $G(e, p)=\{(u_i, o_i'):i\in I\}$.

In the classical set, the hierarchy is characterized through set containment. However, in case of other generalizations of the classical set like the fuzzy set, soft set, or hesitant fuzzy set, it is characterized through different ways. Alkhazalah and Salleh [5] defined soft expert subsets by using the classical set containment approach in which grading of opinions is not considered. Taking into consideration the opinions of experts, we define the notion of a subset for GSE sets in a more generalized way as below.

Definition 20: For a GSE set (F, A, Y) over U and for any $e, e'\in A$, $p, p'\in Y$, if

$$F(e, p)=\{(u_i, o_i):i\in I\} \text{ and } F(e', p')=\{(u_i', o_i'):i\in I\},$$

then $F(e, p)$ is said to be contained in $F(e', p')$ [or equivalently $F(e, p)$ is subset of $F(e', p')$, denoted by $F(e, p)\subseteq F(e', p')$], if

$$o_i\le o_i' \text{ for each } i\in\{1, 2, 3,\ldots,n\}.$$

The above condition states that the degree of each alternative in $F(e, p)$ is less than the corresponding degree in $F(e', p')$.

Example 2: In Example 1, $F(e_1, b)=\{(u_1, 0.2), (u_2, 0.5), (u_3, 0.4), (u_4, 0.5), (u_5, 0.6)\}\subseteq F(e_4, c)=\{(u_1, 0.3), (u_2, 0.8), (u_3, 0.5), (u_4, 0.5), (u_5, 0.6)\}$ because the opinion for each u_i in $F(e_1, b)$ is less than or equal to its corresponding value in $F(e_4, c)$.

Definition 21: For two GSE sets (F, A, Y) and (G, B, Z) over U, (F, A, Y) is called subset of (G, B, Z), denoted by $(F, A, Y)\subseteq(G, B, Z)$, if
1. $A\subseteq B$.
2. $Y\subseteq Z$.
3. $F(e, p)\subseteq G(e, p)$ for all $e\in A$, $p\in Y$.

In this case, (G, B, Z) is called a superset of (F, A, Y) denoted by $(G, B, Z)\supseteq(F, A, Y)$.

By Definition 21, we can see that the comparison of two GSE sets is pointwise, which means that the values of the two GSE sets are compared for each pair of values separately. In case of soft expert sets, containment, as defined in Ref. [5], is a global property that ignores individual opinions completely. Also, in that case, two soft expert sets can be compared but there is no way to compare their respective values separately.

Definition 22: Two GSE sets (F, A, Y) and (G, B, Z) over U are said to be equal, denoted by $(F, A, Y)=(G, B, Z)$, if $A=B$, $Y=Z$, and $F(e, p)=G(e, p)$ for all $e\in A$ $(=B)$, $p\in Y$ $(=Z)$.

Proposition 1: *For two GSE sets* (F, A, Y) *and* (G, B, Z) *over* U, *if* $(F, A, Y)\subseteq(G, B, Z)$ *and* $(G, B, Z)\subseteq(F, A, Y)$, *then* $(F, A, Y)=(G, B, Z)$.

Proof. It can easily be proved using Definitions 22 and 21. □

This is one of the most significant results for GSE sets. The inclusion here is based on graded values or opinions as in hesitant fuzzy sets; however, the above result does not hold in case of hesitant fuzzy sets. To overcome this shortcoming, many inclusion measures and criteria have been developed. Hesitant equality has also been introduced. However, all these attempts were more or less useless in practical implementations.

Xia and Xu [33] defined the score function of hesitant fuzzy element h, that is, $s(h)=\left(\sum_{\gamma\in h}\gamma\right)/\#h,$ where $s(\cdot)$ is the score function and $\#h$ is the number of elements in h. This score function serves as a measure to compare two hesitant fuzzy sets. Following the same technique, we define score function for a GSE set as below.

Definition 23: For a given GSE set (F, A, Y) over $U=\{u_1, u_2, ..., u_n\}$, where A contains m criteria, the score function for any u_i $(i=1, 2,\ldots,n)$ with respect to the opinions of an expert $p\in Y$ is denoted by $s(u_i, p)$ and is defined as

$$s(u_i, p)=\left(\sum_{j=1}^{m}o_j\right)/m,$$

where $o_1, o_2,\ldots,o_m$ are the respective opinions of the expert p for the alternative u_i with respect to the criteria $e_1, e_2,\ldots,e_n$.

Theorem 2: *For any two GSE sets* (F, A, Y) *and* (G, B, Z) *over* U, *we have*

$$(F, A, Y)\cap(G, B, Z)\subseteq(F, A, Y), (G, B, Z);$$
$$(F, A, Y), (G, B, Z)\subseteq(F, A, Y)\cup(G, B, Z).$$

Proof. For any GSE sets (F, A, Y) and (G, B, Z), let $(F, A, Y)\cap(G, B, Z)=(H, A\cap B, Y\cap Z)$. As $A\cap B\subseteq A, B$ and $Y\cap Z\subseteq Y, Z$ and for any $e\in A\cap B$, $p\in Y\cap Z$ using Definition 19, we have

$$H(e, p)=\{(u_i,\ \min\{o_i, o_i'\}):i\in I\},$$

where $F(e, p)=\{(u_i, o_i):i\in I\}$ and $G(e, p)=\{(u_i, o_i'):i\in I\}$. Thus, by Definition 20, $H(e, p)\subseteq\{(u_i, o_i):i\in I\}=F(e, p)$ and $H(e, p)\subseteq\{(u_i, o_i'):i\in I\}=G(e, p)$. This shows that $(F, A, Y)\cap(G, B, Z)\subseteq(F, A, Y), (G, B, Z)$.

Also, let $(F, A, Y)\cup(G, B, Z)=(J, A\cup B, Y\cup Z)$. As $A\subseteq A\cup B$ and $Y\subseteq Y\cup Z$, for any $e\in A$, $p\in Y$, using Definition 18, we have

$$J(e, p)=\begin{cases}\{(u_i,\ \max\{o_i, o_i'\}):i\in I\} & \text{if } (e, p)\in(A\cap B,\ Y\cap Z)\\ \{(u_i,\ o_i):i\in I\} & \text{if } (e, p)\in(A,\ Y)\backslash(B,\ Y)\end{cases}$$

In both cases, using Definition 20, we have $F(e, p)\subseteq J(e, p)$. Similarly, $G(e, p)\subseteq J(e, p)$. Thus, (F, A, Y), $(G, B, Z)\subseteq(F, A, Y)\cup(G, B, Z)$. □

Theorem 3: *Let* U *be the universe set. For all GSE sets* (F, A, Y), (G, B, Z), *and* (H, C, X) *over* U, *the following properties hold:*

1. *Idempotent:* $(F, A, Y)\cap(F, A, Y)=(F, A, Y)$, $(F, A, Y)\cup(F, A, Y)=(F, A, Y)$;
2. *Commutative:* $(F, A, Y)\cap(G, B, Z)=(G, B, Z)\cap(F, A, Y)$, $(F, A, Y)\cup(G, B, Z)=(G, B, Z)\cup(F, A, Y)$;

3. *Associative:* $(F, A, Y)\cap((G, B, Z)\cap(H, C, X))=((F, A, Y)\cap(G, B, Z))\cap(H, C, X)$, $(F, A, Y)\cup((G, B, Z)\cup(H, C, X))=((F, A, Y)\cup(G, B, Z))\cup(H, C, X)$;
4. *Distributive:* $(F, A, Y)\cap((G, B, Z)\cup(H, C, X))=((F, A, Y)\cap(G, B, Z))\cup((F, A, Y)\cap(H, C, X))$, $(F, A, Y)\cup((G, B, Z)\cap(H, C, X))=((F, A, Y)\cup(G, B, Z))\cap((F, A, Y)\cup(H, C, X))$;
5. *De Morgan's laws:* $((F, A, Y)\cap(G, B, Z))^c=(F, A, Y)^c\cup(G, B, Z)^c$, $((F, A, Y)\cup(G, B, Z))^c=(F, A, Y)^c\cap(G, B, Z)^c$;
6. *Double-negation law:* $((F, A, Y)^c)^c=(F, A, Y)$.

Proof. These can be derived directly from Definitions 18, 19, 17, and 22. □

In general, absorption laws do not hold for hesitant fuzzy sets. However, these laws hold in case of the GSE set, as can be seen in the next result.

Theorem 4: *For any two GSE sets* (F, A, Y) *and* (G, B, Z) *over* U, *the following absorption laws hold:*

$$(F, A, Y)\cap((F, A, Y)\cup(G, B, Z))=(F, A, Y);$$

$$(F, A, Y)\cup((F, A, Y)\cap(G, B, Z))=(F, A, Y).$$

Proof. By Definitions 18 and 19, we have $(F, A, Y)\cap((F, A, Y)\cup(G, B, Z))=(H, A\cap(A\cup B), Y\cap(Y\cup Z))=(H, A, Y)$ such that for any $e\in A$ and $p\in Y$, we have

$$H(e, p)=\begin{cases} F(e, p)\cap(F(e, p)\cup G(e, p)) & \text{if } (e, p)\in(A\cap B, Y\cap Z)\\ F(e, p)\cap(F(e, p) & \text{if } (e, p)\in(A, Y)\setminus(B, Z).\end{cases}$$

In the first case when $(e, p)\in(A\cap B, Y\cap Z)$, $F(e, p)=\{(u_i, o_i): i\in I\}$ and $G(e, p)=\{(u_i, o_i'): i\in I\}$, using Definitions 18, 19, and 20 we get

$$\begin{aligned} F(e, p)\cap(F(e, p)\cup G(e, p)) &= \{(u_i, o_i): i\in I\}\cap(\{(u_i, o_i): i\in I\}\cup\{(u_i, o_i'): i\in I\})\\ &= \{(u_i, o_i): i\in I\}\cap\{(u_i, \max\{o_i, o_i'\}): i\in I\}\\ &= \{(u_i, \min\{o_i, \max\{o_i, o_i'\}\}): i\in I\}\\ &\subseteq \{(u_i, o_i): i\in I\}=F(e, p)\\ &\subseteq \{(u_i, \max\{o_i, \min\{o_i, o_i'\}\}): i\in I\}\\ &= \{(u_i, \min\{o_i, \max\{o_i, o_i'\}\}): i\in I\}\\ &= F(e, p)\cap(F(e, p)\cup G(e, p)). \end{aligned}$$

The above arguments give us our required result for the first case.

In the second case, when $(e, p)\in(A, Y)\setminus(B, Z)$, using Definition 18, we have

$$(F, A, Y)\cap((F, A, Y)\cup(G, B, Z))=(F, A, Y)\cap(F, A, Y)=(F, A, Y),$$

which is our required result for this case as well. Thus, in both cases, we have

$$(F, A, Y)\cap((F, A, Y)\cup(G, B, Z))=(F, A, Y).$$

Similarly, we can prove that

$$(F, A, Y)\cup((F, A, Y)\cap(G, B, Z))=(F, A, Y).$$ □

4 Decision Making with the Aid of GSE Sets

Decision-making problems have extensively been studied using hesitant fuzzy sets in which there are several experts who have to decide among various alternatives [22, 23, 34, 42]. For that purpose, the most common approach is to aggregate the opinions first for each criteria and alternative. Then, alternatives are ranked by aggregating the average criteria.

As already mentioned, the experts' individual opinions have been ignored while modeling decisions by hesitant fuzzy sets. Experts may have different expertise regarding different criteria. To overcome this shortcoming, GSE sets can be used to give due weightage to the opinions of experts individually.

In this section, we develop an algorithm with the aid of GSE sets for decision analysis in which experts will be given weightage according to their area of expertise. Let $\{u_1, u_2, \ldots, u_n\}$ be a finite set of n alternatives and $E=\{e_1, e_2, \ldots, e_m\}$ be a set of m criteria. Further, we take X as a set of experts and O as a set of possible opinions. Our goal is to decide among the various alternatives subject to expert's opinion regarding given criteria. This is a decision-making problem. To handle such type of problems by using GSE sets, we propose following algorithmic steps.

Algorithm 1

- **Step 1:** Utilize the evaluations of experts in the form of GSE sets to determine the opinions regarding given alternatives and criteria.
- **Step 2:** Find the weighted average of opinions for each pair (u_i, e_j) $(i=1, 2, \ldots, n, j=1, 2, \ldots, m)$ by assigning suitable weights to the experts according to their area of expertise.
- **Step 3:** Using Definition 23, calculate the scores $s(u_i)$ of u_i $(i=1, 2, \ldots, n)$ considering the aggregate values of experts in Step 2.
- **Step 4:** Rank all the alternatives according to $s(u_i)$ in descending order.
- **Step 5:** End.

This algorithm allows us to handle situations in which several experts have to make a decision on a given set of alternatives by selecting the most relevant one. Step 2 of the algorithm is designed to give due weightage to the experts according to their area of expertise. Score function is then used to aggregate the criteria in Step 3, and Step 4 ranks the alternatives according to their scores.

Example 3: A person wants to start a small business with low capital. He is considering five different businesses; u_1 is a computer and mobile repair business, u_2 is a baby sitting and child care business, u_3 is a dairy products business, u_4 is a real-estate agency business, and u_5 is an artist freelance business. Let us denote the set of these business types (alternatives) by U.

Let $Q=\{e_1=\text{high profit}, e_2=\text{market area}, e_3=\text{revenue and profitability}, e_4=\text{ownership and taxes}\}$ be the set of criteria. Let $Y=\{a, b, c\}$ be the set of experts. Expert a is selected for acknowledged expertise in evaluating e_1 and e_4; expert b in evaluating e_1, e_2, and e_3; and expert c in evaluating e_2, e_3, and e_4. Also, we take $O=\{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ as the set of possible opinions of experts regarding risk factor.

- **Step 1:** Utilize the evaluations of experts in the form of GSE sets for the given problem. For ease of calculation, these can also be written in tabular form as in Tables 1–3.
  $F(e_1, a)=\{(u_1, 0.3), (u_2, 0.4), (u_3, 0.2), (u_4, 0.5), (u_5, 0.8)\}$, $F(e_1, b)=\{(u_1, 0.2), (u_2, 0.5), (u_3, 0.4), (u_4, 0.5), (u_5, 0.6)\}$, $F(e_1, c)=\{(u_1, 0.4), (u_2, 0.5), (u_3, 0.3), (u_4, 0.6), (u_5, 0.7)\}$, $F(e_2, a)=\{(u_1, 0.9), (u_2, 0.0), (u_3, 0.2), (u_4, 0.3),$

Table 1: Opinions of Expert a.

	(e_1, a)	(e_2, a)	(e_3, a)	(e_4, a)
u_1	0.3	0.9	0.5	0.6
u_2	0.4	0.0	0.3	0.8
u_3	0.2	0.2	0.9	0.5
u_4	0.5	0.3	0.7	0.7
u_5	0.8	0.6	0.2	0.6

Table 2: Opinions of Expert *b*.

	(e_1, b)	(e_2, b)	(e_3, b)	(e_4, b)
u_1	0.2	0.8	0.4	0.5
u_2	0.5	0.1	0.4	0.6
u_3	0.4	0.4	0.7	0.4
u_4	0.5	0.1	0.5	0.6
u_5	0.6	0.4	0.3	0.3

Table 3: Opinions of Expert *c*.

	(e_1, c)	(e_2, c)	(e_3, c)	(e_4, c)
u_1	0.4	0.7	0.5	0.3
u_2	0.5	0.3	0.3	0.8
u_3	0.3	0.3	0.9	0.5
u_4	0.6	0.3	0.7	0.5
u_5	0.7	0.5	0.2	0.6

$(u_5, 0.6)\}$, $F(e_2, b)=\{(u_1, 0.8), (u_2, 0.1), (u_3, 0.4), (u_4, 0.1), (u_5, 0.4)\}$, $F(e_2, c)=\{(u_1, 0.7), (u_2, 0.3), (u_3, 0.3), (u_4, 0.3), (u_5, 0.5)\}$, $F(e_3, a)=\{(u_1, 0.5), (u_2, 0.3), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_3, b)=\{(u_1, 0.4), (u_2, 0.4), (u_3, 0.7), (u_4, 0.5), (u_5, 0.3)\}$, $F(e_3, c)=\{(u_1, 0.5), (u_2, 0.3), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_4, a)=\{(u_1, 0.6), (u_2, 0.8), (u_3, 0.5), (u_4, 0.7), (u_5, 0.6)\}$, $F(e_4, b)=\{(u_1, 0.5), (u_2, 0.6), (u_3, 0.4), (u_4, 0.6), (u_5, 0.3)\}$, $F(e_4, c)=\{(u_1, 0.3), (u_2, 0.8), (u_3, 0.5), (u_4, 0.5), (u_5, 0.6)\}$.

– **Step 2:** Find the weighted average of opinions for each pair (u_i, e_j) $(i=1, 2, 3, 4, 5, j=1, 2, 3, 4)$ by assigning weight 2 to expert *a* for e_1 and e_4 and 1 for e_2 and e_3. Similarly, assign weight 2 to expert b each for e_1, e_2, and e_3 and 1 for e_4 and assign weight 2 to expert c each for e_2, e_3, and e_4 and 1 for e_1. Thus, opinions of experts have been aggregated in this step and results have been displayed in Table 4.
For example, for the pair u_1, e_1, the weighted average has been calculated as

$$[2(0.3)+2(0.2)+1(0.4)]/(2+2+1)=0.28.$$

The rest of the entries can be calculated in a similar way.

– **Step 3:** Using Definition 23 for aggregate experts' opinions instead of individual values, calculate scores $s(u_i)$ $(i=1, 2, 3, 4, 5)$ to get

$$s(u_1)=0.495,\ s(u_2)=0.43,\ s(u_3)=0.48,\ s(u_4)=0.49,\ s(u_5)=0.49.$$

– **Step 4:** Rank all the business types u_i $(i=1, 2, 3, 4, 5)$ in accordance with their scores $s(u_i)$ to get the preference relation $u_2 \succ u_3 \succ u_4 \approx u_5 \succ u_1$ (the alternative with the lowest overall risk factor is the most preferred one, while the one with highest overall risk factor is the least preferred). Thus, the most appropriate business is u_2.

Table 4: Weighted average of opinions.

	e_1	e_2	e_3	e_4
u_1	0.28	0.78	0.46	0.46
u_2	0.46	0.16	0.34	0.76
u_3	0.30	0.32	0.82	0.48
u_4	0.52	0.22	0.62	0.60
u_5	0.70	0.48	0.24	0.54

4.1 Comparative Analysis

We compare our algorithm with the following algorithm, which is based on the most widely used weighted averaging operator [10].

Algorithm 2: Let $U=\{u_1, u_2, \ldots, u_n\}$ be the set of alternatives, $E=\{e_1, e_2, \ldots, e_l\}$ be the set of attributes, and $X=\{x_1, x_2, \ldots, x_m\}$ be the set of experts.

Further, we take opinion of experts in the form of GSE elements.

- **Step 1:** Utilize the evaluations of experts in the form of GSE sets.
- **Step 2:** Separate the opinions of each expert.
- **Step 3:** Assign weights to each expert according to their area of expertise.
- **Step 4:** Aggregate the attributes by using a GSE-weighted average operator.
- **Step 5:** Find the average of these alternatives.
- **Step 6:** Arrange these alternatives according to situation.
- **Step 7:** Choose the best alternative.

Now, for a comparative analysis, we apply the above algorithm to Example 3.

- **Step 1:** Utilize the evaluations of experts in the form of GSE sets:
  $F(e_1, a)=\{(u_1, 0.3), (u_2, 0.4), (u_3, 0.2), (u_4, 0.5), (u_5, 0.8)\}$, $F(e_1, b)=\{(u_1, 0.2), (u_2, 0.5), (u_3, 0.4), (u_4, 0.5), (u_5, 0.6)\}$, $F(e_1, c)=\{(u_1, 0.4), (u_2, 0.5), (u_3, 0.3), (u_4, 0.6), (u_5, 0.7)\}$, $F(e_2, a)=\{(u_1, 0.9), (u_2, 0.0), (u_3, 0.2), (u_4, 0.3), (u_5, 0.6)\}$, $F(e_2, b)=\{(u_1, 0.8), (u_2, 0.1), (u_3, 0.4), (u_4, 0.1), (u_5, 0.4)\}$, $F(e_2, c)=\{(u_1, 0.7), (u_2, 0.3), (u_3, 0.3), (u_4, 0.3), (u_5, 0.5)\}$, $F(e_3, a)=\{(u_1, 0.5), (u_2, 0.3), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_3, b)=\{(u_1, 0.4), (u_2, 0.4), (u_3, 0.7), (u_4, 0.5), (u_5, 0.3)\}$, $F(e_3, c)=\{(u_1, 0.5), (u_2, 0.3), (u_3, 0.9), (u_4, 0.7), (u_5, 0.2)\}$, $F(e_4, a)=\{(u_1, 0.6), (u_2, 0.8), (u_3, 0.5), (u_4, 0.7), (u_5, 0.6)\}$, $F(e_4, b)=\{(u_1, 0.5), (u_2, 0.6), (u_3, 0.4), (u_4, 0.6), (u_5, 0.3)\}$, $F(e_4, c)=\{(u_1, 0.3), (u_2, 0.8), (u_3, 0.5), (u_4, 0.5), (u_5, 0.6)\}$.
- **Step 2:** Separate the opinion of each expert to obtain Tables 1–3.
- **Step 3:** Assign weights $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)^t$ to each expert according to their area of expertise.
- **Step 4:** Find the weighted average of opinions for each pair (u_i, e_j) $(i=1, 2, 3, 4, 5, j=1, 2, 3, 4)$. Thus, the opinions of experts have been aggregated in this step and results have been displayed in Table 5.
  For example, for the pair (u_1, e_1), the weighted average has been calculated as

$$1-(1-0.3)^{(2\div 3)}(1-0.2)^{(2\div 3)}(1-0.4)^{(1\div 3)}=0.42697.$$

- **Step 5:** Find the average of each alternative:

1. $u_1=0.67958$;
2. $u_2=0.55252$;
3. $u_3=0.62771$;
4. $u_4=0.66825$;
5. $u_5=0.65579$.

Table 5: Aggregated opinions by using GSE weighted average operator.

	e_1	e_2	e_3	e_4
u_1	0.42697	0.95068	0.64431	0.69634
u_2	0.64431	0.17232	0.50203	0.89142
u_3	0.45567	0.45567	0.95519	0.64431
u_4	0.70760	0.34748	0.81101	0.8069
u_5	0.87571	0.69348	0.3693	0.68465

- **Step 6:** Arrange these alternatives to get the preference relation $u_2 \succ u_3 \succ u_5 \succ u_4 \succ u_1$ (the alternative with the lowest overall risk factor is the most preferred one, while the one with the highest overall risk factor is the least preferred).
- **Step 7:** Thus, the most appropriate business is u_2.
  It can be seen that the results using Algorithms 1 and 2 are highly compatible.

5 Conclusions

In this paper, the GSE set has been discussed, which can be treated as a generalization of the hesitant fuzzy set. Some basic operations associated with the structure have been defined and analyzed. For comparison purposes, the notions of "subset" and "score" have also been defined. Some important results have been proved, which failed to hold in the case of hesitant fuzzy sets. For example, the notion of containment in hesitant fuzzy sets is an open problem. One of the most widely used measures of containment was given by Xia and Xu [33]. However, in that case, the inclusion of two hesitant fuzzy elements in each other does not imply their equality. This issue can be resolved by using the proposed structure. In addition, a decision-making algorithm with the aid of the GSE set is developed. There are so many techniques to solve decision-making problems through hesitant fuzzy sets. However, the suggested technique has an advantage over the existing methods in that it considers the relative importance of the experts according to their area of expertise. A practical risk decision-making example is presented to reveal the significance of the algorithm. As a future work, we aim to study and define appropriate aggregation operators, distance, and similarity measures for GSE sets.

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