

Research Article

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Approximate recovery of the Sturm–Liouville problem on a half-line from the Weyl function

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Abstract: The inverse problem of the reconstruction of the Sturm–Liouville problem on a half-line from its Weyl function is considered. Given the Weyl function, we obtain the potential in the Schrödinger equation and the boundary condition at the origin. If the boundary condition is known, this problem is equivalent to the inverse scattering problem of the recovery of the potential, which is zero on a half-line, from a given reflection coefficient. We develop a simple and direct method for solving the inverse problem. The method consists of two steps. First, the Jost solution is computed from the Weyl function, by solving a homogeneous Riemann boundary value problem. Second, a system of linear algebraic equations is constructed for the coefficients of series representations of three solutions of the Schrödinger equation. The potential and the boundary condition are then recovered from the first component of the solution vector of the system. A numerical illustration of the functionality of the method is presented.

Keywords: Weyl function, inverse problem, Neumann series of Bessel functions, Jost solution, inverse scattering

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1 Introduction

We consider the Schrödinger equation

$$-y'' + q(x)y = \rho^2 y, \quad x > 0, \quad (1.1)$$

on a half-line with a short-range ($(1+x)q(x) \in \mathcal{L}_1(0, \infty)$) real-valued potential $q(x)$ and a homogeneous boundary condition at the origin

$$y'(0) - hy(0) = 0,$$

containing constant $h \in \mathbb{R}$. It is well known that for all $\rho \in \overline{\mathbb{C}_+}$ ($\mathbb{C}_+ := \{\rho \in \mathbb{C} : \text{Im } \rho \geq 0\}$), the Schrödinger equation possesses a unique, so-called Jost solution $e(\rho, x)$, which asymptotically, when $x \rightarrow \infty$, behaves as the exponential function $e^{i\rho x}$. The Weyl (or Titchmarsh-Weyl) function $M(\rho)$ is introduced as a quotient

$$M(\rho) := \frac{e(\rho, 0)}{e'(\rho, 0) - he(\rho, 0)}.$$

The problem of reconstructing $q(x)$ and h from a given $M(\rho)$ has attracted considerable interest (see, e.g., [4, 12, 26–28, 31]) because of its varied applications in physical models and its close relation to other inverse spectral problems. In particular, below we show that when h is known, this inverse problem is equivalent to the recovery of a potential (extended onto $x < 0$ by zero) from the reflection coefficient (an inverse scattering problem). The reflection coefficient and the Weyl function are related by a simple algebraic expression (equality (2.13) below). In several publications (in particular, [25–27]) A. G. Ramm considered the recovery of $q(x)$ from the I -function, which is defined as $\frac{1}{M(\rho)}$, when $h = 0$. He proved the uniqueness result as well as close relations of

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this inverse problem to other inverse spectral problems, including the recovery from the spectral function. The main approaches to the recovery of $q(x)$ and h from $M(\rho)$ include the use of a Gelfand–Levitan-type integral equation, in which the spectral data are obtained from $M(\rho)$; a nonlinear integro-differential equation deduced by B. Simon [12, 28]; the method of spectral mappings developed by V. A. Yurko [30, 31]; the boundary control method [4].

The purpose of this study is to present a simple method for the numerical solution of the inverse problem of the recovery of $q(x)$ and h from a given $M(\rho)$. It is based on a solution of a Riemann boundary value problem in the first step, for recovering $e(\rho, 0)$ and $\Delta(\rho) := e'(\rho, 0) - he(\rho, 0)$ and on certain series representations for solutions of (1.1). In addition to the Jost solution $e(\rho, x)$, we consider a couple of solutions satisfying certain initial conditions at the origin. For them, we have convenient series representations in the form of Neumann series of Bessel functions (NSBF) (for the definition and basic properties of the Neumann series we refer to [29, Chapter XVI]), which were obtained in [22], see also [14, 16]. The Jost solution $e(\rho, x)$ admits a power series representation in terms of a parameter z related to ρ via a Möbius mapping (relation (3.6) below). Substituting the series representations for the three solutions into a simple identity relating them to each other, we obtain a system of linear algebraic equations for the series coefficients. It is remarkable that the first component of the solution vector of the system is sufficient for recovering both the potential $q(x)$ and the constant h .

We notice that the NSBF series representations for solutions and the series representation for $e(\rho, x)$ were used separately for solving a variety of coefficient inverse problems. For example, in [17] and subsequently in [5–8, 18–20], with the aid of the NSBF series representations there were developed efficient numerical methods for solving coefficient inverse problems on finite intervals and compact quantum trees, while in [10, 13, 15, 16, 21], the power series representation for $e(\rho, x)$ was used for solving inverse spectral and scattering problems on a half-line and on the whole axis. However, this is the first work where an interaction between these different series representations is involved and is essential for the overall approach. The main system of linear algebraic equations from the last step involves both the NSBF representations for the solutions satisfying conditions at the origin and the power series representation for the Jost solution satisfying conditions at the infinity. A numerical illustration developed in the present work confirms the viability of this approach for practical computation.

Besides this introduction the paper contains five sections. In Section 2 we introduce necessary definitions and results, including the problem statement and the relation between the Weyl function and the reflection coefficient. In Section 3 we introduce the series representations for the solutions and some of their properties. In Section 4 we present the method for solving the inverse problem. In Section 4.1 we solve a Riemann boundary value problem for recovering $e(\rho, 0)$ and $\Delta(\rho)$. Here we need to know the location of zeros of $M(\rho)$ (if they exist) and the values of the function $M(\rho)$ for $\rho \geq 0$. In Section 4.2 we deduce the system of linear algebraic equations for the coefficients of the three series representations. The potential $q(x)$ and the constant h are then recovered from the first component of the solution vector of the system. Section 5 presents a numerical illustration developed in Matlab, which for an explicitly solvable potential confirms the viability of the approach.

2 Weyl function, reflection coefficient and inverse problem setting

2.1 Definitions and inverse problem setting

Let $q(x)$ be a real-valued function, Lebesgue-measurable and such that

$$(1 + x)q(x) \in \mathcal{L}_1(0, \infty). \quad (2.1)$$

We consider the Schrödinger equation

$$-y'' + q(x)y = \lambda y, \quad x > 0, \quad (2.2)$$

where $\lambda = \rho^2$, $\rho \in \overline{\mathbb{C}_+}$ is a spectral parameter and $\mathbb{C}_+ := \{\rho \in \mathbb{C} : \text{Im } \rho > 0\}$. The solution $y(x)$ is sought to satisfy the boundary condition

$$y'(0) - hy(0) = 0, \quad (2.3)$$

where h is some fixed real number. Under the conditions imposed, the Sturm–Liouville problem (2.2)–(2.3) possesses at most a finite set of simple eigenvalues, which are the values ρ^2 , which admit nontrivial solutions of (2.2)–(2.3) belonging to $\mathcal{L}_2(0, \infty)$. The eigenvalues, if they exist, are all negative (see, e.g., [23, 24, 26, 31]). By $\varphi(\rho, x)$ and $S(\rho, x)$ we denote the solutions of (2.2) satisfying the initial conditions

$$\varphi(\rho, 0) = 1, \quad \varphi'(\rho, 0) = h, \quad (2.4)$$

and

$$S(\rho, 0) = 0, \quad S'(\rho, 0) = 1. \quad (2.5)$$

For all $\rho \in \overline{\mathbb{C}_+}$, equation (2.2) possesses a unique solution $e(\rho, x)$ (the Jost solution) such that for $v = 0, 1$ the following asymptotic relations hold:

$$e^{(v)}(\rho, x) = (i\rho)^v e^{i\rho x} (1 + o(1)), \quad x \rightarrow \infty \quad (2.6)$$

uniformly in $\overline{\mathbb{C}_+}$ (see, e.g., [9, 26, 31]). Thus, for every $\rho \in \mathbb{C}_+$ fixed, the function $e(\rho, x)$ belongs to $\mathcal{L}_2(0, \infty)$. When $\operatorname{Im} \rho = 0$, it is only bounded, and $e(-\rho, x) = \overline{e(\rho, x)}$. For $\rho = i\tau$, $\tau > 0$, the Jost solution $e(\rho, x)$ is real valued. Moreover, for large values of $|\rho|$ the following asymptotic relation holds:

$$e^{(v)}(\rho, x) = (i\rho)^v e^{i\rho x} \left(1 + \frac{\omega(x)}{i\rho} + o\left(\frac{1}{\rho}\right) \right), \quad |\rho| \rightarrow \infty \quad (2.7)$$

uniformly with respect to $x \geq 0$ (see [31, p. 129]), where

$$\omega(x) := -\frac{1}{2} \int_x^{\infty} q(t) dt.$$

For all $x \in [0, \infty)$ fixed, both $e(\rho, x)$ and $e'(\rho, x)$ are analytic in \mathbb{C}_+ and continuous in $\overline{\mathbb{C}_+}$ (see [31, p. 206]).

Denote

$$\Delta(\rho) := e'(\rho, 0) - h e(\rho, 0)$$

and

$$\Phi(\rho, x) = \frac{e(\rho, x)}{\Delta(\rho)}. \quad (2.8)$$

The function $\Phi(\rho, x)$ (called the Weyl solution) satisfies equation (2.2), as well as the conditions

$$\Phi'(\rho, 0) - h\Phi(\rho, 0) = 1, \quad \Phi(\rho, x) = O(e^{i\rho x}), \quad x \rightarrow \infty. \quad (2.9)$$

Note that $\Phi(\rho, x)$ is defined uniquely by (2.2) and (2.9).

Denote

$$M(\rho) := \frac{e(\rho, 0)}{\Delta(\rho)}. \quad (2.10)$$

We call $M(\rho)$ the Weyl function. Note that for $\rho \in \mathbb{R}$, $M(-\rho) = \overline{M(\rho)}$.

The singularities of $M(\rho)$, $\rho \in \mathbb{C}_+$, if they exist, coincide with the square roots of the eigenvalues of (2.2)–(2.3), while zeros of $M(\rho)$, $\rho \in \mathbb{C}_+$, if they exist, coincide with the square roots of the eigenvalues of (2.2) with the Dirichlet boundary condition

$$y(0) = 0. \quad (2.11)$$

In both cases the eigenvalues are strictly negative numbers.

We have an obvious relation, which is a consequence of (2.4), (2.5) and (2.8),

$$\Phi(\rho, x) = M(\rho)\varphi(\rho, x) + S(\rho, x). \quad (2.12)$$

We are interested in the following inverse problem.

Problem (IP1). Given $M(\rho)$, find $q(x)$ and h .

A fairly complete theory of this problem can be found in [31]. The main result of the present work is a practical method for the reconstruction of $q(x)$ and h from the knowledge of $M(\rho)$ for $\rho \geq 0$ and of its zeros. The problem is of interest in many applications. Depending on a concrete physical model, the Weyl function, for example,

may represent a ratio of the components of an electromagnetic field inciding perpendicularly onto a layered medium (see [26, p. 3]) or be defined by a reflection coefficient in a scattering problem (see below). Moreover, $M(\rho)$ is closely related to the spectral function and to the scattering data in such a way that it can be computed from each of these data (see the work of A.G. Ramm who considers the I -function, which is $1/M(\rho)$ for $h = 0$ [26, 27] and the book of V. A. Yurko [31]). For solving (IP1), V. A. Yurko developed a method of spectral mappings which allows one to reduce the problem to the solution of an integral equation involving integration along an infinite contour in a complex plane. To our best knowledge the method has not been realized numerically.

2.2 Relation of (IP1) to inverse scattering problem

Note that $M(\rho)$ is closely related to the reflection coefficient. Consider equation (2.2) on a whole axis $(-\infty, \infty)$ with a potential $q(x)$, which is identical zero for $x < 0$, and satisfies (2.1). Then for a plane wave e^{ipx} incoming from $-\infty$, the solution of the scattering problem has the form

$$u(\rho, x) = \begin{cases} e^{ipx} + R(\rho)e^{-ipx}, & x < 0, \\ T(\rho)e(\rho, x), & x > 0, \end{cases}$$

where $R(\rho)$ and $T(\rho)$ are the reflection and transmission coefficients, respectively. Additionally, the continuity of $u(\rho, x)$ and $u'(\rho, x)$ at the origin leads to the relations

$$1 + R(\rho) = T(\rho)e(\rho, 0)$$

and

$$i\rho(1 - R(\rho)) = T(\rho)e'(\rho, 0).$$

Thus,

$$M(\rho) = \frac{e(\rho, 0)}{\Delta(\rho)} = \frac{1 + R(\rho)}{i\rho(1 - R(\rho)) - h(1 + R(\rho))}, \quad (2.13)$$

and hence

$$R(\rho) = \frac{(i\rho - h)M(\rho) - 1}{(i\rho + h)M(\rho) + 1}.$$

Thus, if h is known, the problem of the reconstruction of $q(x)$ from the Weyl function is equivalent to the reconstruction from the reflection coefficient. The uniqueness result for this problem was obtained in [2].

In particular, when $h = 0$, we have

$$M(\rho) = \frac{1 + R(\rho)}{i\rho(1 - R(\rho))}, \quad R(\rho) = \frac{i\rho M(\rho) - 1}{i\rho M(\rho) + 1}.$$

Remark 2.1. Directly from (2.10) or (2.13), we deduce the following relation:

$$M_h(\rho) = \frac{M_0(\rho)}{1 - hM_0(\rho)},$$

where the subindex refers to the value of h in the definition of the Weyl function. As a consequence we observe that even if the discrete spectrum of (2.2)–(2.3) is empty for $h = 0$, any point $\rho_* = i\tau$, $\tau > 0$, can be made a singular point of (2.2)–(2.3) with the same potential $q(x)$, by choosing $h = \frac{1}{M_0(\rho_*)}$. Similarly, $\rho = 0$ can be made a singularity of $M_h(\rho)$ for some $h \in \mathbb{R}$.

3 Series representations for solutions

In this section we recall convenient series representations for the solutions $\varphi(\rho, x)$, $S(\rho, x)$ and $e(\rho, x)$ which play an essential role in our approach for solving (IP1).

Theorem 3.1 ([22]). *The solutions $\varphi(\rho, x)$ and $S(\rho, x)$ of (2.2) admit the series representations*

$$\varphi(\rho, x) = \cos \rho x + \sum_{n=0}^{\infty} g_n(x) \mathbf{j}_{2n}(\rho x), \quad (3.1)$$

$$S(\rho, x) = \frac{\sin \rho x}{\rho} + \frac{1}{\rho} \sum_{n=0}^{\infty} s_n(x) \mathbf{j}_{2n+1}(\rho x), \quad (3.2)$$

where $\mathbf{j}_k(z)$ stands for the spherical Bessel function of order k ($\mathbf{j}_k(z) := \sqrt{\frac{\pi}{2z}} J_{k+\frac{1}{2}}(z)$, see, e.g., [1]). For every $\rho \in \mathbb{C}$ the series converge pointwise. For every $x > 0$ the series converge uniformly in any strip of the complex plane of the variable ρ , parallel to the real axis. In particular, the remainders of their partial sums

$$\varphi_N(\rho, x) := \cos(\rho x) + \sum_{n=0}^N g_n(x) \mathbf{j}_{2n}(\rho x) \quad \text{and} \quad S_N(\rho, x) := \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho} \sum_{n=0}^N s_n(x) \mathbf{j}_{2n+1}(\rho x)$$

admit the estimates

$$|\varphi(\rho, x) - \varphi_N(\rho, x)| \leq \frac{\varepsilon_N(x) \sinh(ax)}{a} \quad \text{and} \quad |\rho S(\rho, x) - \rho S_N(\rho, x)| \leq \frac{\varepsilon_N(x) \sinh(ax)}{a} \quad (3.3)$$

for any $\rho \in \mathbb{C}$ belonging to a strip $|\operatorname{Im} \rho| \leq a$, $a \geq 0$, where $\varepsilon_N(x)$ is a positive function tending to zero when $N \rightarrow \infty$. The first coefficients $g_0(x)$, $s_0(x)$ have the form

$$g_0(x) = \varphi(0, x) - 1, \quad s_0(x) = 3 \left(\frac{S(0, x)}{x} - 1 \right), \quad (3.4)$$

and the rest of the coefficients can be calculated following a simple recurrent integration procedure.

For more results concerning representations (3.1) and (3.2) we refer to [22] and [16].

Theorem 3.2 ([15]). *The solution $e(\rho, x)$ of (2.2) admits the series representation*

$$e(\rho, x) = e^{i\rho x} \left(1 + (z + 1) \sum_{n=0}^{\infty} a_n(x) z^n \right), \quad (3.5)$$

where

$$z := \frac{\frac{1}{2} + i\rho}{\frac{1}{2} - i\rho}. \quad (3.6)$$

The series is convergent in the unit disk of the complex plane of the variable z . For any $x \geq 0$, the series $\sum_{n=0}^{\infty} a_n^2(x)$ converges. The first coefficient has the form

$$a_0(x) = e \left(\frac{i}{2}, x \right) e^{\frac{x}{2}} - 1,$$

and the rest of the coefficients can be calculated following a simple recurrent integration procedure [11].

The coefficients $\{g_n(x), s_n(x), a_n(x)\}$ in series representations (3.1), (3.2) and (3.5) are real valued. Notice that (3.6) is a Möbius mapping of the upper half-plane of the complex variable ρ onto the unit disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$. It maps the point $\rho = 0$ to $z = 1$, the ray $\rho = i\tau$, $\tau > 0$ (corresponding to $\lambda < 0$) to the interval $(-1, 1)$ in terms of z , and when τ runs from 0 to $+\infty$, z runs from 1 to -1. When ρ (and hence λ) runs from 0 to $+\infty$ along the real line, z runs from 1 to -1 along the upper unitary half-circle.

For the remainder of the series (3.5) the following estimates are valid. If $\operatorname{Im} \rho > 0$, we have

$$|e(\rho, x) - e_N(\rho, x)| \leq \varepsilon_N(x) \frac{e^{-\operatorname{Im} \rho x}}{\sqrt{2 \operatorname{Im} \rho}} \quad (3.7)$$

where $e_N(\rho, x) := e^{i\rho x} (1 + (z + 1) \sum_{n=0}^N a_n(x) z^n)$ and $\varepsilon_N(x) := (\sum_{n=1}^{\infty} |a_n(x)|^2)^{\frac{1}{2}}$. If $\rho \in \mathbb{R}$, the equality holds

$$\|e(\cdot, x) - e_N(\cdot, x)\|_{\mathcal{L}_2(-\infty, \infty)} = \sqrt{2\pi} \varepsilon_N(x). \quad (3.8)$$

For more results concerning representation (3.5) we refer to [16] and [21].

4 Solution of (IP1)

4.1 Reconstruction of $e(\rho, 0)$ and $\Delta(\rho)$

Given $M(\rho)$ for $\rho \geq 0$ and the location of its zeros (h is unknown), in a first step, we are interested in recovering $e(\rho, 0)$ and $\Delta(\rho)$. Denote

$$G(\rho) := -\frac{\rho|M(\rho)|^2}{\operatorname{Im} M(\rho)}, \quad \rho \in \mathbb{R},$$

and

$$w(\rho) := \begin{cases} 1 & \text{if } M(\rho) \neq 0 \text{ in } \overline{\mathbb{C}_+}, \\ \prod_{k=1}^K \frac{\rho - i\tau_k}{\rho + i\tau_k} & \text{if } M(\rho) \text{ has } K \text{ zeros in } \mathbb{C}_+. \end{cases}$$

Here $\rho_k = i\tau_k$, $\tau_k > 0$, denote zeros of $M(\rho)$, if they exist.

Theorem 4.1. *If $M(0) \neq 0$, the following equalities are valid:*

$$e(\rho, 0) = w(\rho) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G(t)}{t - \rho} dt\right), \quad \operatorname{Im} \rho > 0, \quad (4.1)$$

and

$$e(\rho, 0) = w(\rho) \exp\left(\frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{\log G(t)}{t - \rho} dt + \frac{1}{2} \log G(\rho)\right), \quad \rho \in \mathbb{R},$$

where the integral is understood in the sense of the Cauchy principal value.

If $M(0) = 0$, the following equalities are valid:

$$e(\rho, 0) = \frac{\rho}{\rho + i\tau} w(\rho) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G_0(t)}{t - \rho} dt\right), \quad \operatorname{Im} \rho > 0,$$

and

$$e(\rho, 0) = \frac{\rho}{\rho + i\tau} w(\rho) \exp\left(\frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{\log G_0(t)}{t - \rho} dt + \frac{1}{2} \log G_0(\rho)\right), \quad \rho \in \mathbb{R},$$

where

$$G_0(\rho) := \frac{\rho^2 + \tau^2}{\rho^2} G(\rho),$$

where $\tau > 0$ is such that $\tau \neq \tau_k$, $k = 1, \dots, K$.

Proof. Note that

$$\frac{1}{|\Delta(\rho)|^2} = -\frac{\operatorname{Im} M(\rho)}{\rho}, \quad \rho \in \mathbb{R}. \quad (4.2)$$

Indeed, for $\rho \in \mathbb{R}$ consider

$$\begin{aligned} \operatorname{Im} M(\rho) &= \frac{1}{2i} (M(\rho) - \overline{M(\rho)}) = \frac{1}{2i} \left(\frac{e(\rho, 0)}{\Delta(\rho)} - \frac{e(-\rho, 0)}{\Delta(-\rho)} \right) \\ &= \frac{1}{2i} \left(\frac{e(\rho, 0)\Delta(\rho) - e(-\rho, 0)\Delta(\rho)}{|\Delta(\rho)|^2} \right) = \frac{1}{2i} \left(\frac{W\{e(\rho, 0), e(-\rho, 0)\}}{|\Delta(\rho)|^2} \right), \end{aligned}$$

where $W\{\cdot, \cdot\}$ stands for a Wronskian. Since the Wronskian of any two solutions of (2.2) is constant, using (2.6), we obtain $W\{e(\rho, 0), e(-\rho, 0)\} = -2i\rho$, and hence (4.2) is valid.

Multiplying (4.2) by $|e(\rho, 0)|^2$, we obtain

$$|e(\rho, 0)|^2 = -\frac{\rho|M(\rho)|^2}{\operatorname{Im} M(\rho)}, \quad \rho \in \mathbb{R}. \quad (4.3)$$

This equality can be written as

$$e(\rho, 0) = G(\rho) \frac{1}{e(-\rho, 0)}, \quad \rho \in \mathbb{R}, \quad (4.4)$$

where from (4.3) we have that $G(\rho) > 0$ for all $\rho \in \mathbb{R} \setminus \{0\}$, and $G(\pm\infty) = 1$ (see (2.7)). From (4.3) it can also be seen that $\rho = 0$ can be a zero of second order of $G(\rho)$.

Let us suppose, first, that $M(\rho)$ does not have zeros in $\overline{\mathbb{C}_+}$, or equivalently, $e(\rho, 0)$ does not have zeros in $\overline{\mathbb{C}_+}$. Then (4.4) can be written as

$$h_+(\rho) = G(\rho)h_-(\rho), \quad \rho \in \mathbb{R}, \quad (4.5)$$

where $h_+(\rho)$ and $h_-(\rho)$ are limit values from above and below, respectively, of a function $h(\rho)$, which is analytic and different from zero in \mathbb{C}_+ and \mathbb{C}_- . Moreover, $G(\rho)$ is a bounded, nonvanishing function. Thus, we have a classical homogeneous Riemann problem on a half-plane, and hence

$$h(\rho) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G(t)}{t - \rho} dt\right). \quad (4.6)$$

Thus, if $M(\rho) \neq 0$ in $\overline{\mathbb{C}_+}$, then $e(\rho, 0)$ is recovered from $M(\rho)$ by the formula

$$e(\rho, 0) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G(t)}{t - \rho} dt\right), \quad \operatorname{Im} \rho > 0.$$

Let us assume now that $M(\rho)$ has $K \geq 1$ zeros in \mathbb{C}_+ . They are of the form $\rho_k = i\tau_k$, $\tau_k > 0$, $k = 1, \dots, K$. Multiplying both sides of (4.4) by $\frac{1}{w(\rho)}$ with

$$w(\rho) := \prod_{k=1}^K \frac{\rho - i\tau_k}{\rho + i\tau_k}$$

we arrive at (4.5) with

$$h_+(\rho) := \frac{e(\rho, 0)}{w(\rho)} \quad \text{and} \quad h_-(\rho) := \frac{1}{w(\rho)e(-\rho, 0)} = \frac{w(-\rho)}{e(-\rho, 0)}. \quad (4.7)$$

Now $h_+(\rho)$ and $h_-(\rho)$ are limit values from above and below, respectively, of a function $h(\rho)$, which is analytic and different from zero in \mathbb{C}_+ and \mathbb{C}_- . Hence $h(\rho)$ has the form (4.6), and consequently,

$$e(\rho, 0) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G(t)}{t - \rho} dt\right) \prod_{k=1}^K \frac{\rho - i\tau_k}{\rho + i\tau_k}, \quad \operatorname{Im} \rho > 0.$$

Thus,

$$e(\rho, 0) = w(\rho) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G(t)}{t - \rho} dt\right), \quad \operatorname{Im} \rho > 0.$$

On the real axis, with the aid of Plemelj–Sokhotski formulas, one has

$$e(\rho, 0) = w(\rho) \exp\left(\frac{1}{2\pi i} \operatorname{p.v.} \int_{-\infty}^{\infty} \frac{\log G(t)}{t - \rho} dt + \frac{1}{2} \log G(\rho)\right), \quad \rho \in \mathbb{R},$$

where the integral is understood in the sense of the Cauchy principal value.

Finally, if $M(0) = 0$, we choose any $\tau > 0$, such that $\tau \neq \tau_k$, $k = 1, \dots, K$, and multiply equality (4.5) (with $h_+(\rho)$ and $h_-(\rho)$ defined in (4.7)) by $\frac{\rho^2 + \tau^2}{\rho^2}$:

$$\frac{\rho + i\tau}{\rho} h_+(\rho) = \frac{\rho^2 + \tau^2}{\rho^2} G(\rho) \left(\frac{\rho}{\rho - i\tau} h_-(\rho) \right), \quad \rho \in \mathbb{R}.$$

Denoting

$$H_+(\rho) := \frac{\rho + i\tau}{\rho} h_+(\rho), \quad H_-(\rho) := \frac{\rho}{\rho - i\tau} h_-(\rho), \quad G_0(\rho) := \frac{\rho^2 + \tau^2}{\rho^2} G(\rho),$$

we obtain

$$H_+(\rho) = G_0(\rho)H_-(\rho), \quad \rho \in \mathbb{R}.$$

Here $H_+(\rho)$ and $H_-(\rho)$ are limit values from above and below, respectively, of a function $H(\rho)$, which is analytic in \mathbb{C}_+ and \mathbb{C}_- , $G_0(\rho)$ is a bounded, nonvanishing function. Thus,

$$H(\rho) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G_0(t)}{t - \rho} dt\right).$$

Consequently,

$$e(\rho, 0) = \frac{\rho}{\rho + i\tau} w(\rho) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log G_0(t)}{t - \rho} dt\right), \quad \operatorname{Im} \rho > 0,$$

and

$$e(\rho, 0) = \frac{\rho}{\rho + i\tau} w(\rho) \exp\left(\frac{1}{2\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{\log G_0(t)}{t - \rho} dt + \frac{1}{2} \log G_0(\rho)\right), \quad \rho \in \mathbb{R}.$$

This finishes the proof. \square

Thus, $e(\rho, 0)$ is recovered from the knowledge of $M(\rho)$ for $\rho \geq 0$ and of the location of its zeros. In its turn, $\Delta(\rho)$ is obtained from

$$\Delta(\rho) = \frac{e(\rho, 0)}{M(\rho)}. \quad (4.8)$$

4.2 Reconstruction of $q(x)$ and h

For recovering $q(x)$ and h from $e(\rho, 0)$ and $\Delta(\rho)$, we make use of identity (2.12). It can be written as

$$e(\rho, x) = e(\rho, 0)\varphi(\rho, x) + \Delta(\rho)S(\rho, x). \quad (4.9)$$

Substituting series representations (3.1), (3.2) and (3.5), we obtain the equality

$$\begin{aligned} e(\rho, 0) \sum_{n=0}^{\infty} g_n(x) \mathbf{j}_{2n}(\rho x) + \frac{\Delta(\rho)}{\rho} \sum_{n=0}^{\infty} s_n(x) \mathbf{j}_{2n+1}(\rho x) - e^{i\rho x}(z+1) \sum_{n=0}^{\infty} a_n(x) z^n \\ = e^{i\rho x} - e(\rho, 0) \cos(\rho x) - \frac{\sin(\rho x)\Delta(\rho)}{\rho} \quad \text{for all } \rho \in \overline{\mathbb{C}_+} \text{ and } x \geq 0. \end{aligned} \quad (4.10)$$

Here $e(\rho, 0)$ and $\Delta(\rho)$ are known, computed in the first step.

Now, let us fix $x \geq 0$ and consider a sufficiently large number K of the points $\rho_k \in \overline{\mathbb{C}_+}$. To compute approximate values of a number of the coefficients $\{g_n(x), s_n(x)\}_{n=0}^{N_1}$, $\{a_n(x)\}_{n=0}^{N_2}$, from (4.10) we obtain a finite system of linear algebraic equations

$$\begin{aligned} e(\rho_k, 0) \sum_{n=0}^{N_1} g_n(x) \mathbf{j}_{2n}(\rho_k x) + \frac{\Delta(\rho_k)}{\rho_k} \sum_{n=0}^{N_1} s_n(x) \mathbf{j}_{2n+1}(\rho_k x) - e^{i\rho_k x}(z+1) \sum_{n=0}^{N_2} a_n(x) z^n \\ = e^{i\rho_k x} - e(\rho_k, 0) \cos(\rho_k x) - \frac{\sin(\rho_k x)\Delta(\rho_k)}{\rho_k}, \quad k = 1, \dots, K. \end{aligned} \quad (4.11)$$

Note that the number of equations K can be taken arbitrarily large, while the number of the unknowns $(2N_1 + 1) + N_2 + 1$, due to estimates (3.3), (3.7), (3.8) can be kept relatively small, especially for a convenient choice of the points ρ_k belonging to a strip $0 \leq \operatorname{Im} \rho \leq C$, with $C > 0$ being not too large. Solving this system for a sufficiently dense set of points $x_j \in (0, b]$ on an arbitrary interval of interest $(0, b] \subset (0, \infty)$, we find $g_0(x)$ in $[0, b]$ ($g_0(0) = 0$). Now, due to the first equality from (3.4), we have that $\varphi(0, x) = g_0(x) + 1$, and thus,

$$q(x) = \frac{g_0''(x)}{g_0(x) + 1} \quad \text{and} \quad h = g_0'(0). \quad (4.12)$$

Remark 4.2. For potentials which additionally to (2.1) belong to the Sobolev class $\mathcal{H}^1[0, b]$, other series representations, from [18], can be used for the solutions $\varphi(\rho, x)$ and $S(\rho, x)$. They also have the form of Neumann series of Bessel functions, however unlike (4.12), in the last step, the recovery of $q(x)$ does not need any differentiation.

5 Numerical illustration

The proposed approach can be implemented directly using an available numerical computing environment. The reported computation was performed in Matlab R2023b on an Intel i7-1360P equipped laptop computer and took no more than several seconds.

Example 1. Consider the potential

$$q(x) = \frac{1200e^{10x}}{(6e^{10x} - 1)^2}.$$

From [3, Example 6.1] we have

$$e(\rho, x) = e^{i\rho x} \left(1 + \frac{10i}{\rho + 5i} \frac{1}{6e^{10x} - 1} \right)$$

and hence for $h = 0$ we obtain that

$$e(\rho, 0) = 1 + \frac{2i}{\rho + 5i}, \quad \Delta(\rho) = \frac{i\rho(\rho + 7i) - 24i}{\rho + 5i},$$

and

$$M(\rho) = \frac{\rho + 7i}{i\rho(\rho + 7i) - 24i}.$$

It is easy to see that no eigenvalue exists, and $M(0) \neq 0$.

In the first step we recovered $e(\rho, 0)$ and $\Delta(\rho)$ from $M(\rho)$ numerically, by using (4.1). For this, the integral from (4.1) was computed by the modified 6 point Newton–Cotes integration formula (see [22] for details) on a large interval $(-T, T)$ with $T = 3000$ and for $K = 501$ points ρ_k with $\text{Im } \rho_k = 1$ and $\text{Re } \rho_k$ distributed logarithmically equally spaced on the segment $[0.001, 1001]$. That is, $\text{Re } \rho_k = 10^{\alpha_k}$ with $\alpha_k, k = 1, \dots, K$, being distributed uniformly on $[\log(0.001), \log(1001)] \approx [-3, 3]$. Thus, $e(\rho, 0)$ was computed with the aid of (4.1), and then $\Delta(\rho)$ was computed by (4.8).

In the second step, system (4.11) was considered at 251 points $x \in [0, \pi]$ and for the same set of the points ρ_k . Note that, since all the unknown $\{g_n(x), s_n(x)\}_{n=0}^{N_1}$, $\{a_n(x)\}_{n=0}^{N_2}$ are real, for each ρ_k , equation (4.11) gives us two equations: with real and imaginary parts of matrix coefficients. Finally, numbers N_1 and N_2 were chosen automatically. For every $x_j \in [0, \pi]$ we compute the rank of each of the three thirds of the matrix of the system (matrix consisting of the first $N_1 + 1$ columns, matrix consisting of the columns $N_1 + 2, \dots, 2N_1 + 2$, and matrix consisting of the columns $2N_1 + 3, \dots, 2N_1 + N_2 + 3$), which is the number of singular values of the matrix that are larger than a default tolerance (which was chosen as 10^{-8}). Then the number of the columns remained in each third of the matrix is taken equal to the corresponding rank, this means that for different points x_j , a different number of the unknown NSBF coefficients is computed, so that, in fact, system (4.11) is considered in the form

$$\begin{aligned} & \sum_{n=0}^{N_1(x)} g_n(x) \operatorname{Re}(e(\rho_k, 0) j_{2n}(\rho_k x)) + \sum_{n=0}^{N_1(x)} s_n(x) \operatorname{Re} \left(\frac{\Delta(\rho_k) j_{2n+1}(\rho_k x)}{\rho_k} \right) - \sum_{n=0}^{N_2(x)} a_n(x) \operatorname{Re}(e^{i\rho_k x} (z+1) z^n) \\ &= \operatorname{Re} \left(e^{i\rho_k x} - e(\rho_k, 0) \cos(\rho_k x) - \frac{\sin(\rho_k x) \Delta(\rho_k)}{\rho_k} \right), \\ & \sum_{n=0}^{N_1(x)} g_n(x) \operatorname{Im}(e(\rho_k, 0) j_{2n}(\rho_k x)) + \sum_{n=0}^{N_1(x)} s_n(x) \operatorname{Im} \left(\frac{\Delta(\rho_k) j_{2n+1}(\rho_k x)}{\rho_k} \right) - \sum_{n=0}^{N_2(x)} a_n(x) \operatorname{Im}(e^{i\rho_k x} (z+1) z^n) \\ &= \operatorname{Im} \left(e^{i\rho_k x} - e(\rho_k, 0) \cos(\rho_k x) - \frac{\sin(\rho_k x) \Delta(\rho_k)}{\rho_k} \right), \quad k = 1, \dots, K. \end{aligned}$$

Comparison of $e(\rho_k, 0)$ computed numerically in the first step with those obtained from the exact formula showed the maximum absolute error of $1 \cdot 10^{-7}$. Solution of the system of linear algebraic equations with thus computed $e(\rho_k, 0)$ and $\Delta(\rho_k)$ led to $g_0(x_j)$ computed with the maximum absolute error of $2.6 \cdot 10^{-5}$. Finally, $q(x)$ was computed by (4.12) with the maximum absolute error of 0.025 (see Figure 1), while h was recovered with the absolute error $4 \cdot 10^{-4}$.

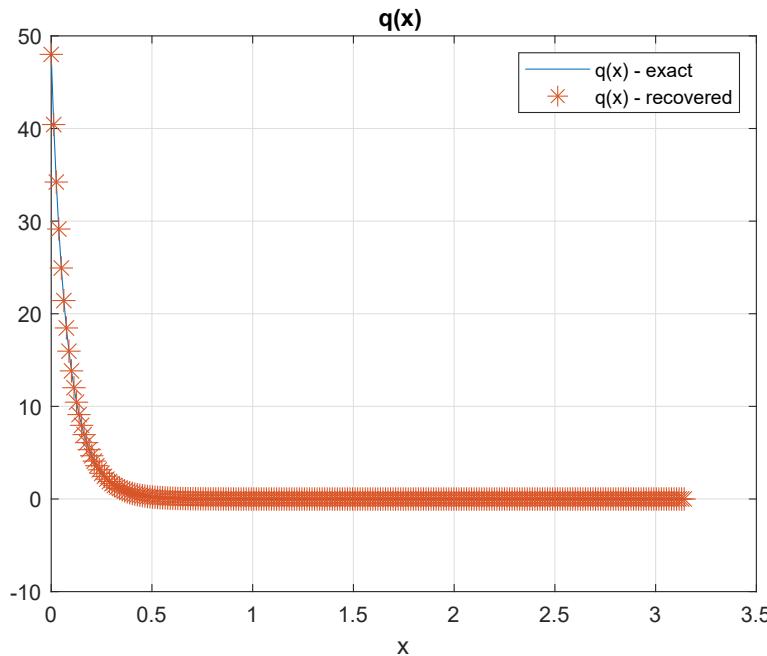


Figure 1: Potential from Example 1, recovered with a maximum absolute error 0.025.

For the second test we considered the same potential, but $h = -4$. In this case there is one eigenvalue $\rho_*^2 = -1$, which however does not affect any calculation. With all the same parameters chosen as above, $q(x)$ was computed with the maximum absolute error of 0.028, while h was recovered with the absolute error $3 \cdot 10^{-4}$.

6 Conclusions

An approach to approximate recovery of the Sturm–Liouville problem on a half-line from the Weyl function is developed. More precisely, one needs to know the location of zeros of the Weyl function (if they exist) and the values of the Weyl function on the half-line $\rho \geq 0$. First, two characteristic functions are computed (the Jost function and the characteristic function of the Sturm–Liouville problem). For this, explicit formulas are obtained (deduced by solving a Riemann boundary value problem). Second, by solving a system of linear algebraic equations, an auxiliary function is obtained, from which both the potential $q(x)$ and the constant h from the boundary condition of the Sturm–Liouville problem are recovered. A numerical illustration is presented, confirming the practical viability of the approach.

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