DE GRUYTER J. Inverse Ill-Posed Probl. 2025; 33(3): 401–428

Research Article

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Integrating the probe and singular sources methods: III. Mixed obstacle case

https://doi.org/10.1515/jiip-2024-0055
Received August 14, 2024; accepted March 4, 2025

Abstract: The main purpose of this paper is to develop further the integrated theory of the probe and singular sources methods (IPS) which may work for a group of inverse obstacle problems. Here as a representative and typical member of the group, an inverse obstacle problem governed by the Helmholtz equation with a fixed wave number in a bounded domain is considered. It is assumed that the solutions of the Helmholtz equation outside the set of unknown obstacles satisfy the homogeneous Dirichlet or Neumann boundary conditions on each surface of obstacles. This is the case when two extreme types of obstacles are embedded in a medium. By considering this case, not only a concise technique for IPS is introduced but also a general correspondence principle from IPS to the probe method is suggested. Besides, as a corollary it is shown that the probe method together with the singular sources method reformulated in terms of the probe method has the Side B under a smallness conditions on the wave number k, which is the blowing up property of a sequence computed from the associated Dirichlet-to-Neumann map.

Keywords: Inverse obstacle problem, probe method, singular sources method, Helmholtz equation, Dirichlet boundary condition, Neumann boundary condition, third indicator function, IPS function

MSC 2020: 35R30, 78A46, 35J05, 35E05

1 Introduction

Both the probe method of Ikehata [7, 8] (later reformulated in [11]) and singular sources method of Potthast [19, 20] now become well-known classical analytical methods for reconstruction issue of *inverse obstacle problems* governed by partial differential equations. This paper is concerned with the *integrated theory* of the probe and singular sources methods (IPS), which is initiated by the author himself in [16, 17]. In particular, we focus on the role of IPS in deriving the probe and singular sources methods together with introducing a technique to treat some kind of inverse obstacle problems governed by partial differential equations. For the purpose we consider a prototype inverse obstacle problem governed by the Helmholtz equation with a fixed wave number.

Now let us formulate the prototype problem. Let Ω be a bounded domain of $\mathbb{R}^3$ with Lipschitz-boundary [6]. We denote by D a mathematical model of *discontinuity* embedded in the background medium Ω. We assume that D takes the form $D = D_n \cup D_d$, where D_n and D_d are open subsets of $\mathbb{R}^3$ with Lipschitz-boundary with $\overline{D_n} \cap \overline{D_d} = \emptyset$, $\overline{D_n} \cup \overline{D_d} \subset \Omega$ and that $\Omega \setminus (\overline{D_n} \cup \overline{D_d})$ is connected. We denote by ν the unit outward normal vector to not only $\partial\Omega$ but also ∂D. On the surfaces of D_n and D_d two boundary conditions of different type are imposed as specified below.

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Let $k \geq 0$. Given an arbitrary $f \in H^{\frac{1}{2}}(\partial\Omega)$, let $u = u(x)$ in $H^1(\Omega \setminus \overline{D})$ be the weak solution of

$$\begin{cases} \Delta u + k^2 u = 0, & x \in \Omega \setminus \overline{D}, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial D_n, \\ u = 0, & x \in \partial D_d, \\ u = f, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

This means that, $u = f$ on $\partial\Omega$, $u = 0$ on ∂D_d in the sense of the trace and, for all $\varphi \in H^1(\Omega \setminus \overline{D})$ with $\varphi = 0$ on $\partial\Omega$ and $\varphi = 0$ on ∂D_d in the sense of the trace, we have

$$-\int_{\Omega\setminus\overline{D}} \nabla u \cdot \nabla\varphi\, dx + \int_{\Omega\setminus\overline{D}} k^2 u\varphi\, dx = 0.$$

Then the bounded linear functional $\frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$ is well defined via the formula

$$\left\langle \frac{\partial u}{\partial \nu}|_{\partial\Omega}, g \right\rangle = \int_{\Omega\setminus\overline{D}} \nabla u \cdot \nabla\phi\, dx - \int_{\Omega\setminus\overline{D}} k^2 u\phi\, dx, \quad g \in H^{\frac{1}{2}}(\partial\Omega),$$

where $\phi \in H^1(\Omega \setminus \overline{D})$ such that $\phi = g$ on $\partial\Omega$ and $\phi = 0$ on ∂D_d in the sense of the trace. Note that unless otherwise specified the functions appearing in this paper are always real-valued; the symbol ν denotes the unit outward normal vector field on $\partial\Omega$ and $\partial D = \partial D_n \cup \partial D_d$.

In this paper, by considering the prototype inverse obstacle problem mentioned below, we further develop a technique to the integrated theory of the probe and singular sources methods.

Problem. Extract information about the geometry of D_n and D_d from the $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ corresponding to *infinitely many* f.

For the problem to have meaning we impose a restriction on k:

Assumption 1. The boundary value problem (1.1) with $f = 0$ has only a trivial solution.

Under Assumption 1 it is well known that the weak solution u of (1.1) exists and unique. Then the map

$$\Lambda_D : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$$

is well defined by

$$\Lambda_D f = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

This is called the Dirichlet-to-Neumann map. So Problem becomes the extraction problem of information about the geometry of D_n and D_d from the graph of the Dirichlet-to-Neumann map Λ_D or its partial knowledge.

It follows from the definition we have the symmetry: for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$,

$$\langle \Lambda_D f, g \rangle = \langle \Lambda_D g, f \rangle.$$

Besides, if $f \in H^{\frac{3}{2}}(\partial\Omega)$ and both $\partial\Omega$ and ∂D are C^2, then $u \in H^2(\Omega \setminus \overline{D})$ and thus $\Lambda_D f = \frac{\partial u}{\partial \nu}|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$ in the sense of the trace [6]. Then, integration by parts (e.g., [6, Lemma 1.5.3.7]) yields the surface integral expression of $\langle \Lambda_D f, g \rangle$ for all $f \in H^{\frac{3}{2}}(\partial\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$,

$$\langle \Lambda_D f, g \rangle = \int_{\partial\Omega} \Lambda_D f(z) g(z)\, dS(z).$$

In this paper, we always consider k such that Assumption 1 is satisfied and, unless otherwise stated the C^2-regularity of $\partial\Omega$ and ∂D are assumed. Note that $k = 0$ satisfies Assumption 1

In [16] by considering the case that $D_d = \emptyset$ in (1.1) and $k = 0$, the author introduced the integrated theory of the probe and singular sources methods. In [17] IPS has been applied to an inverse obstacle problem governed by the Stokes system. Therein a technique to treat a system is introduced. In this paper we pursuit IPS further by considering the case when $D_d \neq \emptyset$ and $k \geq 0$. Especially, with application to elastic bodies in mind, it would be interesting to consider such a case. It is expected that this situation causes some problems due to the coexistence of two different boundary conditions and $k \neq 0$. Besides, it cannot be said that the IPS concept was thoroughly developed in [16, 17] in the sense that the treatment of the probe method therein is independent from IPS. This time we would like to show that not only the singular sources method but also the probe method itself is derived from IPS.

It should be pointed out that, using the *original* probe method [9], this type of problem itself has been considered by Cheng, Liu, Nakamura and Wang [2]. However, they do not have the view point of the IPS developed in this paper. See also Remark 3.4 for more detailed comparison. Note that, for mixed obstacles placed in the *whole space* there are some applications of the *factorization method* [3, 5, 18], *monotonicity method* [1] and both [4].

1.1 The IPS function

The IPS in this paper starts with introducing a family of singular solutions for the back ground medium.

Let $\mathcal{G} = \{G(\,\cdot\,, x)\}_{x\in\Omega}$ be a family of distributions in Ω indexed with $x \in \Omega$ having the form

$$G(y, x) = G(y - x) + H(y, x), \tag{1.2}$$

where

$$G(y - x) = \frac{\cos k|y - x|}{4\pi|y - x|}$$

and $H(\,\cdot\,, x) \in H^2(\Omega)$ is a real-valued solution of the Helmholtz equation in Ω such that, for each $\epsilon > 0$,

$$\sup_{x\in\Omega,\, \mathrm{dist}(x,\partial\Omega)>\epsilon} \|H(\,\cdot\,, x)\|_{H^2(\Omega)} < \infty. \tag{1.3}$$

Note that $G(\,\cdot\, - x)$ coincides with the real part of the standard (complex-valued) fundamental solution of the Helmholtz equation

$$\Phi(y - x) = \frac{e^{ik|y-x|}}{4\pi|y - x|}.$$

Since the imaginary part of $\Phi(\,\cdot\, - x)$ has the unique extension to the whole space as the entire solution of the Helmholtz equation, the function $G(y - x)$ also satisfies

$$\Delta G(\,\cdot\, - x) + k^2 G(\,\cdot\, - x) + \delta(\,\cdot\, - x) = 0$$

as the distribution of $y \in \mathbb{R}^3$ for each fixed $x \in \mathbb{R}^3$.

Definition 1.1. Given $\mathcal{G}$ and $x \in \Omega \setminus \overline{D}$, let $W = W_x(y; \mathcal{G}) = W(y)$ in $H^2(\Omega \setminus \overline{D})$ be the solution of

$$\begin{cases} \Delta W + k^2 W = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial W}{\partial \nu} = -\dfrac{\partial}{\partial \nu} G(y, x), & y \in \partial D_n, \\ W = -G(y, x), & y \in \partial D_d, \\ W = G(y, x), & y \in \partial\Omega. \end{cases} \tag{1.4}$$

We call the function $\Omega \setminus \overline{D} \ni x \mapsto W_x(x; \mathcal{G})$ the IPS function based on $\mathcal{G}$ for obstacle D.

Hereafter we simply write $W_x(y; \mathcal{G}) = W_x(y)$. Needless to say, Assumption 1 ensures the unique solvability of equations (1.4) in the class $H^2(\Omega \setminus \overline{D})$. The sudden appearance of the system (1.4) seems strange, however, it is a natural extension of the corresponding one firstly introduced in [16] in the case when $D_d = \emptyset$ and $k = 0$. Besides, we will see in Section 3 that the IPS function generates the indicator function (see Definition 3.2) for the probe method.

Since the system (1.4) is linear, we have the natural and trivial decomposition of the solution as

$$W_x(y) = w_x(y) + w_x^1(y), \quad y \in \Omega \setminus \overline{D}, \tag{1.5}$$

where $w_x = w_x(y; \mathcal{G}) = w(y)$ in $H^2(\Omega \setminus \overline{D})$ solves

$$\begin{cases} \Delta w + k^2 w = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} = -\dfrac{\partial}{\partial \nu} G(y,x), & y \in \partial D_n, \\ w = -G(y,x), & y \in \partial D_d, \\ w = 0, & y \in \partial \Omega, \end{cases} \tag{1.6}$$

and $w_x^1 = w_x^1(y; \mathcal{G}) = w(y)$ in $H^2(\Omega \setminus \overline{D})$ solves

$$\begin{cases} \Delta w + k^2 w = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} = 0, & y \in \partial D_n, \\ w = 0, & y \in \partial D_d, \\ w = G(y,x), & y \in \partial \Omega. \end{cases} \tag{1.7}$$

The unique solvability of the boundary value problems (1.6) and (1.7) are also a consequence of Assumption 1.

Needless to say, from (1.5) we have

$$W_x(x) = w_x(x) + w_x^1(x), \quad x \in \Omega \setminus \overline{D}. \tag{1.8}$$

We call this the *outer decomposition* or natural decomposition of IPS function.

The IPS for Problem is based on the discovery of the following two representation formulae.

Theorem 1.1. *Let $x \in \Omega \setminus \overline{D}$.*

(1) *We have the expression focused on the Neumann obstacle*

$$\begin{aligned} W_x(x) &= \|\nabla G(\,\cdot\,,x)\|_{L^2(D_n)}^2 - k^2 \|G(\,\cdot\,,x)\|_{L^2(D_n)}^2 + \|\nabla(w_x + (\epsilon_x)_n)\|_{L^2(\Omega\setminus\overline{D})}^2 - k^2\|w_x + (\epsilon_x)_n\|_{L^2(\Omega\setminus\overline{D})}^2 \\ &\quad - \int_{\partial\Omega} G(z,x) \frac{\partial}{\partial \nu} G(z,x)\, dS(z) + \|\nabla w_x^1\|_{L^2(\Omega\setminus\overline{D})}^2 - k^2 \|w_x^1\|_{L^2(\Omega\setminus\overline{D})}^2 - \|\nabla(\epsilon_x)_n\|_{L^2(\Omega\setminus\overline{D})}^2 \\ &\quad + k^2 \|(\epsilon_x)_n\|_{L^2(\Omega\setminus\overline{D})}^2 - \|\nabla G(\,\cdot\,,x)\|_{L^2(D_d)}^2 + k^2\|G(\,\cdot\,,x)\|_{L^2(D_d)}^2, \end{aligned} \tag{1.9}$$

where $(\epsilon_x)_n = (\epsilon_x)_n(y; \mathcal{G}) = \epsilon(y)$ in $H^2(\Omega \setminus \overline{D})$ solves

$$\begin{cases} \Delta \epsilon + k^2 \epsilon = 0, & y \in \Omega \setminus \overline{D}, \\ \epsilon = 0, & y \in \partial D_n, \\ \epsilon = G(y,x), & y \in \partial D_d, \\ \epsilon = 0, & y \in \partial \Omega. \end{cases} \tag{1.10}$$

(2) *We have the expression focused on the Dirichlet obstacle*

$$\begin{aligned} W_x(x) &= -\|\nabla G(\,\cdot\,,x)\|_{L^2(D_d)}^2 + k^2 \|G(\,\cdot\,,x)\|_{L^2(D_d)}^2 - \|\nabla(w_x + (\epsilon_x)_d)\|_{L^2(\Omega\setminus\overline{D})}^2 + k^2\|w_x + (\epsilon_x)_d\|_{L^2(\Omega\setminus\overline{D})}^2 \\ &\quad - \int_{\partial\Omega} G(z,x) \frac{\partial}{\partial \nu} G(z,x)\, dS(z) + \|\nabla w_x^1\|_{L^2(\Omega\setminus\overline{D})}^2 - k^2 \|w_x^1\|_{L^2(\Omega\setminus\overline{D})}^2 + \|\nabla(\epsilon_x)_d\|_{L^2(\Omega\setminus\overline{D})}^2 \\ &\quad - k^2 \|(\epsilon_x)_d\|_{L^2(\Omega\setminus\overline{D})}^2 + \|\nabla G(\,\cdot\,,x)\|_{L^2(D_n)}^2 - k^2\|G(\,\cdot\,,x)\|_{L^2(D_n)}^2, \end{aligned} \tag{1.11}$$

where $(\epsilon_x)_d = (\epsilon_x)_d(y; \mathcal{G}) = \epsilon(y)$ in $H^2(\Omega \setminus \overline{D})$ solves

$$\begin{cases} \Delta \epsilon + k^2 \epsilon = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial \epsilon}{\partial \nu} = \dfrac{\partial}{\partial \nu} G(y,x), & y \in \partial D_n, \\ \dfrac{\partial \epsilon}{\partial \nu} = 0, & y \in \partial D_d, \\ \epsilon = 0, & y \in \partial \Omega. \end{cases} \tag{1.12}$$

Introducing the functions $(\epsilon_x)_n$ and $(\epsilon_x)_d$ helps us to write the IPS function in terms of energy integrals (1.9) and (1.11). We call this technique the *method of complementing function*. A clear advantage is that: roughly speaking, we can immediately see that $(\epsilon_x)_n$ is bounded in $H^2(\Omega \setminus \overline{D})$ if x is close to a point on ∂D_n; $(\epsilon_x)_d$ is bounded in $H^2(\Omega \setminus \overline{D})$ if x is close to a point on ∂D_d.

It shoulder be pointed out that the two expressions (1.9) and (1.11) contain the common terms of integrals

$$-\int_{\partial\Omega} G(z,x)\frac{\partial}{\partial \nu}G(z,x)\,dS(z) + \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})}.$$

Except for those, the expression of the right-hand side on (1.11) coincides with the one on (1.9) multiplied by (−1) and replaced (n, d) with (d, n).

The following corollary is a direct consequence of the facts listed below:

(a) the well-posedness of the boundary value problems (1.6), (1.7), (1.10) and (1.12),
(b) the expressions (1.9) and (1.11),
(c) the property that for any finite cone V with vertex at the origin $x = 0$

$$\int_V |\nabla G(z-x)|^2\,dz = \infty.$$

Corollary 1.1. *The IPS function $W_x(x)$ satisfies* (i), (ii) *and* (iii) *listed below:*

(i) $\lim_{x\to a\in\partial D_n} W_x(x) = \infty$.
(ii) $\lim_{x\to b\in\partial D_d} W_x(x) = -\infty$.
(iii) *For each $\epsilon_i > 0$, $i = 1, 2$,*

$$\sup_{x\in\Omega\setminus\overline{D},\,\mathrm{dist}(x,\partial D)>\epsilon_1,\,\mathrm{dist}(x,\partial\Omega)>\epsilon_2} |W_x(x)| < \infty.$$

Proof. In what follows we denote by $C_1, C_2, \ldots$ positive numbers independent of x. Using (1.9), from (a) and (b) together with (1.2) and (1.3) we have: as $x \to a \in \partial D_n$,

$$W_x(x) \geq \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_n)} - 2k^2\|w\|^2_{L^2(\Omega\setminus\overline{D})} - C_1.$$

Here, by Lemma 2 in Appendix we have, as $x \to a \in \partial D_n$,

$$W_x(x) \geq \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_n)} - C_1 - C_2.$$

Then (c) together with Fatou's lemma yields (i). Next consider the case when $x \to b \in \partial D_d$. It follows from (1.11)

$$-W_x(x) \geq \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_d)} - 2k^2\|w\|^2_{L^2(\Omega\setminus\overline{D})} - C_3.$$

Again, Lemma 2 yields, as $x \to b \in \partial D_d$,

$$-W_x(x) \geq C_4\|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_d)} - C_3 - C_5.$$

This yields the validity of (ii). The validity of statement (iii) is almost clear. □

Therefore, using the asymptotic behavior of IPS function $W_x(x)$ as x approaches ∂D, one can distinguish the soft obstacle D_d and hard obstacle D_n. In particular, we know that IPS function does not have a definite sign unlike a single type of obstacle case [16].

Remark 1.1. In this paper, about the choice of the family $\{H(\,\cdot\,,x)\}_{x\in\Omega}$ in (1.2) we consider only the two cases. The first is the case when $H(y,x) \equiv 0$. In this case we denote $\mathcal{G}$ by $\mathcal{G}^0$. Then $G(\,\cdot\,,x)$ coincides with $G(\,\cdot - x)$. The second is: we impose the boundary condition

$$H(y,x) = -\frac{\cos k|y-x|}{4\pi|y-x|}, \quad y \in \partial\Omega. \tag{1.13}$$

Under the assumption that k^2 is not a Dirichlet eigenvalue for the minus Laplacian $-\Delta$ in Ω, for each $x \in \Omega$ the $H(y,x)$ exists and is unique, and satisfies (1.3). The function $G(\,\cdot\,,x)$ is nothing but the Green function for the domain Ω with the source point at $x \in \Omega$. In this case we denote $\mathcal{G}$ by $\mathcal{G}^*$. Then $W_x = 0$ on $\partial\Omega$ for $x \in \Omega \setminus \overline{D}$ and $w_x^1 \equiv 0$. Hereafter unless otherwise stated, we always impose the condition on k^2 mentioned above when considering $\mathcal{G}^*$.

Theorem 1.1 yields the following corollary.

Corollary 1.2. *Choose $\mathcal{G} = \mathcal{G}^0$. Let $w_x = w_x(\,\cdot\,;\mathcal{G}^0)$ and $w_x^1 = w_x^1(\,\cdot\,;\mathcal{G}^0)$.*

(i) *We have the expression focused on the Neumann obstacle*

$$\begin{aligned}W_x(x) = {}&\|\nabla G(\,\cdot - x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot - x)\|^2_{L^2(D_n)} + \|\nabla(w_x + (\epsilon_x)_n)\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x + (\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})}\\ &- \int_{\partial\Omega} G(z-x)\frac{\partial}{\partial\nu}G(z-x)\,dS(z) + \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})}\\ &- \|\nabla(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla G(\,\cdot - x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot - x)\|^2_{L^2(D_d)},\end{aligned}$$

where $(\epsilon_x)_n = (\epsilon_x)_n(\,\cdot\,;\mathcal{G}^0)$.

(ii) *We have the expression focused on the Dirichlet obstacle*

$$\begin{aligned}W_x(x) = {}&-\|\nabla G(\,\cdot - x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot - x)\|^2_{L^2(D_d)} - \|\nabla(w_x + (\epsilon_x)_d)\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|(w_x + (\epsilon_x)_d)\|^2_{L^2(\Omega\setminus\overline{D})}\\ &- \int_{\partial\Omega} G(z-x)\frac{\partial}{\partial\nu}G(z-x)\,dS(z) + \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})}\\ &+ \|\nabla(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot - x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot - x)\|^2_{L^2(D_n)},\end{aligned}$$

where $(\epsilon_x)_d = (\epsilon_x)_d(\,\cdot\,;\mathcal{G}^0)$.

Remark 1.2. In particular, if $k = 0$, then for all $x \in \Omega$ and $y \in \Omega$ one can rewrite

$$-\int_{\partial\Omega} G(z-x)\frac{\partial}{\partial\nu}G(z-y)\,dS(z) = \int_{\mathbb{R}^3\setminus\overline{\Omega}} \nabla G(z-x)\cdot\nabla G(z-y)\,dz. \tag{1.14}$$

Note that the integrand of this right-hand side is absolutely integrable. Thus, the formulae in Corollary 1.2 become

$$\begin{aligned}W_x(x) = {}&\|\nabla G(\,\cdot - x)\|^2_{L^2(D_n)} + \|\nabla(w_x + (\epsilon_x)_n)\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot - x)\|^2_{L^2(\mathbb{R}^3\setminus\overline{\Omega})}\\ &+ \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla G(\,\cdot - x)\|^2_{L^2(D_d)}\end{aligned}$$

and

$$\begin{aligned}W_x(x) = {}&-\|\nabla G(\,\cdot - x)\|^2_{L^2(D_d)} - \|\nabla(w_x + (\epsilon_x)_d)\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot - x)\|^2_{L^2(\mathbb{R}^3\setminus\overline{\Omega})}\\ &+ \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot - x)\|^2_{L^2(D_n)}.\end{aligned}$$

And also we have:

Corollary 1.3. *Choose $\mathcal{G} = \mathcal{G}^*$. Then $w_x^1(\,\cdot\,;\mathcal{G}^*) = 0$ and $W_x(\,\cdot\,;\mathcal{G}^*) = w_x(\,\cdot\,;\mathcal{G}^*)$.*

(i) *We have the expression focused on the Neumann obstacle*

$$\begin{aligned}W_x(x) = {}&\|\nabla G(\,\cdot\,, x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot\,, x)\|^2_{L^2(D_n)} + \|\nabla(w_x + (\epsilon_x)_n)\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|(w_x + (\epsilon_x)_n)\|^2_{L^2(\Omega\setminus\overline{D})}\\ &- \|\nabla(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla G(\,\cdot\,, x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot\,, x)\|^2_{L^2(D_d)},\end{aligned}$$

where $(\epsilon_x)_n = (\epsilon_x)_n(\,\cdot\,;\mathcal{G}^*)$.

(ii) *We have the expression focused on the Dirichlet obstacle*

$$\begin{aligned}W_x(x) = {}&-\|\nabla G(\,\cdot\,, x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot\,, x)\|^2_{L^2(D_d)} - \|\nabla(w_x + (\epsilon_x)_d)\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|(w_x + (\epsilon_x)_d)\|^2_{L^2(\Omega\setminus\overline{D})}\\ &+ \|\nabla(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot\,, x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot\,, x)\|^2_{L^2(D_n)},\end{aligned}$$

where $(\epsilon_x)_d = (\epsilon_x)_d(\,\cdot\,;\mathcal{G}^*)$.

1.2 IPS function to DN map

In this section, we assume that Assumption 1 for the case $D = \emptyset$ is also satisfied. We denote Λ_D by Λ_0 if $D = \emptyset$. In the probe method the form

$$\langle(\Lambda_0 - \Lambda_D)(v|_{\partial\Omega}), v|_{\partial\Omega}\rangle \equiv \int_{\partial\Omega} (\Lambda_0 - \Lambda_D)(v|_{\partial\Omega})(z)\,v(z)\,dS(z) \tag{1.15}$$

plays the central role, where v is an arbitrary solution of the Helmholtz equation $\Delta v + k^2 v = 0$ in Ω.

The idea of the method of complementing function mentioned above suggests us the decomposition formulae for the form (1.15) stated below.

Theorem 1.2. *Let $v \in H^2(\Omega)$ satisfy the Helmholtz equation $\Delta v + k^2 v = 0$ in Ω. We have*

$$\begin{aligned}\langle(\Lambda_0 - \Lambda_D)(v|_{\partial\Omega}), v|_{\partial\Omega}\rangle = \|\nabla v\|^2_{L^2(D_n)} - k^2\|v\|^2_{L^2(D_n)} + \|\nabla(w+\epsilon_n)\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|(w+\epsilon_n)\|^2_{L^2(\Omega\setminus\overline{D})} \\ - \|\nabla\epsilon_n\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|\epsilon_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla v\|^2_{L^2(D_d)} + k^2\|v\|^2_{L^2(D_d)}\end{aligned} \tag{1.16}$$

and

$$\begin{aligned}\langle(\Lambda_0 - \Lambda_D)(v|_{\partial\Omega}), v|_{\partial\Omega}\rangle = -\|\nabla v\|^2_{L^2(D_d)} + k^2\|v\|^2_{L^2(D_d)} - \|\nabla(w+\epsilon_d)\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|(w+\epsilon_d)\|^2_{L^2(\Omega\setminus\overline{D})} \\ + \|\nabla\epsilon_d\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|\epsilon_d\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla v\|^2_{L^2(D_n)} - k^2\|v\|^2_{L^2(D_n)},\end{aligned} \tag{1.17}$$

where w, ϵ_n and ϵ_d are given by the solutions w_x, $(\epsilon_x)_n$ and $(\epsilon_x)_d$ of (1.6), (1.10) *and* (1.12) *with $G(y,x)$ replaced by $v(y)$, respectively.*

The formulae (1.16) and (1.17) are new and useful for establishing the probe method for Problem. Those should be considered as the generalization of the well known decomposition formula in the case when $D_d = \emptyset$ or $D_n = \emptyset$, see [9] for the Helmholtz equation case. And also note that the expression of the right-hand side on (1.17) coincides with the one on (1.16) multiplied by (-1) and replaced (n, d) with (d, n).

It should be emphasized that the proof of Theorem 1.2 given in Section 3.2 is independent of IPS. Besides, the decomposition formulae (1.16) and (1.17) themselves would be valid also in the context of the weak solution. However, formulae (1.16) and (1.17) should be considered as a byproduct of introducing the IPS function *at first*. Without seeking the energy decomposition of IPS as done in [16] one could never find the idea of the method of complementing functions to form (1.15).

Organization of the paper. In Section 2 the proof of Theorem 1.1 is given. The proof is based on integration by parts and clarifies the meaning of introducing the complementary functions $(\epsilon_x)_n$ and $(\epsilon_x)_d$. Section 3 is devoted to the integrated theory of the probe and singular sources methods. In Section 3.1 a representation formula (3.2) of the indicator function for the probe method as a limit of the indicator sequence (see Definition 3.1) in terms of the IPS function is established. It is Theorem 3.1. This together with Theorem 1.1 yields the Side A of the probe method which is concerned with blowing up of the indicator function on the surface of obstacles. In Section 3.2 first the proof of Theorem 1.2 together with its corollary is given. Besides, In Section 3.3 it is shown that Theorem 1.2 yields the Side B of the probe method which is concerned with blowing up of indicator sequence of the probe method and stated as Theorem 3.2. In Section 3.4 we will see that the singular sources method is included in the IPS theory and in Section 3.5 it is shown that the singular sources method has the same side as the Side B of the probe method. Section 3.6 is devoted to a set of additional remarks related to the natural decomposition (1.5). In the last section the conclusion and some possible applications are briefly mentioned.

In Appendix we describe two lemmas which yield the upper estimate of the L^2-norm of the reflected solution w_x and are used in the proof of Corollary 1.1.

2 Proof of Theorem 1.1

First we show that the $w_x(y)$ which is the solution of (1.6) has two expressions. In what follows we always assume that $(x,y) \in (\Omega\setminus\overline{D})^2$.

Lemma 2.1. *It holds that*

$$\begin{aligned}w_x(y) = \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z,y)\,dS(z) + \int_{D_n} \nabla G(z,x)\cdot\nabla G(z,y)\,dz - \int_{D_n} k^2 G(z,x)G(z,y)\,dz \\ + \int_{\Omega\setminus\overline{D}} \nabla w_x(z)\cdot\nabla w_y(z)\,dz - \int_{\Omega\setminus\overline{D}} k^2 w_x(z)w_y(z)\,dz - \int_{D_d} \nabla G(z,x)\cdot\nabla G(z,y)\,dz \\ + \int_{D_d} k^2 G(z,x)G(z,y)\,dz + \int_{\partial D_d} \Big(w_y(z)\frac{\partial}{\partial\nu}w_x(z) + w_x(z)\frac{\partial}{\partial\nu}w_y(z)\Big)\,dS(z)\end{aligned} \tag{2.1}$$

and

$$
\begin{aligned}
w_x(y) = &\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z,y)\, dS(z) - \int_{D_d} \nabla G(z,x) \cdot \nabla G(z,y)\, dz + \int_{D_d} k^2 G(z,x) G(z,y)\, dz \\
&- \int_{\Omega\setminus\overline{D}} \nabla w_x(z) \cdot \nabla w_y(z)\, dz + \int_{\Omega\setminus\overline{D}} k^2 w_x(z) w_y(z)\, dz + \int_{D_n} \nabla G(z,x) \cdot \nabla G(z,y)\, dz \\
&- \int_{D_n} k^2 G(z,x) G(z,y)\, dz - \int_{\partial D_n} \Big(w_y(z) \frac{\partial}{\partial\nu} w_x(z) + w_x(z) \frac{\partial}{\partial\nu} w_y(z) \Big)\, dS(z).
\end{aligned}
\tag{2.2}
$$

Proof. We start with the standard expression

$$
\begin{aligned}
w_x(y) = &\int_{\partial\Omega} \Big(\frac{\partial}{\partial\nu} w_x(z) G(z,y) - w_x(z) \frac{\partial}{\partial\nu} G(z,y) \Big)\, dS(z) - \int_{\partial D_n} \Big(\frac{\partial}{\partial\nu} w_x(z) G(z,y) - w_x(z) \frac{\partial}{\partial\nu} G(z,y) \Big)\, dS(z) \\
&- \int_{\partial D_d} \Big(\frac{\partial}{\partial\nu} w_x(z) G(z,y) - w_x(z) \frac{\partial}{\partial\nu} G(z,y) \Big)\, dS(z).
\end{aligned}
\tag{2.3}
$$

Applying the boundary conditions on $\partial\Omega$, ∂D_d and ∂D_n to (2.3), we obtain

$$
\begin{aligned}
w_x(y) = &\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z,y)\, dS(z) + \int_{\partial D_n} \Big(\frac{\partial}{\partial\nu} G(z,x) G(z,y) - w_x(z) \frac{\partial}{\partial\nu} w_y(z) \Big)\, dS(z) \\
&+ \int_{\partial D_d} \Big(\frac{\partial}{\partial\nu} w_x(z) w_y(z) - G(z,x) \frac{\partial}{\partial\nu} G(z,y) \Big)\, dS(z).
\end{aligned}
\tag{2.4}
$$

Since x and y outside both D_d and D_n, we have, for $* = d, n$

$$
\begin{aligned}
\int_{\partial D_*} \frac{\partial}{\partial\nu} G(z,x) G(z,y)\, dS(z) &= \int_{D_*} \Delta G(z,x) G(z,y)\, dz + \int_{D_*} \nabla G(z,x) \cdot \nabla G(z,y)\, dz \\
&= \int_{D_*} \nabla G(z,x) \cdot \nabla G(z,y)\, dz - \int_{D_*} k^2 G(z,x) G(z,y)\, dz.
\end{aligned}
$$

Thus (2.4) becomes

$$
\begin{aligned}
w_x(y) = &\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z,y)\, dS(z) + \int_{D_n} \nabla G(z,x) \cdot \nabla G(z,y)\, dz - \int_{D_d} \nabla G(z,x) \cdot \nabla G(z,y)\, dz \\
&- \int_{D_n} k^2 G(z,x) G(z,y)\, dz + \int_{D_d} k^2 G(z,x) G(z,y)\, dz \\
&+ \int_{\partial D_d} w_y(z) \frac{\partial}{\partial\nu} w_x(z)\, dS(z) - \int_{\partial D_n} w_x(z) \frac{\partial}{\partial\nu} w_y(z)\, dS(z).
\end{aligned}
\tag{2.5}
$$

Besides, we have

$$
\begin{aligned}
- \int_{\partial D_n} w_x(z) \frac{\partial}{\partial\nu} w_y(z)\, dS(z) - \int_{\partial D_d} w_x(z) \frac{\partial}{\partial\nu} w_y(z)\, dS(z) &= \int_{\Omega\setminus\overline{D}} w_x(z) \Delta w_y(z)\, dz + \int_{\Omega\setminus\overline{D}} \nabla w_x(z) \cdot \nabla w_y(z)\, dz \\
&= \int_{\Omega\setminus\overline{D}} \nabla w_x(z) \cdot \nabla w_y(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_x(z) w_y(z)\, dz
\end{aligned}
\tag{2.6}
$$

and

$$
- \int_{\partial D_n} w_y(z) \frac{\partial}{\partial\nu} w_x(z)\, dS(z) - \int_{\partial D_d} w_y(z) \frac{\partial}{\partial\nu} w_x(z)\, dS(z) = \int_{\Omega\setminus\overline{D}} \nabla w_x(z) \cdot \nabla w_y(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_x(z) w_y(z)\, dz. \tag{2.7}
$$

From (2.6) one has

$$-\int_{\partial D_n} w_x(z)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)=\int_{\Omega\setminus\overline{D}}\nabla w_x(z)\cdot\nabla w_y(z)\,dz-\int_{\Omega\setminus\overline{D}}k^2w_x(z)w_y(z)\,dz+\int_{\partial D_d}w_x(z)\frac{\partial}{\partial\nu}w_y(z)\,dS(z). \tag{2.8}$$

From (2.7) one has

$$\int_{\partial D_d} w_y(z)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)=-\int_{\Omega\setminus\overline{D}}\nabla w_x(z)\cdot\nabla w_y(z)\,dz+\int_{\Omega\setminus\overline{D}}k^2w_x(z)w_y(z)\,dz-\int_{\partial D_n}w_y(z)\frac{\partial}{\partial\nu}w_x(z)\,dS(z). \tag{2.9}$$

Thus one obtains the two representation of a single integral as follows.

From (2.8) we have

$$\begin{aligned}&\int_{\partial D_d} w_y(z)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)-\int_{\partial D_n}w_x(z)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)\\&=\int_{\partial D_d}\Big(w_y(z)\frac{\partial}{\partial\nu}w_x(z)+w_x(z)\frac{\partial}{\partial\nu}w_y(z)\Big)\,dS(z)+\int_{\Omega\setminus\overline{D}}\nabla w_x(z)\cdot\nabla w_y(z)\,dz-\int_{\Omega\setminus\overline{D}}k^2w_x(z)w_y(z)\,dz.\end{aligned} \tag{2.10}$$

From (2.9) we have

$$\begin{aligned}&\int_{\partial D_d} w_y(z)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)-\int_{\partial D_n}w_x(z)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)\\&=-\int_{\partial D_n}\Big(w_y(z)\frac{\partial}{\partial\nu}w_x(z)+w_x(z)\frac{\partial}{\partial\nu}w_y(z)\Big)\,dS(z)-\int_{\Omega\setminus\overline{D}}\nabla w_x(z)\cdot\nabla w_y(z)\,dz+\int_{\Omega\setminus\overline{D}}k^2w_x(z)w_y(z)\,dz.\end{aligned} \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.5), we obtain (2.1) and (2.2). □

Next we show that the w_x^1 has the following expression.

Lemma 2.2. *We have*

$$\begin{aligned}w_x^1(y)=&\int_{\Omega\setminus\overline{D}}\nabla w_x^1(z)\cdot\nabla w_y^1(z)\,dz-\int_{\Omega\setminus\overline{D}}k^2w_x^1(z)w_y^1(z)\,dz\\&-\int_{\partial\Omega}G(z,x)\frac{\partial}{\partial\nu}G(z,y)\,dS(z)-\int_{\partial\Omega}G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z).\end{aligned} \tag{2.12}$$

Proof. To explain the reason for the introduction of the function w_x^1 step by step, let us forget the set of boundary conditions on (1.7).

First same as (2.3) we start with the standard expression

$$\begin{aligned}w_x^1(y)=&\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu}w_x^1(z)G(z,y)-w_x^1(z)\frac{\partial}{\partial\nu}G(z,y)\Big)\,dS(z)-\int_{\partial D_n}\Big(\frac{\partial}{\partial\nu}w_x^1(z)G(z,y)-w_x^1(z)\frac{\partial}{\partial\nu}G(z,y)\Big)\,dS(z)\\&-\int_{\partial D_d}\Big(\frac{\partial}{\partial\nu}w_x^1(z)G(z,y)-w_x^1(z)\frac{\partial}{\partial\nu}G(z,y)\Big)\,dS(z).\end{aligned} \tag{2.13}$$

Here we impose the boundary conditions

$$\begin{cases}\dfrac{\partial}{\partial\nu}w_x^1(z)=0, & z\in\partial D_n,\\ w_x^1(z)=G(z,x), & z\in\partial\Omega.\end{cases} \tag{2.14}$$

Then (2.13) becomes

$$\begin{aligned}w_x^1(y)=&\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu}w_x^1(z)\,w_y^1(z)-G(z,x)\frac{\partial}{\partial\nu}G(z,y)\Big)\,dS(z)+\int_{\partial D_n}w_x^1(z)\frac{\partial}{\partial\nu}G(z,y)\,dS(z)\\&-\int_{\partial D_d}\Big(\frac{\partial}{\partial\nu}w_x^1(z)\,G(z,y)-w_x^1(z)\frac{\partial}{\partial\nu}G(z,y)\Big)\,dS(z).\end{aligned} \tag{2.15}$$

Here we have

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x^1(z)\, w_y^1(z)\, dS(z) = \int_{\partial D_n} \frac{\partial}{\partial\nu} w_x^1(z)\, w_y^1(z)\, dS(z) + \int_{\partial D_d} \frac{\partial}{\partial\nu} w_x^1(z)\, w_y^1(z)\, dS(z) + \int_{\Omega\setminus\overline{D}} \Delta w_x^1(z)\, w_y^1(z)\, dz + \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\, dz$$

$$= \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z) w_y^1(z)\, dz + \int_{\partial D_d} \frac{\partial}{\partial\nu} w_x^1(z)\, w_y^1(z)\, dS(z).$$

Thus (2.15) becomes

$$\begin{aligned} w_x^1(y) = &\int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z) w_y^1(z)\, dz \\ &- \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu} G(z,y)\, dS(z) + \int_{\partial D_n} w_x^1(z)\frac{\partial}{\partial\nu} G(z,y)\, dS(z) \\ &+ \int_{\partial D_d} \frac{\partial}{\partial\nu} w_x^1(z)\, w_y^1(z)\, dS(z) - \int_{\partial D_d} \left(\frac{\partial}{\partial\nu} w_x^1(z)\, G(z,y) - w_x^1(z)\frac{\partial}{\partial\nu} G(z,y)\right) dS(z). \end{aligned} \tag{2.16}$$

Here using the boundary condition of w_y on ∂D_n, one has

$$\begin{aligned} \int_{\partial D_n} w_x^1(z)\frac{\partial}{\partial\nu} G(z,y)\, dS(z) &= -\int_{\partial D_n} w_x^1(z)\frac{\partial}{\partial\nu} w_y(z)\, dS(z) \\ &= \int_{\Omega\setminus\overline{D}} w_x^1(z)\Delta w_y(z)\, dz - \int_{\partial\Omega} w_x^1(z)\frac{\partial}{\partial\nu} w_y(z)\, dS(z) + \int_{\partial D_d} w_x^1(z)\frac{\partial}{\partial\nu} w_y(z)\, dS(z) \\ &\quad + \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y(z)\, dz \\ &= -\int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu} w_y(z)\, dS(z) + \int_{\partial D_d} w_x^1(z)\frac{\partial}{\partial\nu} w_y(z)\, dS(z) \\ &\quad + \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z) w_y(z)\, dz. \end{aligned}$$

Besides we have

$$\begin{aligned} \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y(z)\, dz &= -\int_{\partial D_d} \frac{\partial}{\partial\nu} w_x^1(z)\, w_y(z)\, dS(z) - \int_{\Omega\setminus\overline{D}} \Delta w_x^1(z) w_y(z)\, dz \\ &= \int_{\partial D_d} \frac{\partial}{\partial\nu} w_x^1(z)\, G(z,y)\, dS(z) + \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z) w_y(z)\, dz. \end{aligned}$$

That is,

$$\int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z) w_y(z)\, dz = \int_{\partial D_d} \frac{\partial}{\partial\nu} w_x^1(z)\, G(z,y)\, dS(z). \tag{2.17}$$

Note that we made use of the first boundary condition on ∂D_n of (2.14) and the boundary condition for w_y on ∂D_d of (1.6). Thus one gets

$$\begin{aligned} \int_{\partial D_n} w_x^1(z)\frac{\partial}{\partial\nu} G(z,y)\, dS(z) = &-\int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu} w_y(z)\, dS(z) + \int_{\partial D_d} w_x^1(z)\frac{\partial}{\partial\nu} w_y(z)\, dS(z) \\ &+ \int_{\partial D_d} \frac{\partial}{\partial\nu} w_x^1(z)\, G(z,y)\, dS(z). \end{aligned}$$

Therefore (2.16) becomes

$$\begin{aligned} w_x^1(y) &= \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\,dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z)w_y^1(z)\,dz - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,y)\,dS(z)\\ &\quad - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z) + \int_{\partial D_d} w_x^1(z)\frac{\partial}{\partial\nu}w_y(z)\,dS(z) + \int_{\partial D_d}\frac{\partial}{\partial\nu}w_x^1(z)\,G(z,y)\,dS(z)\\ &\quad + \int_{\partial D_d}\frac{\partial}{\partial\nu}w_x^1(z)\,w_y^1(z)\,dS(z) - \int_{\partial D_d}\Big(\frac{\partial}{\partial\nu}w_x^1(z)\,G(z,y) - w_x^1(z)\frac{\partial}{\partial\nu}G(z,y)\Big)\,dS(z)\\ &= \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\,dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z)w_y^1(z)\,dz - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,y)\,dS(z)\\ &\quad - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z) + \int_{\partial D_d} w_x^1(z)\frac{\partial}{\partial\nu}w_y(z)\,dS(z) + \int_{\partial D_d}\frac{\partial}{\partial\nu}w_x^1(z)\,w_y^1(z)\,dS(z)\\ &\quad + \int_{\partial D_d} w_x^1(z)\frac{\partial}{\partial\nu}G(z,y)\,dS(z). \end{aligned} \tag{2.18}$$

Here we impose the boundary condition of w_x^1 and w_y^1 on ∂D_d:

$$w_x^1(z) = w_y^1(z) = 0, \quad z \in \partial D_d. \tag{2.19}$$

Thus (2.18) yields (2.12). □

Note that the set of boundary conditions (2.14) and (2.19) coincides with that of (1.7).

From (2.1), (2.2) and (2.12) we immediately obtain the following two expressions for $W_x(y)$.

Proposition 2.1. *It holds that*

$$\begin{aligned} W_x(y) &= \int_{D_n} \nabla G(z,x)\cdot\nabla G(z,y)\,dz - \int_{D_n} k^2 G(z,x)G(z,y)\,dz + \int_{\Omega\setminus\overline{D}} \nabla w_x(z)\cdot\nabla w_y(z)\,dz\\ &\quad - \int_{\Omega\setminus\overline{D}} k^2 w_x(z)w_y(z)\,dz - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,y)\,dS(z) + \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\,dz\\ &\quad - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z)w_y^1(z)\,dz + \int_{\partial\Omega} G(z,y)\frac{\partial}{\partial\nu}w_x(z)\,dS(z) - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)\\ &\quad - \int_{D_d} \nabla G(z,x)\cdot\nabla G(z,y)\,dz + \int_{D_d} k^2 G(z,x)G(z,y)\,dz\\ &\quad - \int_{\partial D_d}\Big(G(z,y)\frac{\partial}{\partial\nu}w_x(z) + G(z,x)\frac{\partial}{\partial\nu}w_y(z)\Big)\,dS(z) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} W_x(y) &= -\int_{D_d} \nabla G(z,x)\cdot\nabla G(z,y)\,dz + \int_{D_d} k^2 G(z,x)G(z,y)\,dz - \int_{\Omega\setminus\overline{D}} \nabla w_x(z)\cdot\nabla w_y(z)\,dz\\ &\quad + \int_{\Omega\setminus\overline{D}} k^2 w_x(z)w_y(z)\,dz - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,y)\,dS(z) + \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\,dz\\ &\quad - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z)w_y^1(z)\,dz + \int_{\partial\Omega} G(z,y)\frac{\partial}{\partial\nu}w_x(z)\,dS(z) - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)\\ &\quad + \int_{D_n} \nabla G(z,x)\cdot\nabla G(z,y)\,dz - \int_{D_n} k^2 G(z,x)G(z,y)\,dz\\ &\quad + \int_{\partial D_n}\Big(w_y(z)\frac{\partial}{\partial\nu}G(z,x) + w_x(z)\frac{\partial}{\partial\nu}G(z,y)\Big)\,dS(z). \end{aligned} \tag{2.21}$$

Now let us explain the role of introducing the solutions of (1.10) and (1.12). It is concerned with the last terms on (2.20) and (2.21). We call this technique the *method of complementing function*.

Lemma 2.3. *We have*

$$\begin{aligned}&\int_{\partial D_d} G(z,y)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)+\int_{\partial D_d} G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)\\&\quad=-\int_{\Omega\setminus\overline{D}}(\nabla(\epsilon_y)_n(z)\cdot\nabla w_x(z)+\nabla(\epsilon_x)_n(z)\cdot\nabla w_y(z))\,dz+\int_{\Omega\setminus\overline{D}}k^2((\epsilon_y)_n(z)w_x(z)+(\epsilon_x)_n(z)w_y(z))\,dz\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}&\int_{\partial D_n} w_y(z)\frac{\partial}{\partial\nu}G(z,x)\,dS(z)+\int_{\partial D_n} w_x(z)\frac{\partial}{\partial\nu}G(z,y)\,dS(z)\\&\quad=-\int_{\Omega\setminus\overline{D}}(\nabla w_y(z)\cdot\nabla(\epsilon_x)_d(z)+\nabla w_x(z)\cdot\nabla(\epsilon_y)_d(z))\,dz+\int_{\Omega\setminus\overline{D}}k^2(w_y(z)(\epsilon_x)_d(z)+w_x(z)(\epsilon_y)_d(z))\,dz.\end{aligned}\tag{2.23}$$

Proof. First we rewrite the integral

$$\int_{\partial D_d} G(z,y)\frac{\partial}{\partial\nu}w_x(z)\,dS(z).$$

Using the equation (1.10), we have

$$\begin{aligned}\int_{\partial D_d} G(z,y)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)&=\int_{\partial D_d}(\epsilon_y)_n(z)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)+\int_{\partial D_n}(\epsilon_y)_n(z)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)\\&\quad-\int_{\partial\Omega}(\epsilon_y)_n(z)\frac{\partial}{\partial\nu}w_x(z)\,dS(z)\\&=-\int_{\Omega\setminus\overline{D}}(\epsilon_y)_n\Delta w_x(z)\,dz-\int_{\Omega\setminus\overline{D}}\nabla(\epsilon_y)_n\cdot\nabla w_x(z)\,dz\\&=-\int_{\Omega\setminus\overline{D}}\nabla(\epsilon_y)_n\cdot\nabla w_x(z)\,dz+\int_{\Omega\setminus\overline{D}}k^2(\epsilon_y)_n(z)w_x(z)\,dz.\end{aligned}$$

Interchanging x and y, we obtain the expression (2.22).

Second we rewrite the integral

$$\int_{\partial D_n} w_x(z)\frac{\partial}{\partial\nu}G(z,x)\,dS(z).$$

Using the equation (1.12), we have

$$\begin{aligned}\int_{\partial D_n} w_x(z)\frac{\partial}{\partial\nu}G(z,y)\,dS(z)&=\int_{\partial D_n} w_x(z)\frac{\partial}{\partial\nu}(\epsilon_y)_d(z)\,dS(z)+\int_{\partial D_d} w_x(z)\frac{\partial}{\partial\nu}(\epsilon_y)_d(z)\,dS(z)\\&\quad-\int_{\partial\Omega} w_x(z)\frac{\partial}{\partial\nu}(\epsilon_y)_d(z)\,dS(z)\\&=-\int_{\Omega\setminus\overline{D}} w_x\Delta(\epsilon_y)_d(z)\,dz-\int_{\Omega\setminus\overline{D}}\nabla w_x(z)\cdot\nabla(\epsilon_y)_d(z)\,dz\\&=-\int_{\Omega\setminus\overline{D}}\nabla w_x(z)\cdot\nabla(\epsilon_y)_d(z)\,dz+\int_{\Omega\setminus\overline{D}}k^2 w_x(z)(\epsilon_y)_d(z)\,dz.\end{aligned}$$

Interchanging x and y, we obtain the expression (2.23). □

Thus (2.20) together with (2.22) yields

$$\begin{aligned}W_x(y) = &\int_{D_n} \nabla G(z,x)\cdot\nabla G(z,y)\,dz - \int_{D_n} k^2 G(z,x)G(z,y)\,dz + \int_{\Omega\setminus\overline{D}} \nabla w_x(z)\cdot\nabla w_y(z)\,dz\\
&- \int_{\Omega\setminus\overline{D}} k^2 w_x(z)w_y(z)\,dz - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,y)\,dS(z) + \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\,dz\\
&- \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z)w_y^1(z)\,dz + \int_{\partial\Omega} G(z,y)\frac{\partial}{\partial\nu}w_x(z)\,dS(z) - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)\\
&- \int_{D_d} \nabla G(z,x)\cdot\nabla G(z,y)\,dz + \int_{D_d} k^2 G(z,x)G(z,y)\,dz\\
&+ \int_{\Omega\setminus\overline{D}} (\nabla(\epsilon_y)_n(z)\cdot\nabla w_x(z) + \nabla(\epsilon_x)_n(z)\cdot\nabla w_y(z))\,dz\\
&- \int_{\Omega\setminus\overline{D}} k^2((\epsilon_y)_n(z)w_x(z) + (\epsilon_x)_n(z)w_y(z))\,dz.\end{aligned} \tag{2.24}$$

And (2.21) together with (2.23) yields

$$\begin{aligned}W_x(y) = &-\int_{D_d} \nabla G(z,x)\cdot\nabla G(z,y)\,dz + \int_{D_d} k^2 G(z,x)G(z,y)\,dz - \int_{\Omega\setminus\overline{D}} \nabla w_x(z)\cdot\nabla w_y(z)\,dz\\
&+ \int_{\Omega\setminus\overline{D}} k^2 w_x(z)w_y(z)\,dz - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,y)\,dS(z) + \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y^1(z)\,dz\\
&- \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z)w_y^1(z)\,dz + \int_{\partial\Omega} G(z,y)\frac{\partial}{\partial\nu}w_x(z)\,dS(z) - \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}w_y(z)\,dS(z)\\
&+ \int_{D_n} \nabla G(z,x)\cdot\nabla G(z,y)\,dz - \int_{D_n} k^2 G(z,x)G(z,y)\,dz\\
&- \int_{\Omega\setminus\overline{D}} (\nabla w_y(z)\cdot\nabla(\epsilon_x)_d(z) + \nabla w_x(z)\cdot\nabla(\epsilon_y)_d(z))\,dz\\
&+ \int_{\Omega\setminus\overline{D}} k^2(w_y(z)(\epsilon_x)_d(z) + w_x(z)(\epsilon_y)_d(z))\,dz.\end{aligned} \tag{2.25}$$

Letting $x = y$ in (2.24) and (2.25), we obtain

$$\begin{aligned}W_x(x) = &\|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_n)} + \|\nabla w_x\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x\|^2_{L^2(\Omega\setminus\overline{D})}\\
&- \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,x)\,dS(z) + \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})}\\
&- \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_d)}\\
&+ 2\int_{\Omega\setminus\overline{D}} \nabla(\epsilon_x)_n(z)\cdot\nabla w_x(z)\,dz - 2\int_{\Omega\setminus\overline{D}} k^2 w_x(z)(\epsilon_x)_n(z)\,dz\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}W_x(x) = &-\|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_d)} - \|\nabla w_x\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|w_x\|^2_{L^2(\Omega\setminus\overline{D})}\\
&- \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,x)\,dS(z) + \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})}\\
&+ \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_n)}\\
&- 2\int_{\Omega\setminus\overline{D}} \nabla w_x(z)\cdot\nabla(\epsilon_x)_d(z)\,dz + 2\int_{\Omega\setminus\overline{D}} k^2 w_x(z)(\epsilon_x)_d(z)\,dz.\end{aligned} \tag{2.27}$$

Here rewrite

$$\|\nabla w_x\|^2_{L^2(\Omega\setminus\overline{D})} + 2\int_{\Omega\setminus\overline{D}} \nabla(\epsilon_x)_n \cdot \nabla w_x(z)\,dz = \|\nabla(w_x + (\epsilon_x)_n)\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} \tag{2.28}$$

and

$$\|\nabla w_x\|^2_{L^2(\Omega\setminus\overline{D})} + 2\int_{\Omega\setminus\overline{D}} \nabla w_x \cdot \nabla(\epsilon_x)_d(z)\,dz = \|\nabla(w_x + (\epsilon_x)_d)\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})}. \tag{2.29}$$

Rewrite also as

$$\|w_x\|^2_{L^2(\Omega\setminus\overline{D})} + 2\int_{\Omega\setminus\overline{D}} (\epsilon_x)_n w_x(z)\,dz = \|w_x + (\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} \tag{2.30}$$

and

$$\|w_x\|^2_{L^2(\Omega\setminus\overline{D})} + 2\int_{\Omega\setminus\overline{D}} w_x(\epsilon_x)_d(z)\,dz = \|w_x + (\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} - \|(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})}. \tag{2.31}$$

Then from (2.26), (2.27), (2.28), (2.29), (2.30) and (2.31) we obtain (1.9) and (1.11) of Theorem 1.1.

Remark 2.1. Assume that $D_d = \emptyset$. This the purely Neumann obstacle case. Then the equation (2.17) becomes

$$\int_{\Omega\setminus\overline{D}} \nabla w_x^1(z)\cdot\nabla w_y(z)\,dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z) w_y(z)\,dz = 0. \tag{2.32}$$

It is easy to see that equation (2.32) combined with (i) of Corollary 1.2 makes the representation of IPS function so simple:

$$\begin{aligned} W_x(x) = {} & \|\nabla W_x\|^2_{L^2(\Omega\setminus\overline{D_n})} - k^2\|W_x\|^2_{L^2(\Omega\setminus\overline{D_n})} + \|\nabla G(\,\cdot\, - x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot\, - x)\|^2_{L^2(D_n)} \\ & - \int_{\partial\Omega} G(z-x)\frac{\partial}{\partial\nu}G(z-x)\,dS(z). \end{aligned}$$

This is an extension of the expression of IPS function given in [16, Remark 1.7] to the case $k \neq 0$. It seems, in the case $D_d \neq \emptyset$ one cannot expect such a simple expression.

3 Integrated theory

3.1 IPS to Side A of probe method

In this subsection we derive the probe method via the integrated theory of the probe and singular sources methods. We fix an arbitrary $\mathcal{G} = \{G(\,\cdot\,, x)\}_{x\in\Omega}$ given by (1.2) unless otherwise specified.

First we recall the notion of a needle. Given $x \in \Omega$ let N_x denote the set of all non-self intersecting piecewise linear curves σ connecting a point on $\partial\Omega$ and x such that other points on σ are in Ω. We call each member in N_x a needle with a tip at x.

Definition 3.1. Given $x \in \Omega$ and $\sigma \in N_x$ a sequence $\{v_n\}$ of $H^2(\Omega)$ functions is called a needle sequence for (x, σ) *based on* $\mathcal{G}$ if each v_n satisfies the Helmholtz equation $\Delta v + k^2 v = 0$ in Ω and $\{v_n\}$ converges to $G(\,\cdot\,, x)$ in $H^2_{\mathrm{loc}}(\Omega\setminus\sigma)$. Then the sequence given by

$$\langle(\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}), v_n|_{\partial\Omega}\rangle \equiv \int_{\partial\Omega} (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega})(z) v_n(z)\,dS(z)$$

is called the *indicator sequence* for the probe method.

Note that, hereafter, unless otherwise specified we assume that Assumption 1 for the case $D = \emptyset$ is also satisfied. This ensures not only the well-definedness of Λ_0 but also the existence of the needle sequence for an arbitrary needle [9].

The Side A of the probe method starts with the convergence property of the indicator sequence as described below.

Theorem 3.1. *Let $x \in \Omega \setminus \overline{D}$ and $\sigma \in N_x$. Let $\{v_n\}$ be an arbitrary needle sequence for (x, σ) based on $\mathcal{G}$. If $\sigma \cap \overline{D} = \emptyset$, then we have*

$$\lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}), v_n|_{\partial\Omega}\rangle = I(x), \tag{3.1}$$

where

$$I(x) = W_x(x) - \langle \Lambda_D(G(\,\cdot\,,x)|_{\partial\Omega}), G(\,\cdot\,,x)|_{\partial\Omega}\rangle + \int_{\partial\Omega} \frac{\partial}{\partial\nu} G(z,x)G(z,x)\,dS(z). \tag{3.2}$$

Proof. Fix $x \in \Omega \setminus \overline{D}$. First we show that the limit of the left-hand side on (3.1) exists and its limit has the expression

$$\lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}, v_n|_{\partial\Omega}\rangle = w_x(x) - \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z)G(z,x)\,dS(z). \tag{3.3}$$

Define

$$G_n(z,x) = G(z,x) - v_n(z), \quad z \in \Omega. \tag{3.4}$$

The form of $G_n(\,\cdot\,,x)$ together with Green's theorem yields an expression of $w_n = w(z)$ at $z = x$, which is the solution of

$$\begin{cases} \Delta w + k^2 w = 0, & z \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w}{\partial\nu} = -\dfrac{\partial v_n}{\partial\nu}, & z \in \partial D_n, \\ w = -v_n, & z \in \partial D_d, \\ w = 0, & z \in \partial\Omega. \end{cases}$$

That is,

$$\begin{aligned} w_n(x) = {} & \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_n(z)G_n(z,x)\,dS(z) - \int_{\partial D_n} \Big(\frac{\partial}{\partial\nu} w_n(z)G_n(z,x) - w_n(z)\frac{\partial}{\partial\nu}G_n(z,x)\Big)\,dS(z) \\ & - \int_{\partial D_d} \Big(\frac{\partial}{\partial\nu} w_n(z)G_n(z,x) - w_n(z)\frac{\partial}{\partial\nu}G_n(z,x)\Big)\,dS(z). \end{aligned} \tag{3.5}$$

By Definition 3.1 and the assumption $\sigma \cap \overline{D} = \emptyset$, we have, as $n \to \infty$, $G_n(\,\cdot\,,x) \to 0$ in $H^2(D)$ and the well-posedness, we have $w_n \to w_x$ in $H^2(\Omega \setminus \overline{D})$. Therefore it follows from these and the Sobolev embedding, letting $n \to \infty$ of (3.5), we obtain

$$w_x(x) = \lim_{n\to\infty} \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_n(z)G_n(z,x)\,dS(z) = \lim_{n\to\infty} \bigg(\int_{\partial\Omega} \frac{\partial w_n}{\partial\nu} G(z,x)\,dS(z) - \int_{\partial\Omega} \frac{\partial w_n}{\partial\nu} v_n(z)\,dS(z)\bigg). \tag{3.6}$$

Since $\frac{\partial w_n}{\partial\nu} \to \frac{\partial w_x}{\partial\nu}$ in $H^{\frac{1}{2}}(\partial\Omega)$, the first term of the right-hand side on (3.6) is convergent and the limit is given by

$$\int_{\partial\Omega} \frac{\partial w_x}{\partial\nu} G(z,x)\,dS(z).$$

Therefore the second term of the right-hand side on (3.6) is also convergent and its limit satisfies

$$w_x(x) = \int_{\partial\Omega} \frac{\partial w_x}{\partial\nu} G(z,x)\,dS(z) - \lim_{n\to\infty} \int_{\partial\Omega} \frac{\partial w_n}{\partial\nu} v_n(z)\,dS(z).$$

Using this and the trivial expression

$$\frac{\partial}{\partial\nu} w_n(z) = -(\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega})(z), \quad z \in \partial\Omega,$$

we obtain

$$w_x(x) = \int_{\partial\Omega} \frac{\partial w_x}{\partial\nu}(z)\,G(z,x)\,dS(z) + \lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}), v_n|_{\partial\Omega}\rangle.$$

This is nothing but (3.3).

Next we show that the right-hand side of the formula (3.3) coincides with that of formula (3.2). Recalling (2.12) of Lemma 2.2 with $x = y$, we have

$$-\int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}w_x(z)\,dS(z) = w_x^1(x) - \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} + \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,x)\,dS(z). \tag{3.7}$$

Besides from equation (1.7) we have

$$\langle\Lambda_D G(\,\cdot\,,x)|_{\partial\Omega}, G(\,\cdot\,,x)|_{\partial\Omega}\rangle = \int_{\partial\Omega}\frac{\partial}{\partial\nu}w_x^1(z)\,w_x^1(z)\,dS(z) = \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})}. \tag{3.8}$$

From (3.7) and (3.8) together with (1.8) we see that

$$\begin{aligned} w_x(x) - \int_{\partial\Omega}\frac{\partial}{\partial\nu}w_x(z)G(z,x)\,dS(z) &= w_x(x) + w_x^1(x) - \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} + \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,x)\,dS(z) \\ &= W_x(x) - \langle\Lambda_D G(\,\cdot\,,x)|_{\partial\Omega}, G(\,\cdot\,,x)|_{\partial\Omega}\rangle + \int_{\partial\Omega} G(z,x)\frac{\partial}{\partial\nu}G(z,x)\,dS(z). \end{aligned} \tag{3.9}$$

A combination of (3.3) and (3.9) yields the desired conclusion. □

Definition 3.2. The function $I(x)$ appeared as the limit (3.1) and expressed as (3.2) is called the *indicator function for the probe method* based on $\mathcal{G}$.

The formula (3.1) should be understood as a computation formula of the indicator function by using $\Lambda_0 - \Lambda_D$. Besides, this shows that IPS function $W_x(x)$ can be calculated from Λ_D (and Λ_0 which can be calculated in advance) acting on the needle sequences from the surface $\partial\Omega$ to inside.

The second and third terms of the right-hand side on (3.2) are bounded when x is away from $\partial\Omega$, by virtue of (1.2) and (1.3). Thus it follows from Corollary 1.1 and (3.2) that

(i) $\lim_{x\to a\in\partial D_n} I(x) = \infty$,

(ii) $\lim_{x\to b\in\partial D_d} I(x) = -\infty$.

Then it follows from Theorem 1.1, (3.2) and (3.8) that the $I(x)$ has two expressions:

$$\begin{aligned} I(x) = {}&\|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_n)} + \|\nabla(w_x + (\epsilon_x)_n)\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x + (\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} \\ &- \|\nabla(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|(\epsilon_x)_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_d)} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} I(x) = {}&-\|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_d)} + k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_d)} - \|\nabla(w_x + (\epsilon_x)_d)\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|w_x + (\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} \\ &+ \|\nabla(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|(\epsilon_x)_d\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot\,,x)\|^2_{L^2(D_n)} - k^2\|G(\,\cdot\,,x)\|^2_{L^2(D_n)}. \end{aligned} \tag{3.11}$$

Then we can easily check that, for each $\epsilon_i > 0$, $i = 1, 2$,

$$\sup_{x\in\Omega\setminus\overline{D},\,\mathrm{dist}(x,\partial D)>\epsilon_1,\,\mathrm{dist}(x,\partial\Omega)>\epsilon_2} |I(x)| < \infty. \tag{3.12}$$

So the convergence of the indicator sequence (3.1), blowing up property of the $I(x)$ mentioned (i) and (ii) above and (3.12) establish the Side A of the probe method for the mixed obstacle case.

The point that should be emphasized is: from IPS we obtained that $I(x)$ as the limit of the indicator sequence takes the expressions (3.10) and (3.11). Is should be also pointed out that, the expression (3.11) coincides with (-1)-times the expression (3.10) replaced with (n, d) with (d, n).

Remark 3.1. If $\mathcal{G} = \mathcal{G}^0$, from the well-posedness of boundary value problems (1.6), (1.10) and (1.12) one can relax (3.12) as: for each $\epsilon > 0$,

$$\sup_{x\in\Omega\setminus\overline{D},\,\mathrm{dist}(x,\partial D)>\epsilon} |I(x)| < \infty.$$

Definition 3.3. For an arbitrary point $x \in \Omega \setminus \overline{D}$ define

$$I^1(x) = \langle \Lambda_D(G(\,\cdot\,,x)|_{\partial\Omega}), G(\,\cdot\,,x)|_{\partial\Omega}\rangle - \int_{\partial\Omega} \frac{\partial}{\partial\nu} G(z,x)G(z,x)\,dS(z). \tag{3.13}$$

In principle, it is possible to calculate $I^1(x)$ in advance from given Λ_D without probing. Besides, if $\mathcal{G} = \mathcal{G}^0$, it follows from (3.8) we have the energy integral expression of (3.13):

$$I^1(x) = \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} - \int_{\partial\Omega} \frac{\partial}{\partial\nu} G(z,x)G(z,x)\,dS(z).$$

In particular, if $k = 0$, then by (1.14) this becomes

$$I^1(x) = \|\nabla w_x^1\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla G(\,\cdot - x)\|^2_{L^2(\mathbb{R}^3\setminus\overline{\Omega})}, \quad x \in \Omega \setminus \overline{D}.$$

Finally, by (3.2) and (3.13) we have the *inner decomposition* of IPS function:

$$W_x(x) = I(x) + I^1(x), \quad x \in \Omega \setminus \overline{D}. \tag{3.14}$$

The equations (1.8) and (3.14) give us two ways of decomposition of IPS function.

Remark 3.2. The two types of the decompositions (3.10) and (3.11) suggest the replacement:

$$G(\,\cdot\,,x) \to v,$$

where v is an arbitrary solution of the Helmholtz equation in Ω. Theorem 1.2 can be considered as an example of the validity of this replacement.

3.2 Proof of Theorem 1.2 and a corollary

Proof of Theorem 1.2. First we prove the validity of (1.16). We have

$$\begin{aligned}
\|\nabla(w+\epsilon_n)\|^2_{L^2(\Omega\setminus\overline{D}} - \|\nabla\epsilon_n\|^2_{L^2(\Omega\setminus\overline{D})} &= \|\nabla w\|^2_{L^2(\Omega\setminus\overline{D})} + 2\int_{\Omega\setminus\overline{D}} \nabla\epsilon_n\cdot\nabla w\,dx \\
&= -\int_{\partial D} \frac{\partial w}{\partial\nu} w\,dS + k^2\|w\|^2_{L^2(\Omega\setminus\overline{D})} - 2\int_{\partial D} \epsilon_n \frac{\partial w}{\partial\nu}\,dS + 2\int_{\Omega\setminus\overline{D}} k^2\epsilon w\,dz \\
&= \int_{\partial D_d} \frac{\partial w}{\partial\nu} v\,dS + \int_{\partial D_n} \frac{\partial v}{\partial\nu} w\,dS - 2\int_{\partial D_d} v\frac{\partial w}{\partial\nu}\,dS \\
&\quad + k^2\|w+\epsilon_n\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|\epsilon_n\|^2_{L^2(\Omega\setminus\overline{D})} \\
&= \left(\int_{\partial D_n} \frac{\partial v}{\partial\nu} w\,dS - \int_{\partial D_d} \frac{\partial w}{\partial\nu} v\,dS\right) + k^2\|w+\epsilon_n\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|\epsilon_n\|^2_{L^2(\Omega\setminus\overline{D})}.
\end{aligned} \tag{3.15}$$

Besides we have

$$\begin{aligned}
\langle(\Lambda_0 - \Lambda_D)(v|_{\partial\Omega}), v|_{\partial\Omega}\rangle &= -\int_{\partial\Omega} \frac{\partial w}{\partial\nu} v\,dS \\
&= -\int_{\Omega\setminus\overline{D}} \nabla w\cdot\nabla v\,dz + \int_{\Omega\setminus\overline{D}} k^2 wv\,dz - \int_{\partial D} \frac{\partial w}{\partial\nu} v\,dS \\
&= \int_{\partial D} w\frac{\partial v}{\partial\nu}\,dS - \int_{\partial D} \frac{\partial w}{\partial\nu} v\,dS \\
&= -\int_{\partial D_d} v\frac{\partial v}{\partial\nu}\,dS + \int_{\partial D_n} w\frac{\partial v}{\partial\nu}\,dS - \int_{\partial D_d} \frac{\partial w}{\partial\nu} v\,dS + \int_{\partial D_n} \frac{\partial v}{\partial\nu} v\,dS \\
&= \left(\int_{\partial D_n} \frac{\partial v}{\partial\nu} v\,dS - \int_{\partial D_d} v\frac{\partial v}{\partial\nu}\,dS\right) + \left(\int_{\partial D_n} \frac{\partial v}{\partial\nu} w\,dS - \int_{\partial D_d} \frac{\partial w}{\partial\nu} v\,dS\right).
\end{aligned} \tag{3.16}$$

Here we have

$$\begin{cases} \|\nabla v\|^2_{L^2(D_n)} - k^2\|v\|^2_{L^2(D_n)} = \int\limits_{\partial D_n} \frac{\partial v}{\partial \nu} v\, dS, \\ \|\nabla v\|^2_{L^2(D_d)} - k^2\|v\|^2_{L^2(D_d)} = \int\limits_{\partial D_d} \frac{\partial v}{\partial \nu} v\, dS. \end{cases} \tag{3.17}$$

Now from (3.15), (3.16) and (3.17) we obtain (1.16).

Next we have

$$\begin{aligned} -\|\nabla(w+\epsilon_d)\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla\epsilon_d\|^2_{L^2(\Omega\setminus\overline{D})} &= -\|\nabla w\|^2_{L^2(\Omega\setminus\overline{D})} - 2\int\limits_{\Omega\setminus\overline{D}} \nabla\epsilon_d\cdot\nabla w\, dz \\ &= \int\limits_{\partial D} \frac{\partial w}{\partial \nu} w\, dS + 2\int\limits_{\partial D} \frac{\partial}{\partial \nu}\epsilon_d\, w\, dS - k^2\|w\|^2_{L^2(\Omega\setminus\overline{D})} - 2\int\limits_{\Omega\setminus\overline{D}} k^2\epsilon_d w\, dz \\ &= -\int\limits_{\partial D_d} \frac{\partial w}{\partial \nu} v\, dS - \int\limits_{\partial D_n} \frac{\partial v}{\partial \nu} w\, dS + 2\int\limits_{\partial D_n} \frac{\partial}{\partial \nu} v\, w\, dS \\ &\quad - k^2\|w+\epsilon_d\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|\epsilon_d\|^2_{L^2(\Omega\setminus\overline{D})} \\ &= \left(\int\limits_{\partial D_n} \frac{\partial v}{\partial \nu} w\, dS - \int\limits_{\partial D_d} \frac{\partial w}{\partial \nu} v\, dS\right) - k^2\|w+\epsilon_d\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|\epsilon_d\|^2_{L^2(\Omega\setminus\overline{D})}. \end{aligned} \tag{3.18}$$

Thus from (3.16), (3.17) and (3.18) we obtain (1.17). □

Note that once we have found the equation to prove, the proof is just a calculation. The point of Theorem 3.1 is the introduction of complementing functions ϵ_n and ϵ_d in such a way that the integral

$$\int\limits_{\partial D_n} \frac{\partial v}{\partial \nu} w\, dS - \int\limits_{\partial D_d} \frac{\partial w}{\partial \nu} v\, dS$$

has two energy integral expressions given by (3.15) and (3.18).

As a direct corollary, we obtain:

Corollary 3.1. *Let $x \in \Omega$ and $\sigma \in N_x$. Let $\{v_m\}$ be an arbitrary needle sequence for (x,σ) based on $\mathcal{G}$. We have*

$$\begin{aligned} \langle(\Lambda_0 - \Lambda_D)(v_m|_{\partial\Omega}), v_m|_{\partial\Omega}\rangle &= \|\nabla v_m\|^2_{L^2(D_n)} - k^2\|v_m\|^2_{L^2(D_n)} + \|\nabla(w_m + (\epsilon_m)_n)\|^2_{L^2(\Omega\setminus\overline{D})} \\ &\quad - k^2\|w_m + (\epsilon_m)_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla(\epsilon_m)_n\|^2_{L^2(\Omega\setminus\overline{D})} + k^2\|(\epsilon_m)_n\|^2_{L^2(\Omega\setminus\overline{D})} \\ &\quad - \|\nabla v_m\|^2_{L^2(D_d)} + k^2\|v_m\|^2_{L^2(D_d)} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \langle(\Lambda_0 - \Lambda_D)(v_m|_{\partial\Omega}), v_m|_{\partial\Omega}\rangle &= -\|\nabla v_m\|^2_{L^2(D_d)} + k^2\|v_m\|^2_{L^2(D_d)} - \|\nabla(w_m + (\epsilon_m)_d)\|^2_{L^2(\Omega\setminus\overline{D})} \\ &\quad + k^2\|w_m + (\epsilon_m)_d\|^2_{L^2(\Omega\setminus\overline{D})} + \|\nabla(\epsilon_m)_d\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|(\epsilon_m)_d\|^2_{L^2(\Omega\setminus\overline{D})} \\ &\quad + \|\nabla v_m\|^2_{L^2(D_n)} - k^2\|v_m\|^2_{L^2(D_n)}, \end{aligned} \tag{3.20}$$

where w_m, $(\epsilon_m)_n$ and $(\epsilon_m)_d$ are given by w_x, $(\epsilon_x)_n$ and $(\epsilon_x)_d$ with $G(y,x)$ in (1.6), (1.10) *and* (1.12) *replaced by $v_m(y)$, respectively.*

It should be emphasized that the point is the idea or the principle of the derivation of the things to be proved, like (1.16) and (1.17) or, (3.19) and (3.20). It is not a trivial fact as we have already seen. It is based on the *correspondence principle* mentioned below.

Principle. Replace the singular solution $G(z,x)$ appeared in some identity, say (3.10) and (3.11), involving w_x, $(\epsilon_x)_n$ and $(\epsilon_x)_d$ with $\{v_m\}$ based on $\mathcal{G}$. Then one gets a corresponding identity for $\{v_m\}$ (to be proved independently), say (3.19) and (3.20).

Note that conversely (3.19) and (3.20) yield immediately (3.10) and (3.11) by taking the limit and the formula (3.1), respectively.

3.3 Side B of probe method

The Side B of the probe method is concerned with the blowing up property of the indicator sequence. It is based on Theorem 1.2 or Corollary 3.1 and the blowing up property of the needle sequence stated below.

Proposition 3.1. *Given an arbitrary point $x \in \Omega$ and needle $\sigma \in N_x$ let $\{v_m\}$ be an arbitrary needle sequence for (x, σ) based on $\mathcal{G}$.*
(a) *Let V be an arbitrary finite cone with vertex at x. Then we have*

$$\lim_{m\to\infty} \|\nabla v_m\|^2_{L^2(V\cap\Omega)} = \infty.$$

(b) *Let $z \in \Omega$ be an arbitrary point on $\sigma \setminus \{x\}$ and B open ball centered at z. Then we have*

$$\lim_{m\to\infty} \|\nabla v_m\|^2_{L^2(B\cap\Omega)} = \infty.$$

This fact has been already established in [11].

Once we have (3.19), (3.20) and Proposition 3.1, one gets the following theorem which states the Side B of the probe method.

Theorem 3.2. *Let $k = 0$. Let $x \in \Omega$ and $\sigma \in N_x$. Assume that one of the two cases* (a) *and* (b) *listed below is satisfied:*
(a) $x \in \overline{D}$,
(b) $x \in \Omega \setminus \overline{D}$ *and* $\sigma \cap D \neq \emptyset$.
Then for any needle sequence $\{v_m\}$ for (x, σ) based on $\mathcal{G}$ we have

$$\lim_{m\to\infty} \langle (\Lambda_0 - \Lambda_D)(v_m|_{\partial\Omega}), v_m|_{\partial\Omega}\rangle = \begin{cases} \infty & \text{if } \sigma \cap \overline{D_d} = \emptyset, \\ -\infty & \text{if } \sigma \cap \overline{D_n} = \emptyset. \end{cases}$$

Proof. We describe only the case when $\sigma \cap \overline{D_d} = \emptyset$. In this case we have $x \in \Omega \setminus \overline{D_d}$ and the convergence $v_m \to G(\,\cdot\,, x)$ in $H^2(D_d)$ yields the boundedness of the sequence $\{(\epsilon_m)_n\}$ in $H^2(\Omega \setminus \overline{D})$. Thus it follows from (3.19) that

$$\langle (\Lambda_0 - \Lambda_D)(v_m|_{\partial\Omega}), v_m|_{\partial\Omega}\rangle \geq \|\nabla v_m\|_{L^2(D_n)} - C,$$

where C is a positive constant. Here, by Proposition 3.1 under (a) and (b) above we have $\|\nabla v_m\|_{L^2(D_n)} \to \infty$. This completes the proof. □

Remark 3.3. Note that Theorem 3.2 does not cover all the possible cases for (x, σ). For example, if both of the conditions $\sigma \cap \overline{D_d} \neq \emptyset$ and $x \in \overline{D_n}$ are satisfied, it would be difficult to state something about the behavior of the indicator sequence.

The problem is the case when $k \neq 0$. For this, even in the case when $D_d = \emptyset$ we have only a result in [11] under a smallness condition on k. See also [15, Section 2.3.1] for a concise explanation.

Here, applying the idea described therein to formulae (3.19) and (3.20), we show a result.

We assume that D_n and D_d have the form

$$D_n = \bigcup_{j=1}^{N} D_{n,j} \quad D_d = \bigcup_{l=1}^{M} D_{d,l},$$

where $D_{n,j}$, $j = 1, \ldots, N$, and $D_{d,l}$, $l = 1, M$, are connected components of D_n and D_d, respectively and satisfy $\overline{D_{n,j}} \cap \overline{D_{n,j'}} = \emptyset$ if $j \neq j'$; $\overline{D_{d,l}} \cap \overline{D_{d,l'}} = \emptyset$ if $l \neq l'$.

The assumption on k is as follows: k satisfies all the inequalities listed below:

$$C(\Omega \setminus \overline{D})^2 k^2 \leq 1, \tag{3.21}$$

$$\max_{j=1,\ldots,N} 8C(D_{n,j})^2 k^2 < 1, \tag{3.22}$$

$$\max_{l=1,\ldots,M} 8C(D_{d,j})^2 k^2 < 1. \tag{3.23}$$

Here the constants $C(\Omega \setminus \overline{D})$, $C(D_{n,j})$ and $C(D_{d,l})$ denote the Poincaré constants [21] in the following sense,

respectively:
(i) Constant $C(\Omega \setminus \overline{D})$ satisfies, for all $w \in H^1(\Omega \setminus \overline{D})$ with $w = 0$ on $\partial\Omega$,

$$\|w\|_{L^2(\Omega\setminus\overline{D})} \le C(\Omega \setminus \overline{D})\|\nabla w\|_{L^2(\Omega\setminus\overline{D})}.$$

(ii) Constant $C(D_{n,j})$ satisfies, for all $v \in H^1(D_{n,j})$ with $\int_{D_{n,j}} v\,dz = 0$,

$$\|v\|_{L^2(D_{n,j})} \le C(D_{n,j})\|\nabla v\|_{L^2(D_{n,j})}.$$

(iii) Constant $C(D_{d,l})$ satisfies, for all $v \in H^1(D_{d,l})$ with $\int_{D_{d,l}} v\,dz = 0$,

$$\|v\|_{L^2(D_{d,l})} \le C(D_{d,l})\|\nabla v\|_{L^2(D_{d,l})}.$$

Theorem 3.3. *All the statements of Theorem 3.2 for $k \neq 0$ are valid under the smallness condition* (3.21), (3.22) *and* (3.23).

Proof. Consider the case σ satisfy $\sigma \cap \overline{D_d} = \emptyset$. Applying the inequality (3.21) to the function $w = w_m + (\epsilon)_m$, we have

$$\|\nabla(w_m + (\epsilon_m)_n)\|^2_{L^2(\Omega\setminus\overline{D})} - k^2\|w_m + (\epsilon_m)_n\|^2_{L^2(\Omega\setminus\overline{D})} \ge 0.$$

Thus (3.19) yields

$$\langle(\Lambda_0 - \Lambda_D)(v_m|_{\partial\Omega}), v_m|_{\partial\Omega}\rangle \ge (\|\nabla v_m\|^2_{L^2(D_n)} - k^2\|v_m\|^2_{L^2(D_n)}) + R_m, \tag{3.24}$$

where

$$R_m = -\|\nabla(\epsilon_m)_n\|^2_{L^2(\Omega\setminus\overline{D})} - \|\nabla v_m\|^2_{L^2(D_d)}.$$

By the convergence property of $\{v_m\}$ in $H^2(D_d)$, we have $\{\epsilon_m\}$ is bounded in $H^2(\Omega \setminus \overline{D})$. Therefore the sequence $\{R_m\}$ is bounded. Besides, applying the same argument in [11] (and also see [15]) to the first term of the right-hand side on (3.24), we obtain

$$\|\nabla v_m\|^2_{L^2(D_n)} - k^2\|v_m\|^2_{L^2(D_n)} \ge C_1\|\nabla v_m\|^2_{L^2(D_n)} - C_2,$$

here C_1 and C_2 are positive constants independent of m, however, depends on σ, D_n and k satisfying (3.22). Thus the blowing up property of the indicator sequence is reduced to that of $\|\nabla v_m\|^2_{L^2(D_n)}$, that is covered by Proposition 3.1.

The treatment of the case when $\sigma \cap \overline{D_n} = \emptyset$ is the same except for the use of (3.23) and (3.20) instead of (3.22) and (3.19), respectively. □

Remark 3.4. It should be noted that in [2] they considered the probe method [7, 9] for the Helmholtz equation $\Delta u + k^2 u = 0$ in the mixed obstacle case. However, in their paper only the Side A of the probe method is considered and their argument is based on a combination of that of [9] and a detailed singularity analysis of the reflected solution. There is no description about the Side B of the probe method, which has been introduced in [11] and developed in [12]. Besides, even the case when the wave number $k = 0$ their result does not cover Theorem 3.2. This is due to the lack of formulae (1.16) and (1.17) or (3.19) and (3.20).

3.4 Singular sources method included in IPS

The singular sources method consists of three parts listed below.
(a) Given $x \in \Omega \setminus \overline{D}$ and $\sigma \in N_x$ let $\{v_n\}$ be an arbitrary needle sequence for (x, σ) based on $\mathcal{G}$. Then we have formula (3.6), that is,

$$w_x(x) = -\lim_{n\to\infty} < (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}), (G(\,\cdot\,, x) - v_n)|_{\partial\Omega} > . \tag{3.25}$$

(b) It holds that:
 (i) $\lim_{x\to a\in\partial D_n} w_x(x) = \infty$,
 (ii) $\lim_{x\to b\in\partial D_d} w_x(x) = -\infty$.
(c) For each $\epsilon_i > 0$, $i = 1, 2$,

$$\sup_{x\in\Omega\setminus\overline{D},\, \mathrm{dist}(x,\partial D)>\epsilon_1,\, \mathrm{dist}(x,\partial\Omega)>\epsilon_2} |w_x(x)| < \infty. \tag{3.26}$$

The statements (b) and (c) are the direct consequence of Corollary 1.1, outer decomposition (1.8), (1.3) and (1.7).

Note that Remark 3.1 works also for (3.26) in the case when $\mathcal{G} = \mathcal{G}^0$. That is, the condition $\mathrm{dist}(x, \partial\Omega) > \epsilon_2$ in (3.26) is dropped.

Remark 3.5. If $\mathcal{G} = \mathcal{G}^*$, then $G(\,\cdot\,, x) = 0$ on $\partial\Omega$ and by (3.25) one gets

$$w_x(x) = \lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}), v_n|_{\partial\Omega}\rangle$$

provided $\sigma \cap \overline{D} = \emptyset$. From this together with (3.1) and (3.2) we obtain

$$w_x(x) = I(x) = W_x(x).$$

So this is the *completely integrated version* of the probe and singular sources methods. By Theorem 3.2 this version also has the Side B. To distinguish from other cases we denote $w_x(x) = I(x) = W_x(x)$ by $w_x^*(x) = I^*(x) = W_x^*(x)$ if $\mathcal{G} = \mathcal{G}^*$.

3.5 Side B of singular sources method

Given $x \in \Omega$ and $\sigma \in N_x$ let $\{v_n^0\}$ be the needle sequence for (x, σ) based on $\mathcal{G} = \mathcal{G}^0$, that is,

$$v_n^0 \to G(\,\cdot\, - x)$$

in $H^2_{\mathrm{loc}}(\Omega \setminus \sigma)$. Let $H(z) = H(z, x)$ solve

$$\begin{cases} \Delta H + k^2 H = 0, & z \in \Omega, \\ H(z) = -G(z - x), & z \in \partial\Omega. \end{cases}$$

It is clear that the $H(\,\cdot\,, x)$ satisfies (1.3). The function

$$v_n(z) = v_n^0(z) + H(z, x), \quad z \in \Omega,$$

satisfies the Helmholtz equation in Ω and that the sequence $\{v_n\}$ satisfies

$$v_n \to G(\,\cdot\, - x) + H(\,\cdot\,, x)$$

in $H^2_{\mathrm{loc}}(\Omega \setminus \sigma)$. This means that sequence $\{v_n\}$ is a needle sequence for (x, σ) based on $\mathcal{G} = \mathcal{G}^*$ (see also Remark 1.1). Thus, if $\sigma \cap \overline{D} = \emptyset$, then by Remark 3.5 we have

$$\begin{aligned} w_x^*(x) &= I^*(x) = W_x^*(x) \\ &= \lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}), v_n|_{\partial\Omega}\rangle. \end{aligned}$$

Here note that we have

$$v_n(z) = v_n^0(z) - G(z - x), \quad z \in \partial\Omega.$$

Therefore we obtain

$$\begin{aligned} w_x^*(x) &= I^*(x) = W_x^*(x) \\ &= \lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)((v_n^0 - G(\,\cdot\, - x))|_{\partial\Omega}), (v_n^0 - G(\,\cdot\, - x))|_{\partial\Omega}\rangle. \end{aligned}$$

Besides, as a corollary of Theorem 3.2 we obtain

Corollary 3.2. *Let $k = 0$. Let $x \in \Omega$ and $\sigma \in N_x$. Assume that one of the two cases* (a) *and* (b) *listed in Theorem 3.2 is satisfied. Then for any needle sequence $\{v_m^0\}$ for (x, σ) based on $\mathcal{G} = \mathcal{G}^0$ we have*

$$\lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)((v_n^0 - G(\,\cdot\, - x))|_{\partial\Omega}), (v_n^0 - G(\,\cdot\, - x))|_{\partial\Omega}\rangle = \begin{cases} \infty & \text{if } \sigma \cap \overline{D_d} = \emptyset, \\ -\infty & \text{if } \sigma \cap \overline{D_n} = \emptyset. \end{cases}$$

And also as a corollary of Theorem 3.3 we have:

Corollary 3.3. *Let $k \geq 0$ satisfy* (3.21), (3.22) *and* (3.23). *Then the same conclusions as Corollary 3.2 are valid.*

Let $w_x^0(y) = w_x(y; \mathcal{G}^0)$. The w_x^0 solves

$$\begin{cases} \Delta w + k^2 w = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} = -\dfrac{\partial}{\partial \nu} G(y-x), & y \in \partial D_n, \\ w = -G(y-x), & y \in \partial D_d, \\ w = 0, & y \in \partial\Omega. \end{cases}$$

The function $\Omega \setminus \overline{D} \ni x \mapsto w_x^0(x)$ is a natural extension of the *indicator function for the singular sources method* discussed therein to the case $D_d \neq \emptyset$. See also [16, Section 1.3] for an explanation of why $w_x^0(x)$ is called the indicator function for the singular sources method in relation to its original singular sources method of Potthast [20].

Now let $\sigma \cap \overline{D} = \emptyset$. By (3.4) and (3.25) in the case $\mathcal{G} = \mathcal{G}^0$, we have

$$w_x^0(x) = -\lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D) v_n^0|_{\partial\Omega}, G_n(\,\cdot\,, x)|_{\partial\Omega}\rangle,$$

where

$$G_n(z, x) = G(z-x) - v_n^0(z).$$

Here we have the trivial decomposition

$$\begin{aligned} \langle (\Lambda_0 - \Lambda_D) G(\,\cdot\, - x)|_{\partial\Omega}, G(\,\cdot\, - x)|_{\partial\Omega}\rangle = {}& \langle (\Lambda_0 - \Lambda_D) v_n^0|_{\partial\Omega}, v_n^0|_{\partial\Omega}\rangle + \langle (\Lambda_0 - \Lambda_D) G_n(\,\cdot\,, x)|_{\partial\Omega}, G_n(\,\cdot\,, x)|_{\partial\Omega}\rangle \\ & + \langle (\Lambda_0 - \Lambda_D) v_n^0|_{\partial\Omega}, G_n(\,\cdot\,, x)|_{\partial\Omega}\rangle \\ & + \langle (\Lambda_0 - \Lambda_D)(G_n(\,\cdot\,, x)|_{\partial\Omega}), v_n^0|_{\partial\Omega}\rangle. \end{aligned}$$

This together with the symmetry of Dirichlet-to-Neumann maps Λ_0 and Λ_D yields the expression

$$\begin{aligned} -\langle (\Lambda_0 - \Lambda_D) v_n^0|_{\partial\Omega}, G_n(\,\cdot\,, x)|_{\partial\Omega}\rangle = {}& \frac{1}{2}(\langle (\Lambda_0 - \Lambda_D) v_n^0|_{\partial\Omega}, v_n^0|_{\partial\Omega}\rangle + \langle (\Lambda_0 - \Lambda_D) G_n(\,\cdot\,, x)|_{\partial\Omega}, G_n(\,\cdot\,, x)|_{\partial\Omega}\rangle) \\ & - \frac{1}{2}\langle (\Lambda_0 - \Lambda_D) G(\,\cdot\, - x)|_{\partial\Omega}, G(\,\cdot\, - x)|_{\partial\Omega}\rangle. \end{aligned} \tag{3.27}$$

Therefore, using Theorem 3.2 for the choice $\mathcal{G} = \mathcal{G}^0$ and Corollary 3.2, we obtain the side *B* of the singular sources method formulated in [16].

Corollary 3.4. *Let $k = 0$. Let $x \in \Omega$ and $\sigma \in N_x$. Assume that one of the two cases* (a) *and* (b) *listed in Theorem 3.2 is satisfied. Then for any needle sequence $\{v_n^0\}$ for (x, σ) based on $\mathcal{G} = \mathcal{G}^0$ we have*

$$-\lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D) v_n^0|_{\partial\Omega}, (G(\,\cdot\, - x) - v_n^0)|_{\partial\Omega}\rangle = \begin{cases} \infty & \text{if } \sigma \cap \overline{D_d} = \emptyset, \\ -\infty & \text{if } \sigma \cap \overline{D_n} = \emptyset. \end{cases}$$

And also from Theorem 3.3 and Corollary 3.3 we have:

Corollary 3.5. *Let $k \geq 0$ satisfy* (3.21), (3.22) *and* (3.23). *Then we have the same conclusions as Corollary 3.4.*

Corollaries 3.4 and 3.5 could never be obtained using a single methodology, and show us the greatest advantage of the integrated theory.

Remark 3.6. It follows from (3.27) and the existence of the needle sequence which is a consequence of the Runge approximation property for the Helmholtz equation in Ω we have the expression

$$w_x^0(x) = \frac{1}{2}(I^0(x) + I^*(x) - \langle (\Lambda_0 - \Lambda_D) G(\,\cdot\, - x)|_{\partial\Omega}, G(\,\cdot\, - x)|_{\partial\Omega}\rangle),$$

where the $I^0(x)$ denotes the $I(x)$ given by (3.2) (or both of (3.10) and (3.11)) with the case when $\mathcal{G} = \mathcal{G}^0$, that is, $G(\,\cdot\,, x) = G(\,\cdot\, - x)$.

3.6 Additional remarks

3.6.1 Lifting

First for general $\mathcal{G}$, from Lemma 2.1 we obtain

$$w_x(y) - \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z,y)\, dS(z) = w_y(x) - \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_y(z) G(z,x)\, dS(z). \tag{3.28}$$

So define

$$I(x,y) = w_x(y) - \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z,y)\, dS(z), \quad (x,y) \in (\Omega \setminus \overline{D})^2. \tag{3.29}$$

Then (3.28) yields the symmetry

$$I(x,y) = I(y,x).$$

Besides, by (3.1) and (3.3), we have another expression for the indicator function:

$$I(x) = w_x(x) - \int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z,x)\, dS(z).$$

This together with (3.29) yields $I(x) = I(x,y)|_{y=x}$ and in this sense, the $I(x,y)$ is called the *lifting* of $I(x)$.

The inner decomposition (3.14) itself has the lifted version. For general $\mathcal{G}$, by (2.12) we have

$$w_x^1(y) = I^1(x,y) - \int_{\partial\Omega} G(z,x) \frac{\partial}{\partial\nu} w_y(z)\, dS(z), \quad (x,y) \in (\Omega \setminus \overline{D})^2, \tag{3.30}$$

where

$$I^1(x,y) = \int_{\Omega\setminus\overline{D}} \nabla w_x^1(z) \cdot \nabla w_y^1(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_x^1(z) w_y^1(z)\, dz - \int_{\partial\Omega} \frac{\partial}{\partial\nu} G(z,y) G(z,x)\, dS(z).$$

A similar computation to (3.8) yields

$$\langle \Lambda_D(G(\,\cdot\,,y)|_{\partial\Omega}), G(\,\cdot\,,x)|_{\partial\Omega}\rangle = \int_{\Omega\setminus\overline{D}} \nabla w_y^1(z) \cdot \nabla w_x^1(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w_y^1(z) w_x^1(z)\, dz.$$

Thus we have

$$I^1(x,y) = \langle \Lambda_D(G(\,\cdot\,,y)|_{\partial\Omega}), G(\,\cdot\,,x)|_{\partial\Omega}\rangle - \int_{\partial\Omega} \frac{\partial}{\partial\nu} G(z,y) G(z,x)\, dS(z). \tag{3.31}$$

By (3.13), this yields $I^1(x) = I^1(x,y)|_{y=x}$. Thus, $I^1(x,y)$ gives a lifting of $I^1(x)$.

Next rewrite (3.29) as

$$w_x(y) = I(x,y) + \int_{\partial\Omega} G(z,y) \frac{\partial}{\partial\nu} w_x(z)\, dS(z).$$

This together with (3.30) yields

$$W_x(y) = I(x,y) + I^1(x,y) + \int_{\partial\Omega} G(z,y) \frac{\partial}{\partial\nu} w_x(z)\, dS(z) - \int_{\partial\Omega} G(z,x) \frac{\partial}{\partial\nu} w_y(z)\, dS(z). \tag{3.32}$$

This is the lifted version of inner decomposition (3.14). Note also that we have *twisted* decomposition

$$w_y(x) + w_x^1(y) = I(x,y) + I^1(x,y). \tag{3.33}$$

3.6.2 Uniqueness

We consider only two cases: $\mathcal{G} = \mathcal{G}^0, \mathcal{G}^*$. From (3.29) we have

$$I(x,y) = \begin{cases} w_x(y) - \displaystyle\int_{\partial\Omega} \frac{\partial}{\partial\nu} w_x(z) G(z-y)\, dS(z) & \text{if } \mathcal{G} = \mathcal{G}^0, \\ w_x(y) & \text{if } \mathcal{G} = \mathcal{G}^*. \end{cases} \tag{3.34}$$

Note that $w_x(y)$ depends on $\mathcal{G}$ and the symmetry of $I(x,y)$ yields the symmetry $w_x(y) = w_y(x)$ in the case $\mathcal{G} = \mathcal{G}^*$.

The expression (3.34) together with symmetry of $I(x, y)$ yields:

- for each fixed $x \in \Omega \setminus \overline{D}$,
$$\Delta_y I(x, y) + k^2 I(x, y) = 0, \quad y \in \Omega \setminus \overline{D},$$
- for each fixed $y \in \Omega \setminus \overline{D}$,
$$\Delta_x I(x, y) + k^2 I(x, y) = 0, \quad x \in \Omega \setminus \overline{D}.$$

Here the symbols Δ_y and Δ_x denote the Laplacian with respect to y and x, respectively.

By the unique continuation property of the Helmholtz equation, one concludes: indicator function $I(x)$, $x \in \Omega \setminus \overline{D}$ is uniquely determined by $I(x, y)$ given at all $(x, y) \in U \times V$, where U and V are arbitrary nonempty open subsets of $\Omega \setminus \overline{D}$, typically in a small neighborhood of $\partial\Omega$. Besides, using the argument for the proof of (3.3), we obtain also the computation formula of the lifting

$$I(x, y) = \lim_{n\to\infty} \langle (\Lambda_0 - \Lambda_D)(v_n|_{\partial\Omega}), v'_n|_{\partial\Omega} \rangle,$$

where v_n is the same as that of (3.3) and v'_n is an arbitrary needle sequence for (y, σ') based on $\mathcal{G}$ and $\sigma' \in N_y$ satisfying $\sigma' \cap \overline{D} = \emptyset$. So *in principle* or theoretically, it suffices to use only the needle sequences for the needles with tips in $U \times V$, say with $U = V$ and U is given by the intersection of a *small* open ball centered at a point on $\partial\Omega$ with Ω. In that case we can use only the straight needles *explicitly constructed* in [13].

Summing up, we have obtained the following *uniqueness theorem* by using needles localized, say in a *small* neighborhood of $\partial\Omega$ in $\overline{\Omega}$.

Proposition 3.2. *Let $\mathcal{G} = \mathcal{G}^0, \mathcal{G}^*$. Let U be an arbitrary nonempty open subset of $\Omega \setminus \overline{D}$. Assume that we have the data $\Lambda_D(v_n|_{\partial\Omega})$ for all $x \in U$ and a needle $\sigma \in N_x$ with $\sigma \setminus \partial\Omega \subset U$, and a needle sequence $\{v_n\}$ for (x, σ) based on $\mathcal{G}$. Then the obstacles D_d and D_n are uniquely determined by the data.*

The key of the proof is to put the calculation process of the lifting $I(x, y)$ for all $(x, y) \in U^2$ in between. This result could never have been found using a single methodology alone.

3.6.3 Symmetry of $I^1(x, y)$ and implications

For general $\mathcal{G}$ from (3.32) we obtain

$$\frac{W_x(y) + W_y(x)}{2} = I(x, y) + \frac{I^1(x, y) + I^1(y, x)}{2}. \tag{3.35}$$

Note that for general $\mathcal{G}$ the $I^1(x, y)$ is not necessary symmetric with respect to variables x and y. Here we note that the $I^1(x, y)$ is symmetric for $\mathcal{G} = \mathcal{G}^*, \mathcal{G}^0$. In fact, if $\mathcal{G} = \mathcal{G}^*$, then $G(\,\cdot\,, x) = 0$ on $\partial\Omega$ for each $x \in \Omega \setminus \overline{D}$. Thus (3.31) yields $I^1(x, y) = 0 = I^1(y, x)$.

For general $\mathcal{G}$, a similar argument for the proof of the symmetry of Green's function, we have, for $(x, y) \in \Omega^2$ with $x \neq y$,

$$\int_{\partial\Omega} \frac{\partial}{\partial\nu} G(z, y) G(z, x)\, dS(z) = -G(y, x) + \int_{\Omega} \nabla G(z, y) \cdot \nabla G(z, x)\, dz - \int_{\Omega} k^2 G(z, y) G(z, x)\, dz.$$

Note that all the integrands are absolutely integrable since $x \neq y$. Thus $I^1(x, y)$ with $x \neq y$ takes the form

$$\begin{aligned} I^1(x, y) = &\int_{\Omega\setminus\overline{D}} \nabla w^1_x(z) \cdot \nabla w^1_y(z)\, dz - \int_{\Omega\setminus\overline{D}} k^2 w^1_x(z) w^1_y(z)\, dz \\ &+ G(y, x) - \int_{\Omega} \nabla G(z, y) \cdot \nabla G(z, x)\, dz + \int_{\Omega} k^2 G(z, y) G(z, x)\, dz. \end{aligned}$$

Recalling the expression (1.2), we see that $I^1(x, y) = I^1(y, x)$ if and only if $H(y, x) = H(x, y)$. Thus $I^1(x, y)$ is symmetric in the case $\mathcal{G} = \mathcal{G}^0$ since $H \equiv 0$.

Therefore from (3.35) we obtain, for $\mathcal{G} = \mathcal{G}^0, \mathcal{G}^*$,[1]

$$\frac{W_x(y) + W_y(x)}{2} = I(x,y) + I^1(x,y). \tag{3.36}$$

And this together with (3.33) yields *twisted symmetry*:

$$w_x(y) - w_y^1(x) = w_y(x) - w_x^1(y).$$

Besides, the expression (3.31) together with symmetry yields:

- for each fixed $y \in \Omega \setminus \overline{D}$ we have

$$\Delta_x I^1(x,y) + k^2 I^1(x,y) = 0, \quad x \in \Omega \setminus \overline{D};$$

- for each fixed $x \in \Omega \setminus \overline{D}$, we have

$$\Delta_y I^1(x,y) + k^2 I^1(x,y) = 0, \quad y \in \Omega \setminus \overline{D}.$$

Besides, using (3.33) we conclude that, for each fixed $y \in \Omega \setminus \overline{D}$

$$\Delta_x(w_x^1(y)) + k^2 w_x^1(y) = 0, \quad x \in \Omega \setminus \overline{D}, \tag{3.37}$$

and for each fixed $x \in \Omega \setminus \overline{D}$,

$$\Delta_y(w_y(x)) + k^2 w_y(x) = 0, \quad y \in \Omega \setminus \overline{D}.$$

Finally, from (3.36) we obtain, for each fixed $x \in \Omega \setminus \overline{D}$,

$$\Delta_y(W_y(x)) + k^2 W_y(x) = 0, \quad y \in \Omega \setminus \overline{D}.$$

As a conclusion, we have:

Proposition 3.3. *Let $\mathcal{G} = \mathcal{G}^0, \mathcal{G}^*$. Let U be an arbitrary nonempty open subset of $\Omega \setminus \overline{D}$. Then the values $W_x(x)$ and $w_x(x)$ at all $x \in \Omega \setminus \overline{D}$ are uniquely determined by those of $W_x(y)$ and $w_x(y)$ for all $(x,y) \in U^2$, respectively.*

4 Conclusion and remarks

It became clear that the IPS function plays the central role in deriving the probe and singular sources methods. Besides, the *method of complementing function*, which is introduced in the proof of Theorem 1.1, makes everything so clear. Everything about the both methods can be derived from the knowledge of the IPS function $W_x(x)$. As a byproduct, we found the Side B of both the probe and singular sources methods for the mixed obstacle case. This is an advantage of IPS. However, there is a proviso that this comes at the expense of the regularity of the boundaries of the obstacles and whole domain. This seems to be unavoidable in order to establish especially the singular sources method since its is based on Green's theorem.

In this paper, we have considered only the case when the governing equation is given by the Helmholtz equation. However, the method developed here can be applied also to the same type of inverse obstacle problems governed by various partial differential equations, for example, the Navier equation, the Stokes system, the biharmonic equation, and so on. And also it would be interested to consider their time domain versions by the spirit of IPS. Those belong to our next project.

Our theory yields also an *alternative simple proof* of a result on the probe method described in [2], which is nothing but the Side A called in this paper. However, it should be pointed out that the Side B *without smallness* of k is still open at the present time even for the case $D_d = \emptyset$, see [15].

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1 Note that, in particular, if $\mathcal{G} = \mathcal{G}^*$ we have $W_x(y) = I(x,y) = w_x(y)$. This implies the symmetry of $W_x(y) = w_x(y)$.

A Appendix

Note that in this appendix it is assumed that k^2 satisfies Assumption 1.

Lemma 1. *Let $v \in H^2(\Omega)$ be an arbitrary solution of the Helmholtz equation $\Delta v + k^2 v = 0$ in Ω. Let $u \in H^2(\Omega \setminus \overline{D})$ be the solution of* (1.1) *with $f = v$ on $\partial\Omega$. We have*

$$\|u - v\|_{L^2(\Omega\setminus\overline{D})} \le C(\|v\|_{L^2(D_n)} + \|v\|_{L^2(\partial D_d)}),$$

where C is positive constant independent of v.

Proof. Set $w = u - v$. The $w = w(y)$ satisfies

$$\begin{cases} \Delta w + k^2 w = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w}{\partial \nu} = -\dfrac{\partial}{\partial \nu} v(y), & y \in \partial D_n, \\ w = -v(y), & y \in \partial D_d, \\ w = 0, & y \in \partial\Omega. \end{cases}$$

Decompose w as $w = w_1 + w_2$, where the $w_1 = w_1(y)$ solves

$$\begin{cases} \Delta w_1 + k^2 w_1 = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w_1}{\partial \nu} = -\dfrac{\partial}{\partial \nu} v(y), & y \in \partial D_n, \\ w_1 = 0, & y \in \partial D_d, \\ w_1 = 0, & y \in \partial\Omega, \end{cases}$$

and thus $w_2 = w_2(y)$ satisfies

$$\begin{cases} \Delta w_2 + k^2 w_2 = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w_2}{\partial \nu} = 0, & y \in \partial D_n, \\ w_2 = -v(y), & y \in \partial D_d, \\ w_2 = 0, & y \in \partial\Omega. \end{cases}$$

Considering $\Omega \setminus \overline{D}$ as $(\Omega \setminus \overline{D_d}) \setminus \overline{D_n}$ and applying [14, Lemma 2.2] to the case when Ω and D are replaced with $\Omega \setminus \overline{D_d}$ and D_n, respectively, we have

$$\|w_1\|_{L^2(\Omega\setminus\overline{D})} \le C\|v\|_{L^2(D_n)}.$$

So the problem is to show that

$$\|w_2\|_{L^2(\Omega\setminus\overline{D})} \le C\|v\|_{L^2(\partial D_d)}. \tag{A.1}$$

Here we employ a slightly modified argument for the proof of [10, (4.12) in Lemma 4.1]. Solve

$$\begin{cases} \Delta p + k^2 p = w_2, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial p}{\partial \nu} = 0, & y \in \partial D_n, \\ p = 0, & y \in \partial D_d, \\ p = 0, & y \in \partial\Omega. \end{cases}$$

Then we have

$$\begin{aligned} \int_{\Omega\setminus\overline{D}} w_2^2\,dy &= \int_{\Omega\setminus\overline{D}} p w_2\,dy = \int_{\Omega\setminus\overline{D}} (\Delta p + k^2 p) w_2\,dy \\ &= -\int_{D_n} \frac{\partial p}{\partial \nu} w_2\,dS - \int_{D_d} \frac{\partial p}{\partial \nu} w_2\,dS - \int_{\Omega\setminus\overline{D}} \nabla p \cdot \nabla w_2\,dy + \int_{\Omega\setminus\overline{D}} k^2 p w_2\,dy \\ &= \int_{\partial D_d} \frac{\partial p}{\partial \nu} v\,dS + \int_{\partial D_n} \frac{\partial w_2}{\partial \nu} p\,dS + \int_{\partial D_d} \frac{\partial w_2}{\partial \nu} p\,dS = \int_{\partial D_d} \frac{\partial p}{\partial \nu} v\,dS + \int_{\partial D_d} \frac{\partial w_2}{\partial \nu} p\,dS = \int_{\partial D_d} \frac{\partial p}{\partial \nu} v\,dS. \end{aligned}$$

Thus one gets

$$\|w_2\|^2_{L^2(\Omega\setminus\overline{D})} \le \|\nabla p\|_{L^2(\partial D_d)}\|v\|_{L^2(\partial D_d)}.$$

By elliptic regularity up to boundary, we have

$$\|p\|_{H^2(\Omega\setminus\overline{D})} \le C\|w_2\|_{L^2(\Omega\setminus\overline{D})}$$

and the trace theorem yields

$$\|\nabla p\|_{H^{\frac{1}{2}}(\partial D_d)} \le C\|\nabla p\|_{H^1(\Omega\setminus\overline{D})}.$$

Combining these, we obtain (A.1). □

Lemma 2. *The solution $w = w_x$ of* (1.6) *for $x \in \Omega \setminus \overline{D}$ satisfies, for each $\epsilon > 0$,*

$$\sup_{x\in\Omega\setminus\overline{D},\,\epsilon<\mathrm{dist}(x,\partial\Omega)} \|w_x\|_{L^2(\Omega\setminus\overline{D})} < \infty.$$

Proof. There are two ways to validate the statement. The first one is a combination of Lemma 1 and a limiting argument based on the Runge approximation property for the Helmholtz equation in the whole domain Ω provided k^2 is not a Dirichlet eigenvalue for the minus Laplacian $-\Delta$ in Ω. Another one goes back to the idea for establishing [9, estimates (19) and (28)] for the case when $D_n = \emptyset$ and $D_d = \emptyset$, respectively. Since the later one is elementary, we describe here. Decompose w_x as $w_x = (w_1)_x + (w_2)_x$, where the $(w_1)_x = w_1(y)$ solves

$$\begin{cases} \Delta w_1 + k^2 w_1 = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w_1}{\partial \nu} = -\dfrac{\partial}{\partial \nu} G(y, x), & y \in \partial D_n, \\ w_1 = 0, & y \in \partial D_d, \\ w_1 = 0, & y \in \partial\Omega, \end{cases}$$

and thus $(w_2)_x = w_2(y)$ satisfies

$$\begin{cases} \Delta w_2 + k^2 w_2 = 0, & y \in \Omega \setminus \overline{D}, \\ \dfrac{\partial w_2}{\partial \nu} = 0, & y \in \partial D_n, \\ w_2 = -G(y, x), & y \in \partial D_d, \\ w_2 = 0, & y \in \partial\Omega. \end{cases}$$

Considering $\Omega \setminus \overline{D}$ as $(\Omega \setminus \overline{D_d}) \setminus \overline{D_n}$ and applying the argument for the proof of (28) in [9] to the case when Ω and D are replaced with $\Omega \setminus \overline{D_d}$ and D_n, respectively, we have

$$\|(w_1)_x\|_{L^2(\Omega\setminus\overline{D})} \le C\bigg(\int_{\partial D_n} |z - x|^{\frac{1}{2}}\bigg|\frac{\partial G}{\partial \nu}(z, x)\bigg|\, dS(z) + \int_{D_n} |G(z, x)|\, dz\bigg),$$

where C is a positive constant independent of $x \in \Omega \setminus \overline{D}$. It is easy to see that this together with (1.3) yields

$$\sup_{x\in\Omega\setminus\overline{D},\,\epsilon<\mathrm{dist}(x,\partial\Omega)} \|(w_1)_x\|_{L^2(\Omega\setminus\overline{D})} < \infty. \tag{A.2}$$

Note that we are considering general $\mathcal{G}$. Using a similar argument for the proof of [9, (19)], we have the estimate

$$\|(w_2)_x\|_{L^2(\Omega\setminus\overline{D})} \le C\|G(\,,x)\|_{L^{\frac{4}{3}}(\partial D_d)}.$$

Then assumption (1.3) and

$$\sup_{x\in\mathbb{R}^3} \|G(\,\cdot\, - x)\|_{L^{\frac{4}{3}}(\partial D_d)} < \infty,$$

we obtain

$$\sup_{x\in\Omega\setminus\overline{D},\,\epsilon<\mathrm{dist}(x,\partial\Omega)} \|(w_2)_x\|_{L^2(\Omega\setminus\overline{D})} < \infty. \tag{A.3}$$

From (A.2) and (A.3) we obtain the desired conclusion. □

Funding: The author was supported by Grant-in-Aid for Scientific Research (C) (No. 24K06812) of Japan Society for the Promotion of Science.

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