Corrigendum

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An inverse problem in corrosion detection: Stability estimates

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Abstract: In this note we correct the proof of [2, Theorem 2.1].

Keywords: Inverse coefficient problem, corrosion detection, logarithmic stability estimate

MSC 2010: 35R30

Unless otherwise stated, Ω is a C^∞ bounded domain of $\mathbb{R}^2$ so that its boundary Γ is the union of two disjoint closed subsets with nonempty interior, $\Gamma = \Gamma_1 \cup \Gamma_2$.

We considered in [2] the stability issue for the problem of determining the boundary coefficient q, appearing in the BVP

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u + qu = 0 & \text{on } \Gamma_1, \\ \partial_\nu u = f & \text{on } \Gamma_2, \end{cases} \tag{1}$$

from the boundary measurement $u_{|\gamma_2}$, where γ_2 is an open subset of Γ_2.

Our proof of [2, Theorem 2.1] is partially incorrect. We rectify here this proof. We precisely establish a stability estimate of logarithmic type for the inverse problem described above. Contrary to the result announced in [2, Theorem 2.1], we do not know whether Lipschitz stability, even around a particular unknown coefficient, is true. Note that Lipschitz stability around an arbitrary unknown boundary coefficient is false in general as the following counter-example shows. Let $\Omega = \{\frac{1}{2} < |x| < 1\}$, $\Gamma_1 = \{|x| = \frac{1}{2}\}$, $\Gamma_2 = \{|x| = 1\}$ and let, in polar coordinates (r, θ),

$$\begin{aligned} u &= 1 + \ln r, \\ u_k &= u + 2^{-k}k^{-2}(r^k + r^{-k})\cos(k\theta), \quad k \geq 1. \end{aligned}$$

By straightforward computations we check that u and u_k are the solutions of the BVP (1) respectively when

$$\begin{aligned} q &= \frac{2}{1 - \ln 2}, \\ q = q_k &= \frac{2 + k^{-1}(2^{-2k+1} - 2)\sin(k\theta)}{1 - \ln 2 + k^{-2}(2^{-2n} + 1)\sin(k\theta)}, \quad k \geq 1, \end{aligned}$$

and $f = 1$.

By simple calculations, we get $\|u - u_k\|_{L^2(\Gamma_2)} = O(2^{-k}k^{-2})$, while $\|q - q_k\|_{L^2(\Gamma_1)} = O(k^{-1})$.

To our knowledge, the only case where Lipschitz stability holds is when q is assumed to be a priori piecewise constant. We refer to [6] for more details.

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Throughout, the unit ball of a Banach space X is denoted by B_X and

$$L^p_K(D) = \{h \in L^p(D) : \operatorname{supp}(h) \subset K\}, \quad 1 \le p \le \infty.$$

The characteristic function of a set A is denoted by χ_A.

Fix $q_0 \in L^\infty(\Gamma_1)$ nonnegative and nonidentically equal to zero and let $f \in L^2(\Gamma_2)$ be nonidentically equal to zero. Denote by $u_0 \in H^{3/2}(\Omega)$ the solution of the BVP

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u + q_0 u = 0 & \text{on } \Gamma_1, \\ \partial_\nu u = f & \text{on } \Gamma_2. \end{cases}$$

As it is observed in [2],

$$\Gamma_0 = \{x \in \Gamma_1 : u_0(x) \ne 0\} \tag{2}$$

is an open dense subset of Γ_1.

For $(\varphi_1, \varphi_2) \in L^2(\Gamma_1) \oplus L^2(\Gamma_2)$, define $L(\varphi_1, \varphi_2) := y$, where $y \in H^{3/2}(\Omega)$ is the unique weak solution of the BVP

$$\begin{cases} \Delta y = 0 & \text{in } \Omega, \\ \partial_\nu y + q_0 y = \varphi_1 & \text{on } \Gamma_1, \\ \partial_\nu y = \varphi_2 & \text{on } \Gamma_2. \end{cases}$$

An application of Green's formula leads to

$$\int_\Omega |\nabla y|^2 \, dx + \int_{\Gamma_1} q_0 y^2 \, d\sigma = \int_{\Gamma_1} \varphi_1 y \, d\sigma + \int_{\Gamma_2} \varphi_2 y \, d\sigma \le \|(\varphi_1, \varphi_2)\|_{L^2(\Gamma_1)\oplus L^2(\Gamma_2)} \|y\|_{H^1(\Omega)}. \tag{3}$$

Using that

$$h \to \Big(\int_\Omega |\nabla h|^2 \, dx + \int_{\Gamma_1} q_0 h^2 \, d\sigma \Big)^{1/2}$$

defines an equivalent norm on $H^1(\Omega)$, we derive from (3) that

$$\|y\|_{H^1(\Omega)} \le \kappa_0 \|(\varphi_1, \varphi_2)\|_{L^2(\Gamma_1)\oplus L^2(\Gamma_2)}$$

for some constant κ_0 depending only on Ω, q_0 and f.

As y is also the solution of the BVP

$$\begin{cases} \Delta y = 0 & \text{in } \Omega, \\ \partial_\nu y + y = (1 - q_0) y + \varphi_1 & \text{on } \Gamma_1, \\ \partial_\nu y = \varphi_2 & \text{on } \Gamma_2, \end{cases}$$

we get from the usual a priori estimates for nonhomogenous BVPs (see [5]) that there exists a constant κ_1, depending only on Ω, q_0 and f, so that

$$\|y\|_{H^{3/2}(\Omega)} \le \kappa_1 \|(\varphi_1, \varphi_2)\|_{L^2(\Gamma_1)\oplus L^2(\Gamma_2)}.$$

In other words, we have proved that $L \in \mathscr{B}(L^2(\Gamma_1) \oplus L^2(\Gamma_2), H^{3/2}(\Omega))$ and

$$\|L\| := \|L\|_{\mathscr{B}(L^2(\Gamma_1)\oplus L^2(\Gamma_2), H^{3/2}(\Omega))} \le \kappa_1.$$

For $q \in L^2(\Gamma_1)$, define the operator H_q as follows:

$$H_q : H^{3/2}(\Omega) \to H^{3/2}(\Omega), \quad H_q(u) = L(-q u_{|\Gamma_1}, 0).$$

If κ is the norm of the trace operator

$$h \in H^{3/2}(\Omega) \to h_{|\Gamma_1} \in C(\Gamma_1),$$

then

$$\|H_q\|_{\mathscr{B}(H^{3/2}(\Omega))} \le \kappa\|L\|\|q\|_{L^2(\Gamma_1)}.$$

Whence, for any $q \in \mathcal{U} = (2\kappa\|L\|)^{-1}B_{L^2(\Gamma_1)}$, $I - H_q$ is invertible and

$$\|(I - H_q)^{-1}\|_{\mathscr{B}(H^{3/2}(\Omega))} \le 2, \quad q \in \mathcal{U}. \tag{4}$$

Define, for $q \in \mathcal{U}$ and $(\varphi_1, \varphi_2) \in L^2(\Gamma_1) \oplus L^2(\Gamma_2)$,

$$u_q(\varphi_1, \varphi_2) = (I - H_q)^{-1}L(\varphi_1, \varphi_2).$$

In light of the identity

$$u_q(\varphi_1, \varphi_2) = L(-qu_{|\Gamma_1} + \varphi_1, \varphi_2),$$

we derive that $u_q(\varphi_1, \varphi_2) \in H^{3/2}(\Omega)$ is the solution of the BVP

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u + (q_0 + q)u = \varphi_1 & \text{on } \Gamma_1, \\ \partial_\nu u = \varphi_2 & \text{on } \Gamma_2. \end{cases}$$

Note that according to (4),

$$\|u_q(\varphi_1, \varphi_2)\|_{H^{3/2}(\Omega)} \le 2\kappa_1\|(\varphi_1, \varphi_2)\|_{L^2(\Gamma_1)\oplus L^2(\Gamma_2)}. \tag{5}$$

Set $u_q = u_q(0, f)$. That is, u_q is the solution of the BVP

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u + (q_0 + q)u = 0 & \text{on } \Gamma_1, \\ \partial_\nu u = f & \text{on } \Gamma_2. \end{cases}$$

Observe that (5) yields

$$\|u_q\|_{H^{3/2}(\Omega)} \le 2\kappa_1\|f\|_{L^2(\Gamma_2)}.$$

Let K be a compact subset of Γ_0 with nonempty interior so that $\Gamma_1 \setminus K \ne \emptyset$. We can mimic the proof of [2, Propoistion 2.1] to show that the mapping

$$\Phi : q \in \mathcal{U} \cap L^2_K(\Gamma_1) \mapsto \chi_{\Gamma_1}[\partial_\nu u_{q|K}] \in L^2_K(\Gamma_1)$$

is continuously Fréchet differentiable and $\Phi'(0) = N$. Here, for $p \in L^2_K(\Gamma_1)$, $Np = \chi_{\Gamma_1}[\partial_\nu v_{p|K}]$, where v_p is the solution of the BVP

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \partial_\nu v + q_0 v = -p & \text{on } \Gamma_1, \\ \partial_\nu v = 0 & \text{on } \Gamma_2. \end{cases}$$

Similarly to the proof of [2, Lemma 2.1], we prove that N is an isomorphism. Therefore, by the Implicit Function Theorem, there exists $\widetilde{\mathcal{U}} \subset \mathcal{U}$ so that Φ^{-1} is Lipschitz continuous on $\widetilde{\mathcal{V}} = \Phi(\widetilde{\mathcal{U}} \cap L^2_K(\Gamma_1))$ with Lipschitz constant less than or equal to $2\|N^{-1}\|$. That is,

$$\|q_1 - q_2\|_{L^2(\Gamma_1)} \le 2\|N^{-1}\|\|\partial_\nu u_{q_1} - \partial_\nu u_{q_2}\|_{L^2(K)}, \quad q_1, q_2 \in \widetilde{U} \cap L^2_K(\Gamma_1). \tag{6}$$

Let k be a positive integer, $s \in \mathbb{R}$, $1 \le r \le \infty$ and consider the vector space

$$B_{s,r}(\mathbb{R}^k) := \{w \in \mathscr{S}'(\mathbb{R}^k) : (1 + |\xi|^2)^{s/2}\widehat{w} \in L^r(\mathbb{R}^k)\},$$

where $\mathscr{S}'(\mathbb{R}^k)$ is the space of temperated distributions on $\mathbb{R}^k$ and $\widehat{w}$ is the Fourier transform of w. Equipped with the norm

$$\|w\|_{B_{s,r}(\mathbb{R}^k)} := \|(1 + |\xi|^2)^{s/2}\widehat{w}\|_{L^r(\mathbb{R}^k)},$$

$B_{s,r}(\mathbb{R}^k)$ is a Banach space. Note that $B_{s,2}(\mathbb{R}^k)$ is merely the Sobolev space $H^s(\mathbb{R}^k)$. Using local charts and a partition of unity, we construct $B_{s,r}(\Gamma_1)$ from $B_{s,r}(\mathbb{R})$ similarly as $H^s(\Gamma_1)$ is built from $H^s(\mathbb{R})$.

Fix $m > 0$. If $f \in H^{3/2}(\Gamma_2)$ and $q \in mB_{B_{3/2,1}(\Gamma_1)}$, then by [1, Theorem 2.3], $u_q \in H^3(\Omega)$ and

$$\|u_q\|_{H^3(\Omega)} \le C_0. \tag{7}$$

Here and henceforth, C_0 is a constant depending only on Ω, f and m. In dimension two, $H^3(\Omega)$ is continuously embedded in $C^2(\overline{\Omega})$. Whence, estimate (7) entails

$$\|u_q\|_{C^2(\overline{\Omega})} \le C_0.$$

Let

$$\Psi(\rho) = |\ln\rho|^{-1/2} + \rho, \quad \rho > 0,$$

be extended by continuity at 0 by setting $\Psi(0) = 0$.

Let γ_2 be a nonempty open subset of Γ_2. According to [3, Proposition 2.7], there exists a constant $C > 0$, depending only on Ω, f, m and γ_2, so that

$$\|\partial_\nu u_{q_1} - \partial_\nu u_{q_2}\|_{L^2(K)} \le C\Psi(\|u_{q_1} - u_{q_2}\|_{H^1(\gamma_2)}). \tag{8}$$

Set

$$\mathcal{Q}_m = mB_{B_{3/2,1}(\Gamma_1)} \cap \widetilde{\mathfrak{U}} \cap L^2_K(\Gamma_1).$$

Note that $\mathcal{Q}_m \neq \emptyset$ if m is chosen sufficiently large.

We can now combine (6) and (8) in order to obtain

$$\|q_1 - q_2\|_{L^2(\Gamma_1)} \le C\Psi(\|u_{q_1} - u_{q_2}\|_{H^1(\gamma_2)}), \quad q_1, q_2 \in \mathcal{Q}_m.$$

We sum up our analysis in the following theorem, where we use the fact that $H^{3/2}(\Gamma_2)$ is continuously embedded in $C^2(\Gamma_2)$,

Theorem 1. *Let $f \in H^{3/2}(\Gamma_2)$, $f \not\equiv 0$, $0 \le q_0 \in L^\infty(\Gamma_1)$, $q_0 \not\equiv 0$, let K be a compact subset of Γ_0, given by (2), with nonempty interior so that $\Gamma_1 \setminus K \neq \emptyset$ and let γ_2 be a nonempty open subset of Γ_2. There exists a neighborhood $\widetilde{\mathfrak{U}}$ of q_0 in $L^2(\Gamma_1)$, depending on f, Ω and K with the property that, if $m > 0$ is chosen in such a way that*

$$\mathcal{Q}_m = mB_{B_{3/2,1}(\Gamma_1)} \cap \widetilde{\mathfrak{U}} \cap L^2_K(\Gamma_1) \neq \emptyset,$$

we find a constant $C > 0$, depending on f, m, Ω, q_0, K and γ_2, so that

$$\|q_1 - q_2\|_{L^2(\Gamma_1)} \le C\Psi(\|u_{q_1} - u_{q_2}\|_{H^1(\gamma_2)}), \quad q_1, q_2 \in \mathcal{Q}_m.$$

Observe that, as in [2], the last theorem can be extended to the case where $\partial\Gamma_1 \cap \partial\Gamma_2 \neq \emptyset$. Also, for the most general case, in dimensions two and three, we can prove a stability estimate of triple logarithmic type (see [3, Theorem 4.9]).

Remark 1. Note that, in general, Γ_0 given by (2) is strictly contained in Γ_1 for an arbitrary q_0. However, we can construct an example of q_0 for which $\Gamma_0 = \Gamma_1$. To this end, fix $0 < \alpha < 1$ and, for $0 \le f \in C^{1,\alpha}(\Gamma_2)$, denote by $w(f) \in C^{2,\alpha}(\overline{\Omega})$ the solution of the BVP

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_1, \\ \partial_\nu w = f & \text{on } \Gamma_2. \end{cases}$$

According to strong maximum principle's and Hopf's lemma (see for instance [4]), $\partial_\nu w < 0$ on Γ_1. Let $q_0 = -\partial_\nu w(f)_{|\Gamma_1} (> 0)$ and set $u_0 = 1 + w$. Then it is straightforward to check that u_0 is the unique solution of the BVP

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u + q_0 u = 0 & \text{on } \Gamma_1, \\ \partial_\nu u = f & \text{on } \Gamma_2. \end{cases}$$

We see that for this particular choice of q_0, we have $\Gamma_0 = \Gamma_1$.

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