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The proper geometric dimension of the mapping class group of an orientable surface with punctures

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Abstract. We show that the *full* mapping class group of any orientable closed surface with punctures admits a cocompact classifying space for proper actions of dimension equal to its virtual cohomological dimension. This was proved for closed orientable surfaces and for *pure* mapping class groups by Aramayona and Martínez Pérez. As a consequence of our result, we also obtain the proper geometric dimension of *full* spherical braid groups.

1 Introduction

Let S_g be an orientable closed connected surface of genus $g \geq 0$, and for $n \geq 1$, consider a collection $\{p_1, \dots, p_n\}$ of distinguished points in S_g. The *mapping class group* Mod_g^n is the group of isotopy classes of all orientation-preserving homeomorphisms $f\colon S_g^n \to S_g^n$, where $S_g^n := S_g - \{p_1, \dots, p_n\}$. The group Mod_g^n permutes the set $\{p_1, \dots, p_n\}$ and the kernel of this action is the *pure mapping class group* PMod_g^n. We denote the group of isotopy classes of orientation-preserving homeomorphisms of S_g by Mod_g.

In this paper, we compute the minimal dimension $\underline{\mathrm{gd}}(\mathrm{Mod}_g^n)$, often called the *proper geometric dimension*, of a classifying space $\underline{E}\,\mathrm{Mod}_g^n$ for proper actions of Mod_g^n. Recall that, for a discrete group G, the space $\underline{E}G$ is a contractible G-CW-complex on which G acts properly, and such that the fixed point set of a subgroup $H < G$ is contractible if H is finite, and is empty otherwise. Since Mod_g^n is virtually torsion-free, its virtual cohomological dimension $\mathrm{vcd}(\mathrm{Mod}_g^n)$ is a lower bound for $\underline{\mathrm{gd}}(\mathrm{Mod}_g^n)$. Our main result is the following.

Theorem 1.1. *For any $g \geq 0$, $n \geq 1$, the proper geometric dimension of* Mod_g^n *is*

$$\underline{\mathrm{gd}}(\mathrm{Mod}_g^n) = \mathrm{vcd}(\mathrm{Mod}_g^n).$$

Moreover, there exists a cocompact $\underline{E}\,\mathrm{Mod}_g^n$ *of dimension equal to* $\mathrm{vcd}(\mathrm{Mod}_g^n)$.

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The computation of the proper geometric dimension follows from Theorem 3.8 for $g = 0$, Theorem 4.3 for $g = 1$, Theorem 5.1 for $g = 2$, and Theorem 6.2 for $g \geq 3$. These theorems are the main results of Sections 3, 4, 5, and 6, respectively, which provides a fair overview of the structure of the paper. For $2g + n > 2$, the Teichmüller space $\mathscr{T}_g^n$ is known to be a model for $\underline{E}\,\mathrm{Mod}_g^n$; see for instance [15, Proposition 2.3] and [21, Section 4.10]. In fact, Ji and Wolpert proved in [15, Theorem 1.2] that the *truncated* Teichmüller space $\mathscr{T}_g^n(\epsilon)$ is a cocompact classifying space for proper actions for ϵ sufficiently small. Hence the *moreover* part of the main result follows from our computation of $\underline{\mathrm{gd}}(\mathrm{Mod}_g^n)$ and a result of Lück [20, Theorem 13.19] (see Theorem 2.1 below).

1.1 Known results and filling a gap in the literature

Harer [12, Section 2] constructed a *spine* for Teichmüller space, that is a PMod_g^n-equivariant deformation retraction of $\mathscr{T}_g^n$ provided $n > 0$. This spine gives a model for $\underline{E}\,\mathrm{PMod}_g^n$, which is of minimal dimension for the mapping class group Mod_g^1 of an orientable surface with exactly one puncture; see [12, Theorem 2.1] and [7, Introduction].

On the other hand, Aramayona and Martínez Pérez proved that the mapping class group of any closed orientable surface Mod_g (see [2, Theorem 1.1]) admits a cocompact classifying space for proper actions of dimension equal to its virtual cohomological dimension. They use their result and the Birman exact sequence to show that this is also the case for the *pure* mapping class group PMod_g^n (see [2, Corollary 1.3], and also arXiv:1302.4238v.2 and [7, Corollary 3.5]).

However, [2, Theorem 1.1] does not imply the result for the *full* mapping class group Mod_g^n when $n \geq 2$, and [2, Theorem 1.1] has been misquoted in the literature. For instance, the existence of a model of minimal dimension for the *full* mapping class group is used in the proofs of [17, Theorem 1 & Main Theorem], [24, Proposition 5.3 & Theorem 1.4], [1, Theorem 1.1], and [16, Theorem 1.5]. Our Theorem 1.1 fills this *gap*, and together with [2, Theorem 1.1], it gives the necessary ingredients to prove the following result.

Corollary 1.2 ([24, Proposition 5.3]). *Let S be a closed (possibly disconnected) surface with a finite number (possibly zero) of punctures. Then there is a cocompact model for $\underline{E}\,\mathrm{Mod}(S)$ of dimension $\underline{\mathrm{gd}}(\mathrm{Mod}(S)) = \mathrm{vcd}(\mathrm{Mod}(S))$.*

1.2 Proper geometric dimension of spherical braid groups

The *full n-th strand spherical braid group $B_n(S_0)$* is the fundamental group of the n-th unordered configuration space of the sphere S_0. It is the trivial group for

$n = 1$ and a cyclic group of order 2 for $n = 2$. For all $n \geq 1$, these groups are virtually torsion free and their virtual cohomological dimension is

$$\mathrm{vcd}(B_n(S_0)) = \max\{0, n-3\}$$

(see [10, Theorem 5]). For $n \geq 3$, the mapping class group Mod_0^n is related to $B_n(S_0)$ by the following central extension (see for example [11, Section 2.4]):

$$1 \to \mathbb{Z}/2 \to B_n(S_0) \to \mathrm{Mod}(S_0^n) \to 1.$$

Note that any cocompact model for $\underline{E}\,\mathrm{Mod}(S_0^n)$ is, via the above short exact sequence, a cocompact model for $\underline{E}\,B_n(S_0)$. It follows from [21, Lemma 5.8] that

$$\underline{\mathrm{gd}}(B_n(S_0)) = \underline{\mathrm{gd}}(\mathrm{Mod}(S_0^n),$$

and Theorem 1.1 implies the following corollary.

Corollary 1.3. *For any $n \geq 1$,*

$$\underline{\mathrm{gd}}(B_n(S_0)) = \mathrm{vcd}(B_n(S_0)) = \max\{0, n-3\}.$$

Moreover, there is a cocompact model for $\underline{E}\,B_n(S_0)$ of dimension $\mathrm{vcd}(B_n(S_0))$.

1.3 Overview of the proof and the paper

In order to prove our main result Theorem 1.1, we follow the general strategy of [2], and obtain the proper geometric dimension $\underline{\mathrm{gd}}(\mathrm{Mod}_g^n)$ by computing its algebraic counterpart $\underline{\mathrm{cd}}(\mathrm{Mod}_g^n)$. This computation amounts to verifying that, for any finite subgroup F of Mod_g^n, the inequality $\mathrm{vcd}(NF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_g^n)$ holds, and using [2, Theorem 3.3] (see Theorem 2.2 below). Here NF denotes the *normalizer* of F in Mod_g^n and $\lambda(F)$ is the *length* of F. In Section 2.1, we introduce the necessary definitions and details.

An application of Nielsen realization and Maher's Lemma 2.4 implies that $\mathrm{vcd}(NF) = \mathrm{vcd}(\mathrm{Mod}_{g_F}^{n_F})$ for any finite subgroup F of Mod_g^n. The parameters g_F and n_F are invariants of the orbifold quotient S_g^n/F as described in Section 2.2. Most of the present paper deals with either computing or upper-bounding the parameters n_F and g_F. Once we have computed g_F and n_F, we can use Harer's computation of the vcd for mapping class groups (2.1) to obtain $\mathrm{vcd}(\mathrm{Mod}_{g_F}^{n_F})$. On the other hand, $\lambda(F)$ can be bounded by the number of factors in the prime decomposition of $|F|$.

For $g \geq 3$, we obtain Theorem 6.2 as a straightforward application of [2, Proposition 4.4], which we explain in Section 6. However, for genus $g = 0$, $g = 1$, and $g = 2$, an independent and careful analysis of the finite subgroups of Mod_g^n is required. This is done in Sections 3, 4, and 5, respectively.

2 Preliminaries

2.1 Proper geometric and cohomological dimensions

There are several notions of dimension defined for a given group G. We recall in this section the notions that will be used in this paper.

A model for the *classifying space of* G *for proper actions* $\underline{E}G$ is a G-CW-complex X such that the fixed point set X^H of a subgroup $H < G$ is contractible if H is finite, and is empty otherwise. Such a model always exists and is unique up to proper G-homotopy. The *proper geometric dimension of* G is defined to be

$$\underline{\mathrm{gd}}(G) = \min\{n \in \mathbb{Z}_{\geq 0} : \text{there exists an } n\text{-dimensional model for } \underline{E}G\}.$$

On the other hand, let $\mathrm{F\small{IN}}$ be the collection of finite subgroups of G, and consider the restricted orbit category $\mathcal{O}_{\mathrm{FIN}}G$, which has as objects the homogeneous G-spaces G/H, $H \in \mathrm{FIN}$, and morphisms the G-maps. An $\mathcal{O}_{\mathrm{FIN}}G$-module is a contravariant functor from $\mathcal{O}_{\mathrm{FIN}}G$ to the category of abelian groups, and a morphism between two $\mathcal{O}_{\mathrm{FIN}}G$-modules is a natural transformation of the underlying functors. The category $\mathcal{O}_{\mathrm{FIN}}G$-mod of $\mathcal{O}_{\mathrm{FIN}}G$-modules is an abelian category with enough projectives; see for example [23, p. 7].

The *proper cohomological dimension of* G is defined to be

$$\begin{aligned}\underline{\mathrm{cd}}(G) = \min\{n \in \mathbb{Z}_{\geq 0} : {} & \text{there exists a projective resolution} \\ & \text{of the } \mathcal{O}_{\mathrm{FIN}}G\text{-module } \mathbb{Z}_{\mathrm{FIN}} \text{ of length } n\},\end{aligned}$$

where $\mathbb{Z}_{\mathrm{FIN}}$ is the constant $\mathcal{O}_{\mathrm{FIN}}G$-module given by

$$\mathbb{Z}_{\mathrm{FIN}}(G/H) = \mathbb{Z} \quad \text{for all } H \in \mathrm{FIN},$$

sending every morphism of $\mathcal{O}_{\mathrm{FIN}}G$ to the identity function.

The *cohomological dimension* $\mathrm{cd}(H)$ of a group H is the length of shortest projective resolution, in the category of H-modules, for the trivial H-module $\mathbb{Z}$. If G is virtually torsion free, then G contains a torsion free subgroup H of finite index, and the *virtual cohomological dimension of* G is defined to be $\mathrm{vcd}(G) = \mathrm{cd}(H)$. A well-known theorem of Serre establishes that $\mathrm{vcd}(G)$ is well defined and it does not depend on the choice of the finite index torsion free subgroup H of G. For every virtually torsion free group G, we have the following inequalities (see [5, Theorem 2]):

$$\mathrm{vcd}(G) \leq \underline{\mathrm{cd}}(G) \leq \underline{\mathrm{gd}}(G) \leq \max\{3, \underline{\mathrm{cd}}(G)\}.$$

Following the general strategy of [2], we will determine the proper geometric dimension $\underline{\mathrm{gd}}(\mathrm{Mod}_g^n)$ by computing the algebraic counterpart $\underline{\mathrm{cd}}(\mathrm{Mod}_g^n)$. We recall here the two main properties of $\underline{\mathrm{cd}}(G)$ that we will need to obtain Theorem 1.1.

First, the following theorem is a consequence of [20, Theorem 13.19], as explained in [2, Section 3]; see also [5, Theorem 1].

Theorem 2.1. *Let G be a group with $\underline{\mathrm{cd}}(G) = d \geq 3$. Then there is a d-dimensional $\underline{E}G$. Moreover, if G has a cocompact model for $\underline{E}G$, then it also admits a cocompact $\underline{E}G$ of dimension d.*

Now, let F be a finite subgroup of G. We denote by $Z_G(F)$, $N_G(F)$, and $W_G(F) = N_G(F)/F$ the centralizer, normalizer, and *Weyl group of* F, respectively. If there is no risk of confusion, we will omit the parenthesis and subindices, i.e. we will use the notation ZF, NF, and WF. The *length* $\lambda(F)$ of a finite group F is the largest $i \geq 0$ for which there is a sequence

$$1 = F_0 < F_1 < \cdots < F_i = F.$$

The second result used in obtaining our Theorem 1.1 is the following theorem by Aramayona and Martínez Pérez that gives a criterion to compute the proper cohomological dimension. It is a consequence of a result of Connolly and Kozniewski [8, Theorem A].

Theorem 2.2. *Let G be a virtually torsion free group such that, for any finite subgroup $F \leq G$,* $\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(G)$. *Then* $\underline{\mathrm{cd}}(G) = \mathrm{vcd}(G)$.

Remark 1. For F a finite subgroup of G, the normalizer NF and the Weyl group WF are commensurable; therefore, $\mathrm{vcd}(NF) = \mathrm{vcd}(WF)$. In our analysis below, we will obtain upper bounds of $\mathrm{vcd}(NF)$ in order to apply Theorem 2.2.

The following lemma will be useful in our computations below. For a given $g \in G$, we denote (respectively) by $Z(g)$, $N(g)$ and $W(g)$ the centralizer, normalizer and Weyl group of the cyclic subgroup $\langle g \rangle$ in G.

Lemma 2.3. *Let G be a virtually torsion free group and F a finite subgroup. Then*

(1) $\lambda(F) \leq \log_2(|F|)$, *and*

(2) $\mathrm{vcd}(WF) \leq \min\{\mathrm{vcd}(Z(g)) \mid g \in F\}$.

Proof. Item (1) is [13, Lemma 3.1]. For item (2), note that ZF is commensurable with WF; hence they have the same vcd. The conclusion follows from the fact that

$$ZF = \bigcap_{g \in F} Z(g)$$

and the monotonicity of the vcd. □

2.2 Mapping class groups

For any $g \geq 0$ and $n \geq 0$, the group Mod_g^n is virtually torsion free and its virtual cohomological dimension was computed by J. L. Harer in [12], i.e.

$$\mathrm{vcd}(\mathrm{Mod}_g^n) = \begin{cases} 0 & \text{if } g = 0 \text{ and } n < 3, \\ n-3 & \text{if } g = 0 \text{ and } n \geq 3, \\ 1 & \text{if } g = 1 \text{ and } n = 0, \\ 4g-5 & \text{if } g \geq 2 \text{ and } n = 0, \\ 4g-4+n & \text{if } g \geq 1 \text{ and } n \geq 1. \end{cases} \tag{2.1}$$

Let F be a finite subgroup of Mod_g^n. By Nielsen realization (see [9, Theorem 7.2] and [18]), there is a hyperbolic structure for S_g^n on which F acts by orientation-preserving isometries, when $\chi(S_g^n) < 0$.

By a slight abuse of notation, we denote by S_g^n/F both the orbifold and the underlying topological surface. In what follows, we will use the following notation:

- g_F is the genus of S_g^n/F,
- o_F is the number of elliptic (orbifold) points of S_g^n/F,
- n_F is the number o_F of elliptic points plus the number of orbits of punctures of this action,
- $(g_F; p_1^F, \dots, p_{o_F}^F)$ is often called the *signature of* F, where $p_1^F, \dots, p_{o_F}^F$ are the orders of the elliptic points of S_g^n/F.

Note that $n_F \leq n/|F| + o_F$.

The following lemma is due to Maher. Together with Harer's computation (2.1), it gives us a way to compute the vcd of the Weyl group WF of any finite subgroup F of Mod_g^n.

Lemma 2.4 ([22, Proposition 2.3]). *Let $F \leq \mathrm{Mod}_g^n$ be a finite subgroup with g_F and n_F as before. Then NF has finite index in $\mathrm{Mod}_{g_F}^{n_F}$. In particular,*

$$\mathrm{vcd}(WF) = \mathrm{vcd}(NF) = \mathrm{vcd}(\mathrm{Mod}_{g_F}^{n_F}).$$

3 Genus 0

The main result of this section is Theorem 3.8 which proves the case $g = 0$ of Theorem 1.1. In the first half of this section, we study the finite subgroups of Mod_0^n and their actions on S_0^n; our analysis starts from Stukow's classification of maximal finite subgroups of Mod_0^n given in [25]. In the second half of the section, we apply the results in the first half to prove Theorem 3.8. It is worth saying that we dealt with cases $5 \leq n \leq 11$ one by one, and all the computations of $\mathrm{vcd}(NF)$ and $\lambda(F)$ are summarized in Appendix A by means of several tables.

3.1 Finite subgroups of Mod_0^n and their realization

We start by recalling Stukow's classification of maximal finite subgroups of Mod_0^n.

Theorem 3.1 ([25, Theorem 3]). *A finite subgroup F of Mod_0^n is a maximal finite subgroup of Mod_0^n if and only if F is isomorphic to one of the following:*

(1) *a cyclic group $\mathbb{Z}/(n-1)$ if $n \neq 4$,*

(2) *the dihedral group D_{2n} of order $2n$,*

(3) *the dihedral group $D_{2(n-2)}$ of order $2(n-2)$ if $n = 5$ or $n \geq 7$,*

(4) *the alternating group A_4 if $n \equiv 4$ or $10 \pmod{12}$,*

(5) *the symmetric group $\mathfrak{S}_4$ if $n \equiv 0, 2, 6, 8, 12, 14, 18$ or $20 \pmod{24}$,*

(6) *the alternating group A_5 if $n \equiv 0, 2, 12, 20, 30, 32, 42$ or $50 \pmod{12}$.*

Stukow also studied the conjugacy classes of finite subgroups of Mod_0^n.

Theorem 3.2 ([25, Theorem 4, Corollary 5]). *Let F be a finite subgroup of Mod_0^n. Then the set of conjugacy classes of subgroups of Mod_0^n isomorphic to F has two elements if*

(1) *$F \cong \mathbb{Z}/2$ with n even,*

(2) *$F \cong D_{2m}$ with $2m \mid n$ or $2m \mid n-2$,*

and one element otherwise. In particular, any two maximal finite subgroups of Mod_0^n are conjugate if and only if they are isomorphic.

Since, for a given finite subgroup F of Mod_0^n, the number n_F only depends on the conjugacy class of F, we only have to work with a representative of such conjugacy class. Let us describe representatives of the conjugacy classes of maximal finite subgroups of Mod_0^n of types (1)–(3) following the numbering used in Theorem 3.1. For this, let us denote by N and S the north and south poles of S_0, respectively. Also, given any group isomorphic to D_{2m}, recall that there is an index 2 cyclic subgroup of order m; we call this subgroup the *subgroup of rotations* and every element which does not belong to the subgroup of rotations is called a *reflection* (all such elements have order 2).

(1) Up to a self-homeomorphism of S_0^n, the action of $\mathbb{Z}/(n-1)$ can be realized as follows: we place one of the punctures on N and the remaining $n-1$ on the equator equidistantly. The generator of $\mathbb{Z}/(n-1)$ acts on S_0^n by a rotation of angle $2\pi/(n-1)$ that fixes N and S.

(2) Up to a self-homeomorphism of S_0^n, the action of D_{2n} can be realized as follows: we place all the punctures along the equator equidistantly. As in the previous case, we have a natural action of the subgroup of rotations $\mathbb{Z}/n$ on S_0^n, and a reflection can be realized as a half turn that interchanges N and S and leaves the set of punctures invariant.

(3) The realization of $D_{2(n-1)}$ is completely analogous to the previous case, the only difference being that we place 2 punctures on N and S and the remaining $n-2$ on the equator.

The numbering in the following proposition is designed to be compatible with that of Theorem 3.1. We will use the above descriptions of the actions of maximal finite groups in the proof of the following statement.

Proposition 3.3. *Let F be a finite subgroup of* Mod_0^n.

(1) *If $F \cong \mathbb{Z}/m \leq \mathbb{Z}/(n-1)$, then*

$$n_F = 2 + \frac{n-1}{m}.$$

(2.1) *If $F \cong \mathbb{Z}/m$ is a subgroup of the rotation subgroup $\mathbb{Z}/n$ of D_{2n}, then*

$$n_F = 2 + \frac{n}{m}.$$

(2.2) *If $F \cong D_{2m} \leq D_{2n}$ with $m > 1$ and when n is odd, then*

$$n_F = 1 + \frac{n+3m}{2m}.$$

When n is even and $2m \mid n$,

$$n_F = \begin{cases} 1 + \frac{n+2m}{2m}, \\ 1 + \frac{n+4m}{2m}, \end{cases}$$

corresponding to the conjugacy classes of subgroups isomorphic to D_{2m}. When n is even and $2m \nmid n$,

$$n_F = 1 + \frac{n+3m}{2m}.$$

(3.1) *If $F \cong \mathbb{Z}/m$ is a subgroup of the rotation subgroup $\mathbb{Z}/(n-2)$ of $D_{2(n-2)}$, then*

$$n_F = 2 + \frac{n-2}{m}.$$

(3.2) *If* $F \cong D_{2m} \leq D_{2(n-2)}$ *with* $m > 1$ *and when* n *is odd, then*

$$n_F = 1 + \frac{n-2+3m}{2m}.$$

When n *is even and* $2m \mid (n-2)$,

$$n_F = \begin{cases} 1 + \frac{n-2+2m}{2m}, \\ 1 + \frac{n-2+4m}{2m}, \end{cases}$$

corresponding to the conjugacy classes of subgroups isomorphic to D_{2m}.

When n *is even and* $2m \nmid (n-2)$,

$$n_F = 1 + \frac{n-2+3m}{2m}.$$

Proof. We proceed by cases.

(1) We can realize the $\mathbb{Z}/m$-action placing a puncture in N and the remaining $n-1$ equidistantly on the equator. Hence $\{\mathrm{N}\}$ and $\{\mathrm{S}\}$ are both orbits of elliptic points of the $\mathbb{Z}/m$-action, while the orbits of punctures are exactly $(n-1)/m$.

(2.1) The $\mathbb{Z}/m$-action can be realized placing all punctures on the equator equidistantly. The conclusion is completely analogous to case (1).

(2.2) Recall that D_{2m} is generated by the rotation subgroup $\mathbb{Z}/m \leq \mathbb{Z}/n$ and a half turn τ that interchanges N and S. In this case, we have the orbit $\{\mathrm{N}, \mathrm{S}\}$. On the other hand, note that the D_{2n}-action is free away from the poles and the equator. Hence any elliptic point other than N and S is on the equator. Let X be the set of all punctures and all the elliptic points on the equator. Note that, for elements of D_{2m}, the identity element fixes all points of X, the nontrivial rotations fix no points in X, and the m reflections each fix two points in X; hence, by the Burnside lemma, the number of orbits of X is $\frac{|X|+2m}{2m}$. Therefore, if e is the number of elliptic points in X, we have

$$n_F = 1 + \frac{n+e+2m}{2m}.$$

Assume n is odd; then the axis points of τ consist exactly of one puncture p and its antipode $-p$ which is an elliptic point. Therefore, there are m elliptic points, that is, $e = m$.

Assume n is even. Then the axes of reflections in D_{2m} are obtained by $\mathbb{Z}/m$-rotating the axis of τ and taking the bisectors of any two consecutive rotations of the axis of τ.

If $2m$ divides n, n/m is even. Let p be a fixed puncture and let t be a fixed generator of $\mathbb{Z}/m$. Then, between p and tp, there are $n/m - 1$ punctures, that is, an odd number of punctures. If τ fixes two punctures, then each bisector also fixes two punctures. Therefore, there are no elliptic points and $e = 0$. If τ fixes two elliptic points, then each bisector also fixes two elliptic points. Therefore, $e = 2m$.

If $2m$ does not divide n, then n/m is odd and m is even. Let p be a fixed puncture and let t be a fixed generator of $\mathbb{Z}/m$. Then, between p and tp, there are $n/m - 1$ punctures, that is, an even number of punctures. Hence, whenever τ fixes two punctures, each bisector fixes two elliptic points, and whenever τ fixes two elliptic points, each bisector fixes two punctures. In both scenarios, we conclude $e = m$.

Cases (3.1) and (3.2) are analogous to cases (2.1) and (2.2), respectively, and are left as an exercise to the reader. □

3.2 Proof of the main theorem for genus 0

We compute in this subsection, Theorem 3.8 below, the geometric dimension for Mod_0^n.

Lemma 3.4. *If F is a finite subgroup of Mod_0^n, then $\mathrm{vcd}(NF) = \max\{n_F - 3, 0\}$.*

Proof. By Nielsen realization, F acts on S_0^n by orientation-preserving homeomorphisms. As a straightforward application of the Riemann–Hurwitz theorem, we have that S_0^n/F is homeomorphic to a sphere, that is, $g_F = 0$. Now the result is a consequence of Lemma 2.4 and Harer's computation of the virtual cohomological dimension for mapping class groups given in (2.1). □

Proposition 3.5. *Let $n \geq 3$ and let $g \in \mathrm{Mod}_0^n$ be an element of finite order $r \geq 2$. Then $r \leq n$ and*

$$\mathrm{vcd}(W(g)) \leq \frac{n}{r} - 1 \leq \frac{n}{2} - 1.$$

Proof. By [25, Theorem 4, Corollary 5], the element g is contained in a unique (up to isotopy) maximal finite cyclic subgroup of order n, $n-1$, or $n-2$, which by Theorem 3.2 and Theorem 3.1, is the rotation subgroup of one of the maximal subgroups $\mathbb{Z}/(n-1)$, D_{2n}, or $D_{2(n-2)}$. Hence, up to conjugation, g can be realized as a rotation that fixes N and S and the punctures are on the equator and possibly in one or both poles. Therefore, $n_F \leq 2 + n/r$, and the first inequality in our statement follows from Lemma 3.4. The second inequality is clear. □

Proposition 3.6. *Let $n \geq 11$ and let $g \in \mathrm{Mod}_0^n$ be an element of order $r \geq 2$. Then*

$$\mathrm{vcd}(W(g)) + \lambda(\langle g\rangle) \leq \mathrm{vcd}(\mathrm{Mod}_0^n).$$

Proof. By Proposition 3.5 and Lemma 2.3, we get

$$\begin{aligned}\mathrm{vcd}(W(g)) + \lambda(\langle g\rangle) &\leq \frac{n}{r} - 1 + \log_2(r) \leq \frac{n}{r} - 1 + \log_2(n)\\ &\leq \frac{n}{2} + \log_2(n).\end{aligned}$$

Since $\frac{n}{2} + \log_2(n)$ and $\mathrm{vcd}(\mathrm{Mod}_0^n) = n - 3$ are both increasing functions of n, it is easy to verify that $\frac{n}{2} + \log_2(n) \le n - 3$ for $n \ge 11$. □

Theorem 3.7. *Let $n = 5$ or $n \ge 7$, and let F be a finite subgroup of* Mod_0^n. *Then*

$$\mathrm{vcd}(WF) + \lambda(F) \le \mathrm{vcd}(\mathrm{Mod}_0^n) = n - 3.$$

Proof. For $n = 5$ or $7 \le n \le 13$, the conclusion follows from the tables in Appendix A, where $\mathrm{vcd}(WF)$ is computed using Proposition 3.3 and Lemma 3.4. The rest of the proof deals with the case $n \ge 14$. By Theorem 3.2, up to conjugation, F is a subgroup of one of six possibilities.

(1) $F \le \mathbb{Z}/(n-1)$. The result follows directly from Proposition 3.6 for $n \ge 11$.

(2) $F \le D_{2n}$. If F is cyclic, the result follows from Proposition 3.6 for $n \ge 11$. Assume that $F \cong D_{2m}$ with $m \mid n$. Then, by Lemma 2.3 and Proposition 3.5, we get

$$\mathrm{vcd}(WF) \le \mathrm{vcd}(\mathbb{Z}/m) \le \frac{n}{m} - 1 \quad \text{and} \quad \lambda(F) \le \log_2(m) + 1.$$

Thus

$$\mathrm{vcd}(WF) + \lambda(F) \le \frac{n}{m} + \log_2(m) \le \frac{n}{2} + \log_2(n).$$

Now note that $\frac{n}{2} + \log_2(n) \le n - 3$ for $n \ge 14$.

(3) $F \le D_{2(n-2)}$. This case is analogous to the previous one.

(4) $F \le A_4$. We have $\lambda(F) \le \lambda(A_4) = 3$. By Lemma 2.3 and Proposition 3.5, we get

$$\mathrm{vcd}(WF) + \lambda(F) \le \frac{n}{2} + 3.$$

Note that $\frac{n}{2} + 3 \le n - 3$ for $n \ge 10$.

(5) $F \le \mathfrak{S}_4$. Analogous to case (4).

(6) $F \le A_5$. Analogous to case (4). □

Theorem 3.8. *Let $n \ge 0$. Then*

$$\underline{\mathrm{gd}}(\mathrm{Mod}_0^n) = \underline{\mathrm{cd}}(\mathrm{Mod}_0^n) = \mathrm{vcd}(\mathrm{Mod}_0^n) = \max\{n-3, 0\}.$$

Proof. We separate the proof into various cases.

First assume $n = 0, 1, 2, 3$. In this case, Mod_0^n is a finite group and the conclusion follows.

Now assume $n = 4$. Since $\mathrm{vcd}(\mathrm{Mod}_0^4) = 1$, we know that Mod_0^4 is virtually finitely generated free; hence, by a well-known theorem of Stallings, Mod_0^4 acts properly on a tree and this action provides a model for $\underline{E}\,\mathrm{Mod}_0^4$ of dimension 1; therefore, $\underline{\mathrm{gd}}(\mathrm{Mod}_0^4) = \underline{\mathrm{cd}}(\mathrm{Mod}_0^4) = 1$.

Now assume $n = 6$. In this case, by Birman–Hilden theory, we have a central extension

$$1 \to \mathbb{Z}/2 \to \mathrm{Mod}_2 \to \mathrm{Mod}_0^6 \to 1,$$

where the kernel is generated by the hyperelliptic involution of S_2. By [14, Proposition 4.3], there is a model X for $\underline{E}\,\mathrm{Mod}_2$ of dimension 3. Therefore, $X^{\mathbb{Z}/2}$ is a model for $\underline{E}\,\mathrm{Mod}_0^6$ of dimension at most 3. We conclude that $\underline{\mathrm{cd}}(\mathrm{Mod}_0^6) = 3$.

Now assume $n = 5$ or $n \geq 7$. From Theorem 3.7 and Theorem 2.2, we obtain the proper cohomological dimension. By the Eilenberg–Ganea theorem, we have that $\underline{\mathrm{gd}}(\mathrm{Mod}_0^n) = \underline{\mathrm{cd}}(\mathrm{Mod}_0^n)$, except possibly when $n = 5$. To rule out that possibility, we consider the following central extension that arises from Birman–Hilden theory:

$$1 \to \mathbb{Z}/2 \to \mathrm{Mod}_1^2 \to \mathrm{Mod}_0^5 \to 1,$$

where the kernel is generated by the hyperelliptic involution of S_1^2. It follows from [21, Lemma 5.8] that $\underline{\mathrm{gd}}(\mathrm{Mod}_0^5) = \underline{\mathrm{gd}}(\mathrm{Mod}_1^2)$. Hence $\underline{\mathrm{gd}}(\mathrm{Mod}_0^5) = 2$ follows from our computation $\underline{\mathrm{gd}}(\mathrm{Mod}_1^2) = \underline{\mathrm{cd}}(\mathrm{Mod}_1^2) = 2$ in the proof of Theorem 4.3. □

4 Genus 1

In this section, we prove Theorem 4.3 which gives the case $g = 1$ of Theorem 1.1. First we consider finite subgroups of Mod_1^n and their actions on S_1^n.

Theorem 4.1. *Let $n \geq 0$, and let F be a finite subgroup of orientation-preserving diffeomorphisms of S_1^n. Then F is isomorphic to $(\mathbb{Z}/s \times \mathbb{Z}/t) \rtimes \mathbb{Z}/m$, where s and t are positive integers such that $st \mid n$ and $m = 1, 2, 3, 4$ or 6. Moreover, the quotient S_1/F only depends on m, and we have the following descriptions of these orbifold quotients:*

- *for $m = 1$, S_1/F is a torus with no elliptic points,*
- *for $m = 2$, S_1/F is a sphere with four elliptic points of orders $(2, 2, 2, 2)$,*
- *for $m = 3$, S_1/F is a sphere with three elliptic points of orders $(3, 3, 3)$,*
- *for $m = 4$, S_1/F is a sphere with three elliptic points of orders $(2, 4, 4)$, and*
- *for $m = 6$, S_1/F is a sphere with three elliptic points of orders $(2, 3, 6)$.*

Proof. Let F be as in the statement; by capping the punctures, the action of F on S_1^n leads to an action of F on the torus S_1. By the uniformization theorem, there is a metric of constant curvature 0 on S_1 such that the action of F is by isometries. Next we can lift this action to the universal covering, that is, there is a group $\tilde{F}$ acting by orientation-preserving euclidean isometries on $\mathbb{R}^2$ that is a central extension of F by a rank 2 group of translations, and such that $\mathbb{R}^2/\tilde{F}$ is diffeomorphic to S_1/F. Hence $\tilde{F}$ is a wallpaper group without reflections or glide reflections, and therefore $\tilde{F}$ is isomorphic to $\mathbb{Z}^2 \rtimes \mathbb{Z}/m$ for $m = 1, 2, 3, 4,$ or 6. This implies that there are positive integers s and t such that

$$F \cong (\mathbb{Z}^2 \rtimes \mathbb{Z}/m)/(s\mathbb{Z} \oplus t\mathbb{Z}) \cong (\mathbb{Z}/s \times \mathbb{Z}/t) \rtimes \mathbb{Z}/m.$$

Furthermore, note that $\mathbb{Z}/s \times \mathbb{Z}/t$ acts freely on S_1 and preserves the set of punctures in S_1^n, which is a set with n elements; as a conclusion, $st \mid n$.

For the *moreover part* of the statement, observe that S_1/F is diffeomorphic to $\mathbb{R}^2/(\mathbb{Z}^2 \rtimes \mathbb{Z}/m)$, and the latter quotients are well known; see for instance [26, Theorem 13.3.6]. □

Proposition 4.2. *Let $n \geq 2$. Then, for every finite subgroup F of Mod_1^n, we have*

$$\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_1^n) = n.$$

Proof. The conclusion is clear when F is the trivial subgroup. Let F be a nontrivial finite subgroup of Mod_1^n; then, by the Nielsen realization theorem, we can realize F as a finite group of orientation-preserving diffeomorphisms of S_1^n. We proceed by cases following the notation and conclusions in Theorem 4.1.

- $F \cong \mathbb{Z}/s \times \mathbb{Z}/t$ with $st \mid n$. In this case, we have that $n_F = n/st$ and S_1^n/F is diffeomorphic to S_1 with n/st punctures. Hence, by Lemma 2.3 (1), we get

$$\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_1^{n_F}) + \lambda(F) \leq \frac{n}{st} + \log_2(n) \leq \frac{n}{2} + \log_2(n).$$

  Now note that $n/2 + \log_2(n) \leq n$ for all $n \geq 0$.

- $F \cong (\mathbb{Z}/s \times \mathbb{Z}/t) \rtimes \mathbb{Z}/2$. In this case, we have that S_1/F is diffeomorphic to a sphere with 4 elliptic points. Hence we conclude that $n_F \leq 4 + n/st \leq 4 + n/2$, and $\mathrm{vcd}(WF) = n_F - 3$. Therefore,

$$\mathrm{vcd}(WF) + \lambda(F) \leq \frac{n}{2} + 1 + \log_2(2n) \leq \frac{n}{2} + 2 + \log_2(n).$$

  Now note that $n/2 + 2 + \log_2(n) \leq n$ for all $n \geq 5$.

- $F \cong (\mathbb{Z}/s \times \mathbb{Z}/t) \rtimes \mathbb{Z}/m$ with $m = 3, 4, 6$. In this case, we have that S_1/F is diffeomorphic to a sphere with 3 elliptic points. Hence we conclude that $n_F \leq 4 + n/st \leq 3 + n/2$, and $\mathrm{vcd}(WF) = n_F - 3$. Therefore,

$$\mathrm{vcd}(WF) + \lambda(F) \leq \frac{n}{2} + \log_2(6n) \leq \frac{n}{2} + 2 + \log_2(n).$$

  Now note that $n/2 + 2 + \log_2(n) \leq n$ for all $n \geq 5$.

To finish the proof, we only have to verify the statement for $n = 2, 3, 4$. From Theorem 4.1, we can describe explicitly the finite subgroups of Mod_1^n and their quotients, and the analysis can be done case by case; the details are left to the reader. □

Theorem 4.3. *Let* $n \geq 0$. *Then*

$$\underline{\mathrm{gd}}(\mathrm{Mod}_1^n)) = \underline{\mathrm{cd}}(\mathrm{Mod}_1^n) = \mathrm{vcd}(\mathrm{Mod}_1^n) = n.$$

Proof. For $n = 1$, the result follows since $\mathrm{Mod}_1^1 \cong \mathrm{SL}(2, \mathbb{Z})$ is virtually free. For $n \geq 2$, Proposition 4.2 and Theorem 2.2 imply $\underline{\mathrm{cd}}(\mathrm{Mod}_1^n) = \mathrm{vcd}(\mathrm{Mod}_1^n)$.

By the Eilenberg–Ganea theorem, we have that $\underline{\mathrm{gd}}(\mathrm{Mod}_1^n) = \underline{\mathrm{cd}}(\mathrm{Mod}_1^n) = n$, except possibly for $n = 2$. Let us rule out this possibility. We have the following short exact sequence:

$$1 \to B_2(S_1)/Z \to \mathrm{Mod}_1^2 \to \mathrm{Mod}_1 \to 1,$$

where $B_2(S_1)$ is the braid group over the torus on 2 strands, and Z is its center. On the other hand, $B_2(S_1)/Z \cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$; see for instance [3, Proposition 4.4 (i)]. Note that $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ is virtually finitely generated free; hence every finite extension G of this group admits a proper action on a tree, which provides a 1-dimensional model for $\underline{E}G$. The group Mod_1 also admits a 1-dimensional model for $\underline{E}\,\mathrm{Mod}_1$; hence, by [21, Theorem 5.16], there is a 2-dimensional model for $\underline{E}\,\mathrm{Mod}_1^2$. This establishes that $\underline{\mathrm{gd}}(\mathrm{Mod}_1^2) = \underline{\mathrm{cd}}(\mathrm{Mod}_1^2) = 2$. □

5 Genus 2

In order to compute the proper geometric dimension of Mod_2^n, with $n \geq 1$, we use Broughton's complete classification, up to topological equivalence, of finite group actions on a genus 2 surface [6, Theorem 4.1 & Table 4]; see Appendix B. Note that there are only finitely many conjugacy classes of finite groups that act on S_2 by homeomorphism. Hence, by Nielsen's realization theorem, given $n \geq 0$, any finite subgroup F of Mod_2^n can be realized, up to conjugation, by one of these finite possibilities. This makes the genus 2 case different in nature to the cases of genus 0 and 1 where the isomorphism types of finite subgroups of the corresponding mapping class groups depend strongly on n.

Theorem 5.1. *Let* $n \geq 1$. *Then, for every nontrivial finite subgroup* F *of* Mod_2^n, *we have*

$$\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_2^n) = n + 4.$$

In particular, $\underline{\mathrm{cd}}(\mathrm{Mod}_2^n) = \underline{\mathrm{gd}}(\mathrm{Mod}_2^n) = \mathrm{vcd}(\mathrm{Mod}_2^n) = n + 4.$

Proof. Since $\underline{\mathrm{cd}}(\mathrm{Mod}_2^n) \geq \mathrm{vcd}(\mathrm{Mod}_2^n) \geq 5$, we have $\underline{\mathrm{cd}}(\mathrm{Mod}_2^n) = \underline{\mathrm{gd}}(\mathrm{Mod}_2^n)$ for every $n \geq 1$.

Let F be a nontrivial finite subgroup of Mod_2^n. By Nielsen realization, F acts on S_2^n by orientation-preserving homeomorphisms. Recall from Section 2.2 that n_F is bounded above by $n/|F| + o_F$.

By Broughton's classification, the quotient S_2^n/F can only have genus $g_F = 0$ or $g_F = 1$. When $g_F = 0$, we have that $\mathrm{vcd}(WF) = \mathrm{vcd}(\mathrm{Mod}_0^{n_F}) = n_F - 3$, and when $g_F = 1$, then $\mathrm{vcd}(WF) = \mathrm{vcd}(\mathrm{Mod}_1^{n_F}) = n_F$. In Table 11, we recall the classification from [6, Table 4], and we give explicit upper bounds for $\lambda(F)$, n_F, and $\mathrm{vcd}(WF)$. From this, we can see that the inequality

$$\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_2^n) = n + 4$$

holds for almost all finite subgroups F of Mod_2^n and $n \geq 1$. The only exception is the case of $F = \mathrm{GL}_2(4)$ and $n = 1$, which does not occur since $F = \mathrm{GL}_2(4)$ cannot be realized as subgroup of Mod_2^1, indeed since the only puncture will be a fixed point of the action, but as we can read in the signature of this action in Table 11, there are no fixed points. □

Remark 2. From Table 11, we can see that there are several finite subgroups F of Mod_2 such that

$$\mathrm{vcd}(WF) + \lambda(F) \not\leq \mathrm{vcd}(\mathrm{Mod}_2) = 3.$$

For instance, $\mathrm{vcd}(WF) + \lambda(F) = 4$ when $F = D_{2(6)}$. Hence we cannot use this strategy to obtain $\underline{\mathrm{gd}}(\mathrm{Mod}_2)$.

6 Genus at least 3

In this section, we promote the results in [2] for Mod_g^0 with $g \geq 3$ to Mod_g^n for $g \geq 3$ and $n \geq 1$, using a simple argument.

Proposition 6.1 ([2, Proposition 4.4]). *For any $g \geq 3$ and any finite subgroup F of Mod_g^0, we have $\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_g^0)$.*

Theorem 6.2. *For any $g \geq 3$, $n \geq 1$, and any finite subgroup F of Mod_g^n, we have*

$$\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_g^n).$$

In particular, $\underline{\mathrm{gd}}(\mathrm{Mod}_g^n) = \underline{\mathrm{cd}}(\mathrm{Mod}_g^n) = \mathrm{vcd}(\mathrm{Mod}_g^n)$.

Proof. Let us consider the Birman short exact sequence (see [4, Theorem 4.3])

$$1 \to B_n(S_g) \to \mathrm{Mod}_g^n \xrightarrow{\varphi} \mathrm{Mod}_g^0 \to 1,$$

where $B_n(S_g)$ is the full braid group over S_g on n strings.

Let F be any finite subgroup of Mod_g^n. Then, by restriction of the above sequence, we obtain the following short exact sequence:

$$1 \to B_n(S_g) \cap NF \to NF \xrightarrow{\varphi} \varphi(NF) \to 1$$

and note that $\varphi(NF) \leq N\varphi(F)$. Therefore,

$$\begin{aligned}
&\mathrm{vcd}(NF) + \lambda(F) \\
&\quad \leq \mathrm{vcd}(\varphi(NF)) + \mathrm{vcd}(B_n(S_g) \cap NF) + \lambda(F) && \text{(by subadditivity of vcd)} \\
&\quad \leq \mathrm{vcd}(N(\varphi(F))) + \mathrm{vcd}(B_n(S_g)) + \lambda(\varphi(F)) && \text{(by monotonicity of vcd)} \\
&\quad \leq \mathrm{vcd}(\mathrm{Mod}_g^0) + \mathrm{vcd}(B_n(S_g)) && \text{(by Proposition 6.1)} \\
&\quad \leq 4g - 5 + n + 1 = 4g + n - 4 = \mathrm{vcd}(\mathrm{Mod}_g^n) && \text{(by [19, Theorem 1.2]).}
\end{aligned}$$

Now the result follows. The *in particular* part follows directly from Theorem 2.2.

□

A $\lambda(F)$ and $\mathrm{vcd}(WF)$ for finite subgroups of Mod_0^n for $5 \leq n \leq 13$

In this appendix, we describe n_F, $\mathrm{vcd}(WF)$, and $\lambda(F)$ for all the conjugacy classes of finite subgroups F of Mod_0^n when $5 \leq n \leq 13$.

Polyhedral subgroups of Mod_0^n

We keep the numbering from Theorem 3.1 to analyze the polyhedral subgroups of Mod_0^n for $n = 6, 8, 10, 12$.

(3) For $n = 10$, up to conjugacy, there is a maximal finite subgroup $F \cong A_4$ of Mod_0^n, which can be realized as the symmetry group of a tetrahedron. There are three orbits of points with nontrivial stabilizer in F: the centers of faces, the centers of edges, and the vertices with orbits of length 4, 6, and 4, respectively. Note that, for $n = 10$, there must be two orbits of punctures (one of length 4 and another of length 6) and one orbit of elliptic points; hence $n_F = 3$.

(4) For $n = 6$, 8, or 12, up to conjugacy, there is a maximal finite subgroup $F \cong \mathfrak{S}_4$ of Mod_0^n, which can be realized as the symmetry group of a cube (octahedron). There are three orbits of points with nontrivial stabilizer in F: the centers of faces, the centers of edges, and the vertices with stabilizers of order 4, 2, and 3, respectively. Note that, for $n = 6$, we must place the punctures in the centers of the faces (unique orbit of length 6), for $n = 8$, we must place the punctures in the vertices of the cube (unique orbit of length 8), and for $n = 12$, the punctures can only be placed in the centers of the edges (unique orbit of length 12). Hence, in S_0^n/F, there is only one orbit of punctures and two orbits of elliptic points and $n_F = 3$.

n	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
6	$\mathfrak{S}_4$	3	0	4
	A_4	3	0	3
8	$\mathfrak{S}_4$	3	0	4
	A_4	3	0	3
10	A_4	3	0	3
12	$\mathfrak{S}_4$	3	0	4
	A_4	4	1	3
	A_5	3	0	4

Table 1. Polyhedral subgroups of Mod_0^n for $n = 6, 8, 10, 12$.

Now let $F \cong A_4 \leq \mathfrak{S}_4$. It can be realized as subgroup of the symmetry group of a tetrahedron embedded into a cube. For $n = 6$, there are two orbits of vertices, one orbit of centers of faces (where we have placed the punctures), and the centers of edges are no longer elliptic points; hence $n_F = 3$. For $n = 8$, there are two orbits of vertices (where we have placed the punctures), one orbit of centers of faces, and the centers of edges are no longer elliptic points; hence $n_F = 3$. Finally, for $n = 10$, there are two orbits of vertices, one orbit of centers of faces, and one orbit of the centers of edges (where we have placed the punctures); hence $n_F = 4$.

(5) For $n = 12$, up to conjugacy, there is a maximal finite subgroup $F \cong A_5$ of Mod_0^n, which can be realized as the symmetry group of a dodecahedron (icosahedron). There are three orbits of points with nontrivial stabilizer in F: the centers of faces, the centers of edges, and the vertices. For $n = 12$, we must place the punctures in the centers of the faces (unique orbit of length 12). Hence, in S_0^n/F, there is only one orbit of punctures and two orbits of elliptic points and $n_F = 3$.

We summarize the conclusions in Table 1. Note that

$$\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_0^n) = n - 3 \quad \text{for } n = 8, 10, 12.$$

However, $\mathrm{vcd}(W\mathfrak{S}_4) + \lambda(\mathfrak{S}_4) = 4 \not\leq \mathrm{vcd}(\mathrm{Mod}_0^6) = 3$.

Cyclic and dihedral subgroups of Mod_0^n

The following tables for $5 \leq n \leq 13$ come directly from Proposition 3.3. Note that, in all these cases,

$$\mathrm{vcd}(WF) + \lambda(F) \leq \mathrm{vcd}(\mathrm{Mod}_0^n) = n - 3.$$

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/4$	3	0	2
	$\mathbb{Z}/2$	4	1	1
(2.1)	$\mathbb{Z}/5$	3	0	1
(2.2)	$D_{2(5)}$	3	0	2
(3.1)	$\mathbb{Z}/3$	3	0	1
(3.2)	$D_{2(3)}$	3	0	2

Table 2. Cyclic and dihedral subgroups of Mod_0^5.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/5$	3	0	1
(2.1)	$\mathbb{Z}/6$	3	0	2
	$\mathbb{Z}/3$	4	1	1
	$\mathbb{Z}/2$	5	2	1
(2.2)	$D_{2(6)}$	3	0	3
	$D_{2(3)}$	3	0	2
		4	1	2
	$D_{2(2)}$	4	1	2
(3.1)	$\mathbb{Z}/4$	3	0	2
	$\mathbb{Z}/2$	4	1	1
(3.2)	$D_{2(4)}$	3	0	3
	$D_{2(2)}$	3	0	2
		4	1	2

Table 3. Cyclic and dihedral subgroups of Mod_0^6.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/6$	3	0	2
	$\mathbb{Z}/3$	4	1	1
	$\mathbb{Z}/2$	5	2	1
(2.1)	$\mathbb{Z}/7$	3	0	1
(2.2)	$D_{2(7)}$	3	0	2

Table 4. Cyclic and dihedral subgroups of Mod_0^7.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(3.1)	$\mathbb{Z}/5$	3	0	1
(3.2)	$D_{2(5)}$	3	0	2

Table 4 (continued)

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/7$	3	0	1
(2.1)	$\mathbb{Z}/8$	3	0	3
	$\mathbb{Z}/4$	4	1	2
	$\mathbb{Z}/2$	6	3	1
(2.2)	$D_{2(8)}$	3	0	4
	$D_{2(4)}$	3	0	3
		4	1	3
	$D_{2(2)}$	4	1	2
		5	2	2
(3.1)	$\mathbb{Z}/6$	3	0	2
	$\mathbb{Z}/3$	4	1	1
	$\mathbb{Z}/2$	5	2	1
(3.2)	$D_{2(6)}$	3	0	3
	$D_{2(3)}$	3	0	2
		4	1	2
	$D_{2(2)}$	4	1	2

Table 5. Cyclic and dihedral subgroups of Mod_0^8.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/8$	3	0	3
	$\mathbb{Z}/4$	4	1	2
	$\mathbb{Z}/2$	6	3	1
(2.1)	$\mathbb{Z}/9$	3	0	2
	$\mathbb{Z}/3$	5	2	1
(2.2)	$D_{2(9)}$	3	0	3
	$D_{2(3)}$	4	1	2
(3.1)	$\mathbb{Z}/7$	3	0	1
(3.2)	$D_{2(7)}$	3	0	2

Table 6. Cyclic and dihedral subgroups of Mod_0^9.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/9$	3	0	2
	$\mathbb{Z}/3$	5	2	1
(2.1)	$\mathbb{Z}/10$	3	0	2
	$\mathbb{Z}/5$	4	1	1
	$\mathbb{Z}/2$	7	4	1
(2.2)	$D_{2(10)}$	3	0	3
	$D_{2(5)}$	3	0	2
		4	1	2
	$D_{2(2)}$	5	2	2
(3.1)	$\mathbb{Z}/8$	3	0	3
	$\mathbb{Z}/4$	4	1	2
	$\mathbb{Z}/2$	6	3	1
(3.2)	$D_{2(8)}$	3	0	4
	$D_{2(4)}$	3	0	3
		4	1	3
	$D_{2(2)}$	4	0	2
		5	2	2

Table 7. Cyclic and dihedral subgroups of Mod_0^{10}.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/10$	3	0	2
	$\mathbb{Z}/5$	4	1	1
	$\mathbb{Z}/2$	7	4	1
(2.1)	$\mathbb{Z}/11$	3	0	1
(2.2)	$D_{2(11)}$	3	0	2
(3.1)	$\mathbb{Z}/9$	3	0	2
	$\mathbb{Z}/3$	5	2	1
(3.2)	$D_{2(9)}$	3	0	2
	$D_{2(3)}$	4	1	1

Table 8. Cyclic and dihedral subgroups of Mod_0^{11}.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/11$	3	0	1
(2.1)	$\mathbb{Z}/12$	3	0	3
	$\mathbb{Z}/6$	4	1	2
	$\mathbb{Z}/4$	5	2	2
	$\mathbb{Z}/3$	6	3	1
	$\mathbb{Z}/2$	8	5	1
(2.2)	$D_{2(12)}$	3	0	4
	$D_{2(6)}$	3	0	3
		4	1	3
	$D_{2(4)}$	4	1	3
	$D_{2(3)}$	4	1	2
		5	2	2
	$D_{2(2)}$	5	2	2
		6	3	2
(3.1)	$\mathbb{Z}/10$	3	0	2
	$\mathbb{Z}/5$	4	1	1
	$\mathbb{Z}/2$	7	4	1
(3.2)	$D_{2(10)}$	3	0	3
	$D_{2(5)}$	3	0	2
		4	1	2
	$D_{2(2)}$	5	2	2

Table 9. Cyclic and dihedral subgroups of Mod_0^{12}.

Proposition 3.3 type	F	n_F	$\mathrm{vcd}(WF)$	$\lambda(F)$
(1)	$\mathbb{Z}/12$	3	0	3
	$\mathbb{Z}/6$	3	0	3
	$\mathbb{Z}/4$	5	2	2
	$\mathbb{Z}/2$	8	5	1
(2.1)	$\mathbb{Z}/13$	3	0	1
(2.2)	$D_{2(13)}$	3	0	2
(3.1)	$\mathbb{Z}/11$	3	0	1
(3.2)	$D_{2(11)}$	3	0	2

Table 10. Cyclic and dihedral subgroups of Mod_0^{13}.

B $\lambda(F)$ and $\mathrm{vcd}(WF)$ for finite subgroups of Mod_2^n

In this appendix, we use Broughton's classification [6, Theorem 4.1 & Table 4] to give upper bounds for n_F, $\mathrm{vcd}(WF)$, and $\lambda(F)$ for all the conjugacy classes of nontrivial finite subgroups F of Mod_2^n when $n \geq 0$.

F	$\lvert F\rvert$	S_2/F	$n_F \leq$	$\mathrm{vcd}(WF) \leq$	$\lambda(F) \leq$
$\mathbb{Z}/2$	2	$(S_0; 2, 2, 2, 2, 2, 2)$	$\frac{n}{2} + 6$	$\frac{n}{2} + 3$	1
$\mathbb{Z}/2$	2	$(S_1; 2, 2)$	$\frac{n}{2} + 2$	$\frac{n}{2} + 2$	1
$\mathbb{Z}/3$	3	$(S_0; 3, 3, 3, 3)$	$\frac{n}{3} + 4$	$\frac{n}{3} + 1$	1
$\mathbb{Z}/2 \times \mathbb{Z}/2$	4	$(S_0; 2, 2, 2, 2, 2)$	$\frac{n}{4} + 5$	$\frac{n}{4} + 2$	2
$\mathbb{Z}/4$	4	$(S_0; 2, 2, 4, 4)$	$\frac{n}{4} + 4$	$\frac{n}{4} + 1$	2
$\mathbb{Z}/5$	5	$(S_0; 5, 5, 5)$	$\frac{n}{5} + 3$	$\frac{n}{5}$	1
$\mathbb{Z}/6$	6	$(S_0; 3, 6, 6)$	$\frac{n}{6} + 3$	$\frac{n}{6}$	2
$\mathbb{Z}/6$	6	$(S_0; 2, 2, 3, 3)$	$\frac{n}{6} + 4$	$\frac{n}{6}+1$	2
$D_{2(3)}$	6	$(S_0; 2, 2, 3, 3)$	$\frac{n}{6} + 4$	$\frac{n}{6}+1$	2
$\mathbb{Z}/8$	8	$(S_0; 2, 8, 8)$	$\frac{n}{8} + 3$	$\frac{n}{8}$	3
$\widetilde{D_2}$	8	$(S_0; 4, 4, 4)$	$\frac{n}{8} + 3$	$\frac{n}{8}$	2
$D_{2(4)}$	8	$(S_0; 2, 2, 2, 4)$	$\frac{n}{8} + 4$	$\frac{n}{8}+1$	3
$\mathbb{Z}/10$	10	$(S_0; 2, 5, 10)$	$\frac{n}{10} + 3$	$\frac{n}{10}$	2
$\mathbb{Z}/2 \times \mathbb{Z}/6$	12	$(S_0; 2, 6, 6)$	$\frac{n}{12} + 3$	$\frac{n}{12}$	3
$D_{4,3,-1}$	12	$(S_0; 3, 4, 4)$	$\frac{n}{12} + 3$	$\frac{n}{12}$	3
$D_{2(6)}$	12	$(S_0; 2, 2, 2, 3)$	$\frac{n}{12} + 4$	$\frac{n}{12} + 1$	3
$D_{2,8,3}$	16	$(S_0; 2, 4, 8)$	$\frac{n}{16} + 3$	$\frac{n}{16}$	4
$\mathbb{Z}/2 \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/3)$	24	$(S_0; 2, 4, 6)$	$\frac{n}{24} + 3$	$\frac{n}{24}$	4
$\mathrm{SL}_2(3)$	24	$(S_0; 3, 3, 4)$	$\frac{n}{24} + 3$	$\frac{n}{24}$	4
$\mathrm{GL}_2(4)$	48	$(S_0; 2, 3, 8)$	$\frac{n}{48} + 3$	$\frac{n}{48}$	5

Table 11. Finite nontrivial subgroups of Mod_2^n.

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