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Strong indecomposability of the outer automorphism groups of nonabelian free profinite groups

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Abstract. In this paper, we prove that the outer automorphism groups of nonabelian topologically finitely generated free profinite groups are strongly indecomposable. This means that no open subgroup has a nontrivial direct product decomposition.

1 Introduction

One of the most fundamental objects in the category of infinite nonabelian profinite groups is the free profinite group F_m of rank $m \geq 2$. With regard to the noncommutativity of F_m, the following fact [1, Proposition 8] is well known.

Fact 1. *The center of F_m is trivial.*

Here, we note that the "discrete analogue" of Fact 1 (i.e., the center-freeness of free discrete groups of rank m) is relatively easy to prove using explicit descriptions of elements as "words". However, since elements of F_m cannot be described in terms of "words" (of finite length) in a useful way, one cannot simply modify the proof used in the discrete setting to prove Fact 1. In fact, in the proof of [1, Proposition 8], cohomology theory of free pro-p groups is applied. This situation suggests that, in general, the study of free profinite groups is much more difficult than the study of free discrete groups.

In this paper, we study the group structure of the outer (continuous) automorphism group $\mathrm{Out}(F_m)$, which is much more complicated than F_m. We recall that

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$\mathrm{Out}(F_m)$ admits a natural structure of profinite group; see [14, Corollary 4.4.4]. Here, we note that, although F_m is topologically finitely generated, $\mathrm{Out}(F_m)$ is not topologically finitely generated; see [14, Theorem 4.4.9]. This fact lies in stark contrast to the fact that the outer automorphism group of a free discrete group of rank m is finitely generated; see [7, §3.5, Theorem N1]. One of the most interesting aspects of our work is that, in our study of $\mathrm{Out}(F_m)$, we apply results in anabelian geometry for hyperbolic curves over number fields or finite fields. In fact, by applying a result in anabelian geometry for hyperbolic curves over finitely generated fields of characteristic 0, the following fact (which concerns the noncommutativity of $\mathrm{Out}(F_m)$) has been proved by Tamagawa [17, Theorem 7.4].

Fact 2. *The center of* $\mathrm{Out}(F_m)$ *is trivial.*

In the following, let $\overline{\mathbb{Q}}$ be an algebraic closure of the field of rational numbers $\mathbb{Q}$, $K \subseteq \overline{\mathbb{Q}}$ a number field, and Z a hyperbolic curve of genus 0 over K. Write $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, $Z_{\overline{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} Z \times_K \overline{\mathbb{Q}}$, and $\Pi_{Z_{\overline{\mathbb{Q}}}}$ for the étale fundamental group of $Z_{\overline{\mathbb{Q}}}$ relative to a suitable choice of basepoint. We recall that $\Pi_{Z_{\overline{\mathbb{Q}}}}$ is a nonabelian free profinite group; see [16, Corollary 4.6.11]. Write $\mathbf{P}$ for the set of prime numbers. For a profinite group G and a nonempty subset $\Sigma \subseteq \mathbf{P}$, write G^{Σ} for the maximal pro-Σ quotient of G. For every nonempty subset $\Sigma \subseteq \mathbf{P}$, write

$$\rho_Z^{\Sigma}\colon G_K \to \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^{\Sigma})$$

for the natural pro-Σ outer Galois representation; see Definition 4.1. We are now ready to state our main results, also concerning the noncommutativity of $\mathrm{Out}(F_m)$.

Theorem 1.1. *Let* $\Sigma \subseteq \mathbf{P}$ *be a subset such that either* $|\Sigma| = 1$ *or* $|\mathbf{P} \setminus \Sigma|$ *is finite. Let* $G \subseteq \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^{\Sigma})$ *be a closed subgroup such that*

- *G contains an open subgroup of $\rho_Z^{\Sigma}(G_K)$,*
- *there exists a prime number $l \in \Sigma$ such that the image of G via the natural morphism*

$$\phi_l\colon \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^{\Sigma}) \twoheadrightarrow \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^{\{l\}}),$$

  which is necessarily surjective by [14, Proposition 4.5.4 (b)], *is slim, i.e., the center of every open subgroup of $\phi_l(G)$ is trivial.*

Then G is strongly indecomposable, i.e., no open subgroup of G has a nontrivial direct product decomposition.

We recall that, for every integer $m \geq 2$ and every nonempty subset $\Sigma \subseteq \mathbf{P}$, F_m^{Σ} is a free pro-Σ group of rank m.

Corollary 1.2. *Let $m \geq 2$ be an integer and $\Sigma \subseteq \mathbf{P}$ a subset such that either $|\Sigma| = 1$ or $|\mathbf{P} \setminus \Sigma|$ is finite. Then $\mathrm{Aut}(F_m^\Sigma)$ and $\mathrm{Out}(F_m^\Sigma)$ are strongly indecomposable.*

We recall that, for an arbitrary nonempty subset $\Sigma \subseteq \mathbf{P}$, the group F_m^Σ itself is strongly indecomposable; see [14, Theorems 3.6.2, 8.6.5]. It is not clear to the authors at the time of writing whether or not the assumption that $|\Sigma| = 1$ or $|\mathbf{P} \setminus \Sigma|$ is finite can be dropped; compare with Remark 4.5.

Finally, we add the following remark. In light of a well known injectivity result of Belyi [5, Theorem C] and Remark 5.2 below, Theorem 1.1 has the following immediate applications, arising from the special case where $\Sigma = \mathbf{P}$ and $Z = \mathbb{P}^1_K \setminus \{0, 1, \infty\}$, the projective line minus three points.

(i) The profinite Grothendieck–Teichmüller group $\widehat{\mathrm{GT}}$, introduced in [3] and considered in [11, Remark 1.11.1], is strongly indecomposable.

(ii) The absolute Galois groups of number fields are strongly indecomposable.

Assertion (i) gives an affirmative answer to an open problem posed by the first author in [8]. Assertion (ii) is a special case of the fact that the absolute Galois groups of Hilbertian fields are strongly indecomposable – which follows immediately from a theorem of Haran–Jarden [4, Corollary 2.5].

2 Notation and conventions

Rings and schemes

Let A be a commutative (unitary) ring. Then we shall write $\mathrm{char}(A)$ for the characteristic of A. We shall write $A^\times$ for the group of units of A. Let F be a perfect field and $\overline{F}$ an algebraic closure of F. Then we shall write $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$.

Let S be a scheme. Then we shall write $\mathrm{Aut}(S)$ for the group of automorphisms of S. Let K be a field, $K \subseteq L$ a field extension, and S a scheme over K. Then we shall write $S_L \stackrel{\text{def}}{=} S \times_K L$. We shall write $\mathrm{Aut}_K(S)$ for the group of automorphisms of S over K and $\mathbb{P}^1_K$ for the projective line over K.

Let S be a connected locally Noetherian scheme. Then we shall write Π_S for the étale fundamental group of S relative to a suitable choice of basepoint. Note that, for any perfect field F, $\Pi_{\mathrm{Spec}(F)} \cong G_F$.

Profinite groups

Write $\mathbf{P}$ for the set of prime numbers. Let $\Sigma \subseteq \mathbf{P}$ be a nonempty subset. Then we shall say that an integer $n \in \mathbb{Z}_{\geq 1}$ is a Σ-integer if every prime factor of n is contained in Σ. Let G a profinite group. Then we shall write G^Σ for the maximal pro-Σ quotient of G. We shall write G^{ab} for the abelianization of G (i.e., the quotient of G by the closure of the commutator subgroup of G), $\mathrm{Aut}(G)$ for the group of automorphisms of G (in the category of profinite groups), $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. If p is a prime number, then we shall also write $G^p \stackrel{\mathrm{def}}{=} G^{\{p\}}$ and $G^{(p)'} \stackrel{\mathrm{def}}{=} G^{\mathbf{P}\setminus\{p\}}$. If D is a discrete group, then we shall write D^p for the pro-p completion of D.

Suppose that G is topologically finitely generated. Then G admits a basis of characteristic open subgroups; see [14, Proposition 2.5.1 (b)]. Such a basis induces a profinite topology on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$. Let $H \subseteq G$ be a closed subgroup. Then we shall write $\mathrm{Aut}^H(G) \subseteq \mathrm{Aut}(G)$ for the subgroup of $\mathrm{Aut}(G)$ consisting of elements that induce automorphisms of H. We shall write $\mathrm{Inn}^H(G) \subseteq \mathrm{Aut}^H(G)$ for the image of H via the natural surjection $G \twoheadrightarrow \mathrm{Inn}(G)$. Let J be a profinite group. Then we shall refer to a continuous homomorphism $J \to \mathrm{Aut}^H(G)/\mathrm{Inn}^H(G)$ as an H-outer action of J on G.

3 Preliminaries

In this section, we recall some basic notions concerning profinite groups and hyperbolic curves, and we establish Lemmas 3.4 and 3.8 which will be applied in Section 5.

First, from [12, Notations and Conventions] and [12, Definition 3.1], we recall basic notions concerning profinite groups.

Definition 3.1. Let G be a profinite group and $H \subseteq G$ a closed subgroup of G.

(i) We shall write $Z_G(H)$ for the *centralizer* of H in G, i.e., the closed subgroup $\{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$, and $Z(G) \stackrel{\mathrm{def}}{=} Z_G(G)$.

(ii) We shall say that G is *slim* if $Z_G(U) = \{1\}$ for every open subgroup U of G, or, equivalently, $Z(U) = \{1\}$ for every open subgroup U of G.

(iii) We shall say that G is *decomposable* if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $G = H_1 \times H_2$. We shall say that G is *indecomposable* if G is not decomposable. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

Remark 3.2. Let G be a slim profinite group. From [12, §0] and [8, Lemma 1.6], we recall the following well known facts.

(i) Every finite normal closed subgroup of G is trivial.

(ii) Let $H \subseteq G$ be an open subgroup and $\alpha \in \mathrm{Aut}(G)$. Suppose that α induces the identity automorphism on H. Then α is the identity automorphism on G.

For completeness, we prove a basic lemma which will be applied in Section 5.

Lemma 3.3. *Let G be a profinite group and $\{G_i\}_{i\in I}$ a directed subset of the set of characteristic open subgroups of G, where $j \geq i \Leftrightarrow G_j \subseteq G_i$, such that*

$$\bigcap_{i\in I} G_i = \{1\}.$$

Write $\phi_i : \mathrm{Out}(G) \to \mathrm{Out}(G/G_i)$ for the natural homomorphism. Then

$$\bigcap_{i\in I} \mathrm{Ker}(\phi_i) = \{1\}.$$

Proof. We consider $\sigma \in \bigcap_{i\in I} \mathrm{Ker}(\phi_i) \subseteq \mathrm{Out}(G)$ and a lifting $\tilde{\sigma} \in \mathrm{Aut}(G)$. For each $i \in I$, write $\tilde{\sigma}_i \in \mathrm{Aut}(G/G_i)$ for the automorphism induced by $\tilde{\sigma}$. Then, since $\sigma \in \mathrm{Ker}(\phi_i)$, the automorphism $\tilde{\sigma}_i$ is inner. Let $\gamma_i \in G/G_i$ be an element which induces $\tilde{\sigma}_i$ via conjugation. Write $C_i \stackrel{\mathrm{def}}{=} \gamma_i \cdot Z(G/G_i) \subseteq G/G_i$. Here, we note that if $i_1 \geq i_2$ ($i_1, i_2 \in I$), then the natural surjection $G/G_{i_1} \twoheadrightarrow G/G_{i_2}$ induces a map $C_{i_1} \to C_{i_2}$. Observe that, since C_i ($i \in I$) is a finite nonempty set, the inverse limit $\varprojlim_{i\in I} C_i$ is nonempty; see [14, Corollary 1.1.6]. Then $\tilde{\sigma}$ is the inner automorphism given by any element $\gamma \in \varprojlim_{i\in I} C_i$ ($\subseteq \varprojlim_{i\in I} G/G_i = G$). This completes the proof of Lemma 3.3. □

Lemma 3.4. *Let G be a topologically finitely generated profinite group and $S \subseteq \mathbf{P}$ a finite subset. Then the natural homomorphism*

$$\mathrm{Out}(G) \to \prod_{p\in \mathbf{P}\setminus S} \mathrm{Out}(G^{(p)'})$$

is injective.

Proof. Since G is topologically finitely generated, there exists a directed subset $\{G_i\}_{i\in I}$ of the set of characteristic open subgroups of G, where $j \geq i \Leftrightarrow G_j \subseteq G_i$, such that $\bigcap_{i\in I} G_i = \{1\}$; see [14, Proposition 2.5.1 (b)]. Fix such a family. For each $i \in I$, let $p_i \in \mathbf{P} \setminus S$ be such that p_i does not divide the order of the finite group G/G_i. Then the natural surjection $G \twoheadrightarrow G/G_i$ factors through the natural surjection $G \twoheadrightarrow G^{(p_i)'}$. Thus, Lemma 3.4 follows from Lemma 3.3. □

Next, we recall basic notions concerning hyperbolic curves.

Definition 3.5. Let k be a field, $\overline{k}$ an algebraic closure of k, and X a smooth curve (i.e., a one-dimensional, smooth, separated, finite-type, geometrically connected scheme) over k. Write $\overline{X}_{\overline{k}}$ for the smooth compactification of $X_{\overline{k}}$ over $\overline{k}$. Then we shall refer to any element of the underlying set of $\overline{X}_{\overline{k}} \setminus X_{\overline{k}}$ as a *cusp* of $X_{\overline{k}}$. We shall say that X is a *smooth curve of type* (g, r) over k if the genus of $X_{\overline{k}}$ is g, and the cardinality of the set of cusps of $X_{\overline{k}}$ is r. If X is a smooth curve of type (g, r) over k, and $2g - 2 + r > 0$, then we shall say that X is a *hyperbolic curve* over k.

Definition 3.6. Let k be an algebraically closed field, Z a hyperbolic curve over k, Q a profinite group, and $q\colon \Pi_Z \twoheadrightarrow Q$ an epimorphism (in the category of profinite groups).

(i) Write $\overline{Z}$ for the smooth compactification of Z over k, K for the function field of $\overline{Z}$, $\tilde{K}$ for the maximal Galois extension of K that is unramified over Z within a fixed separable closure K^{sep}, $\tilde{\overline{Z}}$ for the normalization of $\overline{Z}$ in $\tilde{K}$, and $\tilde{S}$ for the inverse image of $S \overset{\mathrm{def}}{=} \overline{Z} \setminus Z$ in $\tilde{\overline{Z}}$. Note that we have a natural action of $\Pi_Z \cong \mathrm{Gal}(\tilde{K}/K)$ on $\tilde{\overline{Z}}$. Then, for each point $z \in S$, we shall refer to the stabilizer subgroup $\subseteq \Pi_Z$ of any point in $\tilde{S}$ lying over z as a *cuspidal inertia subgroup of* Π_Z *associated to* z. We shall refer to the image under q of any such cuspidal inertia subgroup of Π_Z as a *cuspidal inertia subgroup of* Q *associated to* z.

(ii) In the set-up of (i), we shall write

$$\mathrm{Out}^{\mathrm{C}}(Q) \subseteq \mathrm{Out}(Q) \quad (\text{respectively, } \mathrm{Out}^{|\mathrm{C}|}(Q) \subseteq \mathrm{Out}(Q))$$

for the subgroup of outer automorphisms of Q that induce permutations (respectively, the identity permutation) on the set of the conjugacy classes of cuspidal inertia subgroups of Q.

We conclude with a lemma concerning outer actions on certain quotients of the group Π_Z. In its proof, we employ methods that are similar to those used in the proof of [18, Lemma 1.2].

Definition 3.7. Let G be a profinite group, Π a topologically finitely generated profinite group such that $Z(\Pi) = \{1\}$, and $G \to \mathrm{Out}(\Pi)$ a continuous homomorphism. Then we shall write

$$\Pi \overset{\mathrm{out}}{\rtimes} G \overset{\mathrm{def}}{=} \mathrm{Aut}(\Pi) \times_{\mathrm{Out}(\Pi)} G.$$

Lemma 3.8. *Let k be an algebraically closed field, Z a hyperbolic curve over k, Q a profinite group, and $q\colon \Pi_Z \twoheadrightarrow Q$ an epimorphism. Suppose that Q is topologically finitely generated and slim. Let $J \subseteq \mathrm{Out}^{\mathrm{C}}(Q)$ be a closed subgroup and $V \subseteq Q$ an open subgroup. In particular, $q^{-1}(V) \subseteq \Pi_Z$ may be naturally identified with the étale fundamental group of a hyperbolic curve over k. Write*

$$\phi_V\colon \mathrm{Aut}(V) \twoheadrightarrow \mathrm{Out}(V), \quad \phi_Q\colon \mathrm{Aut}(Q) \twoheadrightarrow \mathrm{Out}(Q)$$

for the natural surjections. Then, for any sufficiently small open subgroup $M \subseteq J$*, there exist an outer action of* M *on* V *and an open injection* $V \overset{\text{out}}{\rtimes} M \hookrightarrow Q \overset{\text{out}}{\rtimes} J$ *such that*

(a) *the outer action of* M *preserves and induces permutations on the set of the conjugacy classes of cuspidal inertia subgroups of* V*,*

(b) *the outer action of* M *on* V *extends uniquely (by the slimness of* Q*) to a* V*-outer action on* Q *that is compatible with the outer action of* J $(\supseteq M)$ *on* Q*,*

(c) *the injection* $V \overset{\text{out}}{\rtimes} M \hookrightarrow Q \overset{\text{out}}{\rtimes} J$ *is the injection determined by the inclusions* $V \subseteq Q$ *and* $M \subseteq J$*, together with the* V*-outer actions on* V *and* Q*.*

Proof. Write $n \in \mathbb{Z}_{\geq 1}$ for the index of V in Q. Write

$$\mathrm{Aut}^{[V]}(Q) \subseteq \mathrm{Aut}(Q)$$

for the subgroup of $\mathrm{Aut}(Q)$ consisting of elements that induce automorphisms of V, and moreover, induce permutations on the set of the conjugacy classes of cuspidal inertia subgroups of V. First, we note that, since V is an open subgroup of Q, $\mathrm{Inn}^V(Q)$ is an open subgroup of $\mathrm{Inn}(Q)$. In particular, there exists an open subgroup $M_1 \subseteq \mathrm{Aut}(Q)$ such that $M_1 \cap \mathrm{Inn}(Q) = \mathrm{Inn}^V(Q)$. Next, we consider the set

$$\mathcal{Q}_n \overset{\text{def}}{=} \{\text{all open subgroups of } Q \text{ of index } n\}.$$

Here, observe that, since Q is topologically finitely generated, $\mathcal{Q}_n$ is a finite set; see [14, Proposition 2.5.1 (a)]. Thus, if we write $\mathrm{Sym}(\mathcal{Q}_n)$ for the group of permutations on Q_n, then the kernel of the natural homomorphism $\mathrm{Aut}(Q) \to \mathrm{Sym}(\mathcal{Q}_n)$ is an open subgroup of $\mathrm{Aut}(Q)$. Write

$$M_2 \overset{\text{def}}{=} \mathrm{Ker}(\mathrm{Aut}(Q) \to \mathrm{Sym}(\mathcal{Q}_n)) \ (\subseteq \mathrm{Aut}(Q)).$$

In particular, it is easy to see from the relevant definitions that

$$M_{\mathrm{Aut}} \overset{\text{def}}{=} M_1 \cap M_2 \cap \phi_Q^{-1}(J) \ (\subseteq \mathrm{Aut}(Q))$$

is an open subgroup of $\phi_Q^{-1}(J)$ satisfying the following conditions:

(i) $M_{\mathrm{Aut}} \cap \mathrm{Inn}(Q) \subseteq \mathrm{Inn}^V(Q)$,

(ii) $M_{\mathrm{Aut}} \subseteq \mathrm{Aut}^{[V]}(Q)$.

Write

$$M_V \overset{\text{def}}{=} \mathrm{Im}(M_{\mathrm{Aut}} \overset{\text{(ii)}}{\hookrightarrow} \mathrm{Aut}^{[V]}(Q) \to \mathrm{Aut}(V) \overset{\phi_V}{\twoheadrightarrow} \mathrm{Out}(V)),$$

$$M \overset{\text{def}}{=} \mathrm{Im}(M_{\mathrm{Aut}} \hookrightarrow \mathrm{Aut}^{[V]}(Q) \hookrightarrow \mathrm{Aut}(Q) \overset{\phi_Q}{\twoheadrightarrow} \mathrm{Out}(Q)),$$

$$M_{V,\mathrm{Aut}} \overset{\text{def}}{=} \mathrm{Im}(M_{\mathrm{Aut}} \hookrightarrow \mathrm{Aut}^{[V]}(Q) \twoheadrightarrow \mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^V(Q)).$$

Then we have a commutative diagram of profinite groups

$$\begin{array}{ccccc}
\mathrm{Aut}(V) & \longleftarrow & \mathrm{Aut}^{[V]}(Q) & \longrightarrow & \mathrm{Aut}(Q) \\
\downarrow{\scriptstyle \phi_V} & & \downarrow & & \downarrow{\scriptstyle \phi_Q} \\
\mathrm{Out}(V) & \longleftarrow & \mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^V(Q) & \longrightarrow & \mathrm{Out}(Q) \\
\uparrow & & \uparrow & & \uparrow \\
M_V & \longleftarrow & M_{V,\mathrm{Aut}} & \longrightarrow & M,
\end{array} \qquad (\star)$$

where the horizontal arrows in the third line are surjective; see the definitions of M_V, M, $M_{V,\mathrm{Aut}}$. Now we verify the following assertion.

Claim 3.8.A. *The horizontal arrows in the third line of diagram* ($\star$) *are bijective.*

Indeed, we note that, in light of our assumption that Q is slim, the natural morphism $\mathrm{Aut}^{[V]}(Q) \to \mathrm{Aut}(V)$ is injective; see Remark 3.2 (ii). Then it follows from the commutativity of the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & V & \longrightarrow & \mathrm{Aut}(V) & \longrightarrow & \mathrm{Out}(V) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & V & \longrightarrow & \mathrm{Aut}^{[V]}(Q) & \longrightarrow & \mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^V(Q) & \longrightarrow & 1
\end{array}$$

that $\mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^V(Q) \to \mathrm{Out}(V)$ is injective, hence that $M_{V,\mathrm{Aut}} \to M_V$ is bijective. On the other hand, it is easy to see from the above condition (i) that $M_{V,\mathrm{Aut}} \to M$ is injective. This completes the proof of Claim 3.8.A.

In light of Claim 3.8.A, we obtain a natural outer action

$$M \xrightarrow{\sim} M_{V,\mathrm{Aut}} \xrightarrow{\sim} M_V \hookrightarrow \mathrm{Out}(V).$$

Then, since the natural morphism $V \overset{\mathrm{out}}{\rtimes} M \to \mathrm{Aut}(V)$ factors through $\mathrm{Aut}^{[V]}(Q)$, we have a natural open injection $V \overset{\mathrm{out}}{\rtimes} M \hookrightarrow Q \overset{\mathrm{out}}{\rtimes} J$. Finally, it is easy to see from the relevant definitions that the morphisms $M \hookrightarrow \mathrm{Out}(V)$ and $V \overset{\mathrm{out}}{\rtimes} M \hookrightarrow Q \overset{\mathrm{out}}{\rtimes} J$ satisfy conditions (a), (b), (c). This completes the proof of Lemma 3.8. □

4 Computations of various Galois centralizers

In this section, by applying highly nontrivial "Grothendieck Conjecture-type results" [9, Theorem A] and [15, Theorem D] in anabelian geometry, we compute various Galois centralizers. These computations will be applied in Section 5.

Definition 4.1. Let k be a field, $\overline{k}$ an algebraic closure of k, and Z a hyperbolic curve over k. Then we have an exact sequence of profinite groups

$$1 \to \Pi_{Z_{\overline{k}}} \to \Pi_Z \to G_k \to 1.$$

We shall write $\rho_Z \colon G_k \to \mathrm{Out}(\Pi_{Z_{\overline{k}}})$ for the outer representation determined by the above exact sequence. Let $\Sigma \subseteq \mathbf{P}$ be a nonempty subset. Then we shall write

$$\rho_Z^{\Sigma} \colon G_k \to \mathrm{Out}(\Pi_{Z_{\overline{k}}}^{\Sigma})$$

for the outer representation induced by ρ_Z, and

$$\Pi_Z^{[\Sigma]} \stackrel{\mathrm{def}}{=} \Pi_Z / \mathrm{Ker}(\Pi_{Z_{\overline{k}}} \twoheadrightarrow \Pi_{Z_{\overline{k}}}^{\Sigma}).$$

Let p be a prime number. If $\Sigma = \{p\}$, then we shall also write $\rho_Z^p \stackrel{\mathrm{def}}{=} \rho_Z^{\Sigma}$.

Lemma 4.2. *Let l be a prime number and $\Sigma \subseteq \mathbf{P}$ a subset such that $l \in \Sigma$. Let $n \in \mathbb{Z}_{\geq 1}$ be a Σ-integer, $K \subseteq \overline{\mathbb{Q}}$ a number field, and $Z \subseteq \mathbb{P}_K^1 \setminus \{0, 1, \infty\}$ an open subscheme obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}_K^1 \setminus \{0, 1, \infty\}$. In particular, Z is a hyperbolic curve of genus* 0 *over K. Write $(\mathbb{P}_{\overline{\mathbb{Q}}}^1 \supseteq)\ Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}\ (\subseteq \mathbb{P}_{\overline{\mathbb{Q}}}^1)$ for the finite étale Galois covering of $Z_{\overline{\mathbb{Q}}}$ of degree n determined by $t \mapsto t^n$, and*

$$Q \stackrel{\mathrm{def}}{=} \Pi_{Z_{\overline{\mathbb{Q}}}}^{\Sigma} / \mathrm{Ker}(\Pi_{Y_{\overline{\mathbb{Q}}}}^{\Sigma} \twoheadrightarrow \Pi_{Y_{\overline{\mathbb{Q}}}}^{l}).$$

Then the following hold.

(i) *We have a natural homomorphism*

$$\mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}^{\Sigma}) \to \mathrm{Out}^{|\mathrm{C}|}(Q).$$

(ii) *Write ρ for the composite of natural homomorphisms*

$$G_K \xrightarrow{\rho_Z^{\Sigma}} \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{\mathbb{Q}}}}^{\Sigma}) \to \mathrm{Out}^{|\mathrm{C}|}(Q);$$

recall our assumption that all cusps of Z are K-rational. Then

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho)) = \{1\}.$$

(iii) *Let $G \subseteq \mathrm{Out}^{|\mathrm{C}|}(Q)$ be a closed subgroup such that G contains an open subgroup of* $\mathrm{Im}(\rho)$. *Then G is slim.*

Proof. We begin by observing that the normal open subgroup

$$\Pi_{Y_{\overline{\mathbb{Q}}}}^{\Sigma} \subseteq \Pi_{Z_{\overline{\mathbb{Q}}}}^{\Sigma}$$

(determined by the finite étale Galois covering $Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}$) may be characterized as the normal open subgroup topologically generated by the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ that is not associated to the cusps 0, ∞, and the unique closed subgroups of the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ associated to the cusps 0, ∞, of index n. Then it is easy to see from the relevant definitions that any element in $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$ induces an element in $\mathrm{Out}^{|\mathrm{C}|}(Q)$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $\sigma \in Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho))$. Then it follows from the above observation that any lifting in $\mathrm{Aut}(Q)$ of σ induces an automorphism of $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$. Let $\tilde{\sigma} \in \mathrm{Aut}(Q)$ be a lifting of σ such that the automorphism

$$\tilde{\sigma}|_{\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^{l}_{Y_{\overline{\mathbb{Q}}}})$$

induced by $\tilde{\sigma}$ preserves the $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 1. Here, we note that, since $\tilde{\sigma}$ preserves the Q-conjugacy class of cuspidal inertia subgroups of Q associated to the cusp 0 (respectively, ∞), and the finite étale Galois covering $Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}$ is totally ramified over the cusp 0 (respectively, ∞), it follows that $\tilde{\sigma}|_{\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}}$ preserves the $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 0 (respectively, ∞). Write

$$\sigma_Y \colon \Pi^{l}_{Y_{\overline{\mathbb{Q}}}} \xrightarrow{\sim} \Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$$

for the outer automorphism determined by

$$\tilde{\sigma}|_{\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}).$$

Observe that, since the outer action of G_K, together with σ_Y, on $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$ preserves the $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 1, it follows from our assumption that

$$\sigma \in Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho))$$

that σ_Y commutes with the outer action of G_K on $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$. Then it follows from the Grothendieck Conjecture [9, Theorem A] that σ_Y arises from a unique isomorphism $f \colon Y_{\overline{\mathbb{Q}}} \xrightarrow{\sim} Y_{\overline{\mathbb{Q}}}$ of schemes over $\overline{\mathbb{Q}}$. Note that, since $\tilde{\sigma}|_{\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}}$ induces the identity permutation on the set of the $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$-conjugacy classes of cuspidal inertia subgroups of $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusps $0, 1, \infty$, it follows that f induces the identity permutation on the subset $\{0, 1, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}}$. In particular, we conclude that f is the identity automorphism, hence that σ_Y is the identity outer automorphism. Recall that the automorphism

$$\tilde{\sigma}|_{\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^{l}_{Y_{\overline{\mathbb{Q}}}})$$

is the restriction of $\tilde{\sigma} \in \mathrm{Aut}(Q)$. Thus, since Q is slim by [12, Proposition 1.4], it follows from Remark 3.2 (ii) that $\tilde{\sigma}$ is an inner automorphism, hence that σ is the identity outer automorphism. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Since every open subgroup of G contains an open subgroup of $\mathrm{Im}(\rho)$, to verify (iii), it suffices to show that $Z(G) = \{1\}$. But this follows from assertion (ii). This completes the proof of assertion (iii), hence of Lemma 4.2. □

The following notion plays an important role in our work.

Definition 4.3. Let l be a prime number. We shall say that a profinite group G is *almost* $\mathbb{Z}_l$ if there exists an open subgroup $H \subseteq G$ such that H is isomorphic to $\mathbb{Z}_l$.

Lemma 4.4. *Let p be a prime number, $\Sigma \subseteq \mathbf{P}$ a nonempty subset such that $p \notin \Sigma$, and k a finite field of characteristic p. In the notation of Definition 4.1, suppose that Z is a hyperbolic curve of genus 0 over k such that all cusps of Z are k-rational. Write $\rho \stackrel{\mathrm{def}}{=} \rho_Z^\Sigma$. Then the following hold.*

(i) *Suppose that $|\mathbf{P} \setminus \Sigma|$ is finite. Then the natural homomorphism*

$$\mathrm{Aut}(Z_{\overline{k}}) \to \mathrm{Out}(\Pi_{Z_{\overline{k}}}^\Sigma)$$

determines an isomorphism

$$\mathrm{Aut}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{Z_{\overline{k}}}^\Sigma)}(\rho(G_k)).$$

(ii) *Suppose that either $|\Sigma| = 1$ or $|\mathbf{P} \setminus \Sigma|$ is finite. Then, if we write*

$$\chi_\Sigma \colon \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{k}}}^\Sigma) \to (\hat{\mathbb{Z}}^\Sigma)^\times$$

for the pro-Σ cyclotomic character, which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi_{Z_{\overline{k}}}^\Sigma$, then the natural composite

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{k}}}^\Sigma)}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi_{Z_{\overline{k}}}^\Sigma) \xrightarrow{\chi_\Sigma} (\hat{\mathbb{Z}}^\Sigma)^\times$$

is injective.

(iii) *Let l be a prime number $\neq p$. Then $Z_{\mathrm{Out}(\Pi_{Z_{\overline{k}}}^l)}(\rho(G_k))$ is almost $\mathbb{Z}_l$.*

Proof. First, we verify assertion (i). Write $\mathrm{Out}_{G_k}(\Pi_Z^{[\Sigma]})$ for the group of $\Pi_{Z_{\overline{k}}}^\Sigma$-outer automorphisms of $\Pi_Z^{[\Sigma]}$ that lie over G_k; see Definition 4.1. Then, since $\Pi_{Z_{\overline{k}}}^\Sigma$ is center-free by [12, Proposition 1.4], it is well known that the natural homomorphism

$$\mathrm{Out}_{G_k}(\Pi_Z^{[\Sigma]}) \to Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k))$$

is an isomorphism; see [17, Lemma 7.1]. On the other hand, since G_k is abelian, it follows from [15, Theorem D], together with the definition of $\mathrm{Out}_{G_k}(\Pi_Z^{[\Sigma]})$, that

$$\mathrm{Aut}(Z_{\overline{k}}/Z) \xrightarrow{\sim} \mathrm{Out}_{G_k}(\Pi_Z^{[\Sigma]}),$$

where $\mathrm{Aut}(Z_{\overline{k}}/Z) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ denotes the subgroup consisting of automorphisms of $Z_{\overline{k}}$ that induce automorphisms of Z compatible with the natural morphism $Z_{\overline{k}} \to Z$.

Next, we verify the following assertion.

Claim 4.4.A. *The inclusion* $\mathrm{Aut}(Z_{\overline{k}}/Z) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ *is bijective.*

Indeed, let $\alpha \in \mathrm{Aut}(Z_{\overline{k}})$ and $\sigma \in G_k$ $(\hookrightarrow \mathrm{Aut}(Z_{\overline{k}}))$. Then, since G_k is abelian, it follows that

$$\gamma \stackrel{\mathrm{def}}{=} \sigma \circ \alpha \circ \sigma^{-1} \circ \alpha^{-1} \in \mathrm{Aut}_{\overline{k}}(Z_{\overline{k}}).$$

Next, we note that γ induces the identity permutation on the set of cusps of $Z_{\overline{k}}$. Thus, we conclude that $\gamma = 1$, hence that α induces a unique automorphism in $\mathrm{Aut}(Z)$ compatible with the natural morphism $Z_{\overline{k}} \to Z$. This completes the proof of Claim 4.4.A.

Thus, by applying Claim 4.4.A, we obtain a natural isomorphism

$$\phi\colon \mathrm{Aut}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)).$$

This completes the proof of assertion (i).

Next, we verify assertion (ii). If $|\Sigma| = 1$, then the desired conclusion follows from the latter half of the proof of [13, Proposition 2.2.4]. Thus, we may assume without loss of generality that $|\mathbf{P} \setminus \Sigma|$ is finite. Write $\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ for the subgroup of automorphisms of $Z_{\overline{k}}$ that induce the identity permutation on the set of cusps of $Z_{\overline{k}}$. Then ϕ induces a composite

$$\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}}) \xrightarrow{\chi_\Sigma} (\hat{\mathbb{Z}}^{\Sigma})^{\times}.$$

Observe that this composite factors as the composite of the natural injection

$$\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \hookrightarrow G_{\mathbb{F}_p}$$

with the pro-Σ cyclotomic character $G_{\mathbb{F}_p} \to (\hat{\mathbb{Z}}^{\Sigma})^{\times}$. Thus, since this character is injective by [2, Théorème 1], we conclude that the natural composite

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}}) \xrightarrow{\chi_\Sigma} (\hat{\mathbb{Z}}^{\Sigma})^{\times}$$

is injective. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). We begin by observing that, since the image of the l-adic cyclotomic character $G_k \to \mathbb{Z}_l^\times$ is infinite, it follows from [10, Corollary 2.7 (i)] that

$$Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{k}}})}(\rho(G_k)) = Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^l_{Z_{\overline{k}}})}(\rho(G_k)).$$

Moreover, we observe that, since

$$\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{k}}}) \text{ is open in } \mathrm{Out}^{\mathrm{C}}(\Pi^l_{Z_{\overline{k}}}),$$

it follows that

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{k}}})}(\rho(G_k)) \text{ is open in } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^l_{Z_{\overline{k}}})}(\rho(G_k)).$$

Thus, to verify assertion (iii), it suffices to show that

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{k}}})}(\rho(G_k)) \text{ is almost } \mathbb{Z}_l.$$

On the other hand, since $\rho(G_k)$ is an infinite abelian group by [8, Lemma 4.2 (iv)], we conclude from assertion (ii) that

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{k}}})}(\rho(G_k)) \text{ is almost } \mathbb{Z}_l,$$

as desired. This completes the proof of assertion (iii), hence of Lemma 4.4. □

Remark 4.5. It is natural to pose the following question.

Question. In the notation of Lemma 4.4 (i) (respectively, Lemma 4.4 (ii)), can the assumption that $|\mathbf{P} \setminus \Sigma|$ is finite (respectively, $|\Sigma| = 1$ or $|\mathbf{P} \setminus \Sigma|$ is finite) be dropped?

However, at the time of writing, the authors do not know whether the answer to this question is affirmative or not.

Lemma 4.6. *Let l be a prime number and $K \subseteq \overline{\mathbb{Q}}$ a number field. In the notation of Definition 4.1, suppose that $k = K$, and Z is a hyperbolic curve over K. Then every open subgroup $U \subseteq \mathrm{Im}(\rho^l_Z)$ is nonabelian.*

Proof. Recall that, since the image of the l-adic cyclotomic character $G_K \to \mathbb{Z}_l^\times$ is infinite, it follows that $\mathrm{Im}(\rho^l_Z)$ is infinite [8, Lemma 4.2 (iv)]. In particular, U is infinite. Write $K' \subseteq \overline{\mathbb{Q}}$ for the finite extension of K determined by U. Suppose that U is abelian. Then, since

$$U \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(U),$$

the centralizer

$$Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(U)$$

is infinite. However, since $\mathrm{Aut}_{K'}(Z_{K'})$ is finite, this contradicts the Grothendieck Conjecture for hyperbolic curves over number fields [9, Theorem A]. Thus, we conclude that U is nonabelian. This completes the proof of Lemma 4.6. □

5 Strong indecomposability of the outer automorphism groups of nonabelian free profinite groups

In this section, by applying the results obtained in Sections 4 and 5, we prove Theorem 1.1 and Corollary 1.2.

Lemma 5.1. *Let $\Sigma \subseteq \mathbf{P}$ be a nonempty subset and $K \subseteq \overline{\mathbb{Q}}$ a number field. In the notation of Definition 4.1, suppose that $k = K$, and Z is a hyperbolic curve of genus 0 over K. Let*

$$G \subseteq \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$$

be a closed subgroup such that

- *G contains an open subgroup of $\rho^{\Sigma}_{Z}(G_K)$,*
- *there exists a prime number $l \in \Sigma$ such that the image of G via the natural morphism*

$$\phi_l \colon \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \twoheadrightarrow \mathrm{Out}(\Pi^{l}_{Z_{\overline{\mathbb{Q}}}}),$$

  which is necessarily surjective by [14, Proposition 4.5.4 (b)], *is slim,*
- *there exist normal closed subgroups $G_1 \subseteq G$, $G_2 \subseteq G$ such that $G = G_1 \times G_2$.*

Then either

(a) $\phi_l(G_1) = \{1\}$ *and* $G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$, *or*

(b) $\phi_l(G_2) = \{1\}$ *and* $G_2 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$.

Proof. First, by replacing K by a finite extension of K, we may assume without loss of generality that $\rho^{\Sigma}_{Z}(G_K) \subseteq G$, and all cusps of Z are K-rational. Let $\mathfrak{p}$ be a maximal ideal of the ring of integers of K such that

- the characteristic of the residue field at $\mathfrak{p}$ is not equal to l, and
- Z has good reduction at $\mathfrak{p}$.

Let $F \in G_K$ be a lifting of the Frobenius element at $\mathfrak{p}$. We shall write,

- for each $i = 1, 2$, $\mathrm{pr}_i \colon G \twoheadrightarrow G_i$ for the natural projection,
- $J \subseteq G_K$ for the closed subgroup topologically generated by F, where we note that J is isomorphic to $\hat{\mathbb{Z}}$,

- $I \stackrel{\text{def}}{=} \rho_Z^{\Sigma}(J) \subseteq G$,
- $I_1 \stackrel{\text{def}}{=} \mathrm{pr}_1(I) \times \{1\} \subseteq G_1 \times G_2 = G$, $I_2 \stackrel{\text{def}}{=} \{1\} \times \mathrm{pr}_2(I) \subseteq G_1 \times G_2 = G$.

Here, we note that, since I is abelian, it follows that

$$I \subseteq I_1 \times I_2 \subseteq Z_G(I),$$

hence that

$$\phi_l(I) \subseteq \phi_l(I_1) \cdot \phi_l(I_2) \subseteq Z_{\phi_l(G)}(\phi_l(I)) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l(I)).$$

Moreover, we note that, since Z has good reduction at $\mathfrak{p}$, it follows from the theory of specialization isomorphism, together with Lemma 4.4 (iii), and [8, Lemma 4.2 (iv)] that

- $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l(I))$ is almost $\mathbb{Z}_l$, and
- $\phi_l(I)$ is infinite.

In particular, either $\phi_l(I_1)$ is infinite, or $\phi_l(I_2)$ is infinite. We may assume without loss of generality that $\phi_l(I_2)$ is infinite. Observe that, since $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l(I))$ is almost $\mathbb{Z}_l$, it follows that $\phi_l(I_2) \cap \phi_l(I) \subseteq \phi_l(I)$ is an open subgroup. Then, since $G_1 \subseteq Z_G(I_2)$, there exists an open subgroup ${}^{\dagger}I \subseteq I$ such that

$$\phi_l(G_1) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\dagger}I)).$$

Now suppose that $\phi_l(G_1)$ is infinite. Then, since

$$\phi_l(I) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\dagger}I)),$$

and moreover, by Lemma 4.4 (iii),

$$Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\dagger}I)) \text{ is almost } \mathbb{Z}_l,$$

it follows that $\phi_l(G_1) \cap \phi_l(I) \subseteq \phi_l(I)$ is an open subgroup. On the other hand, since $G_2 \subseteq Z_G(G_1)$, there exists an open subgroup ${}^{\ddagger}I \subseteq {}^{\dagger}I$ $(\subseteq I)$ such that

$$\phi_l(G_2) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\ddagger}I)).$$

In summary, we have

$$\rho^l_Z(G_K) = \phi_l(\rho^{\Sigma}_Z(G_K)) \subseteq \phi_l(G) = \phi_l(G_1) \cdot \phi_l(G_2) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\ddagger}I)).$$

Then, since $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\ddagger}I))$ is almost $\mathbb{Z}_l$ by Lemma 4.4 (iii), it follows that there exists an open subgroup $U \subseteq \rho^l_Z(G_K)$ such that U is abelian, a contradiction; see Lemma 4.6. Thus, we conclude that $\phi_l(G_1)$ is finite. Therefore, since $\phi_l(G_1) \subseteq \phi_l(G)$ is a finite normal subgroup, it follows from our assumption that $\phi_l(G)$ is slim that $\phi_l(G_1) = \{1\}$; see Remark 3.2 (i).

Finally, we verify the inclusion $G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$. Write

$$\chi_l\colon \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to \mathbb{Z}_l^{\times}$$

for the l-adic cyclotomic character, which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$. Then, since $\chi_l(I)$ is infinite, it follows from [10, Corollary 2.7 (i)] that

$$I_1 \times I_2 \subseteq Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})}(I) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

In particular, since $\phi_l(I_1) \subseteq \phi_l(G_1) = \{1\}$, we have $I_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$; see [10, Proposition 1.2 (i)]. Thus, we obtain an inclusion

$$I \subseteq I_1 \times I_2^{|\mathrm{C}|} \ (\subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})),$$

where we write

$$I_2^{|\mathrm{C}|} \stackrel{\mathrm{def}}{=} I_2 \cap \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

Here, observe that, since χ_l factors through the restriction of ϕ_l on $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$, it follows that

$$\chi_l(I) \subseteq \chi_l(I_2^{|\mathrm{C}|}) \ (\subseteq \mathbb{Z}_l^{\times}),$$

hence that $\chi_l(I_2^{|\mathrm{C}|})$ is infinite. Therefore, we conclude from [10, Corollary 2.7 (i)] that

$$G_1 \subseteq Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})}(I_2^{|\mathrm{C}|}) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

In particular, since $\phi_l(G_1) = \{1\}$, we have $G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$; see [10, Proposition 1.2 (i)]. This completes the proof of Lemma 5.1. □

Remark 5.2. In the notation of Lemma 5.1, suppose that

$$G \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

Then the second assumption on G (concerning the slimness of $\phi_l(G)$) follows automatically from the first assumption on G. Indeed, to verify the slimness of $\phi_l(G)$, by replacing K by a finite extension of K, we may assume without loss of generality that Z is an open subscheme of $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$ obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$. Then the slimness of $\phi_l(G)$ follows from Lemma 4.2 (iii).

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We begin by observing that every open subgroup Γ of G contains an open subgroup of $\rho^{\Sigma}_Z(G_K)$, and moreover, $\phi_l(\Gamma)$ is slim. In particular, to verify Theorem 1.1, it suffices to prove that G is indecomposable. If $|\Sigma| = 1$, then the indecomposability of G follows from Lemma 5.1. Thus, we may assume without loss of generality that $|\mathbf{P} \setminus \Sigma|$ is finite. Next, by replacing K by a finite extension of K, we may assume without loss of generality that $\rho_Z^{\Sigma}(G_K) \subseteq G$, and all cusps of Z are K-rational. Moreover, we may assume without loss of generality that Z is an open subscheme of $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$ obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}^1_K \setminus \{0, 1, \infty\}$.

Suppose that there exist normal closed subgroups $G_1 \subseteq G$ and $G_2 \subseteq G$ such that $G = G_1 \times G_2$. We shall write,

- for each Σ-integer $n \in \mathbb{Z}_{\geq 1}$, $(\mathbb{P}^1_{\overline{\mathbb{Q}}} \supseteq) \, {}^n Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}} \; (\subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}})$ for the finite étale Galois covering of $Z_{\overline{\mathbb{Q}}}$ of degree n determined by $t \mapsto t^n$,
- $Q_{n,l} \stackrel{\text{def}}{=} \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} / \mathrm{Ker}(\Pi^{\Sigma}_{{}^n Y_{\overline{\mathbb{Q}}}} \twoheadrightarrow \Pi^{l}_{{}^n Y_{\overline{\mathbb{Q}}}})$,
- $\phi_{n,l} \colon \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to \mathrm{Out}^{|\mathrm{C}|}(Q_{n,l})$ for the natural homomorphism; see Lemma 4.2 (i).

Note that ${}^1 Y_{\overline{\mathbb{Q}}} = Z_{\overline{\mathbb{Q}}}$, and $Q_{1,l} = \Pi^{l}_{Z_{\overline{\mathbb{Q}}}}$.

Next, let us observe that, by applying Lemma 5.1, we may assume without loss of generality that

$$\phi_l(G_1) = \{1\} \quad \text{and} \quad G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

In particular, we have a direct product decomposition

$$G^{|\mathrm{C}|} = G_1 \times G_2^{|\mathrm{C}|} \; (\subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})),$$

where we write $G^{|\mathrm{C}|} \stackrel{\text{def}}{=} G \cap \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$ and $G_2^{|\mathrm{C}|} \stackrel{\text{def}}{=} G_2 \cap \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$.

Next, we verify the following assertion.

Claim 1.1.A. *For any Σ-integer $n \in \mathbb{Z}_{\geq 1}$, $\phi_{n,l}(G_1) = \{1\}$.*

Indeed, since $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ is slim by [12, Proposition 1.4], it follows from Lemma 3.8 that we have normal open subgroups $H \subseteq G^{|\mathrm{C}|}$, $H_1 \subseteq G_1$, and $H_2 \subseteq G_2^{|\mathrm{C}|}$ such that

- $H = H_1 \times H_2$,
- there exists an injection $H \hookrightarrow \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{{}^n Y_{\overline{\mathbb{Q}}}})$,
- there exists an injection

$$\Pi^{\Sigma}_{{}^n Y_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} H \hookrightarrow \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G^{|\mathrm{C}|}$$

  that is compatible with the inclusions between respective subgroups

$$\Pi^{\Sigma}_{{}^n Y_{\overline{\mathbb{Q}}}} \subseteq \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$$

  and quotients $H \subseteq G^{|\mathrm{C}|}$.

Then it follows from Lemma 5.1, Remark 5.2, together with [12, Proposition 1.4], that $\phi_{n,l}(H_1) = \{1\}$ or $\phi_{n,l}(H_2) = \{1\}$. Suppose that $\phi_{n,l}(H_2) = \{1\}$. Here, we note that, since $Q^l_{n,l} \xrightarrow{\sim} \Pi^l_{Z_{\overline{\mathbb{Q}}}}$, it follows that ϕ_l factors as the composite of $\phi_{n,l}$ with the natural homomorphism

$$\mathrm{Out}^{|\mathrm{C}|}(Q_{n,l}) \to \mathrm{Out}^{|\mathrm{C}|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}}).$$

In particular, $\phi_l(H_2) = \{1\}$. Then our assumption that $\phi_l(G_1) = \{1\}$ implies that $\phi_l(G_1 \times H_2) = \{1\}$, hence that $\phi_l(\rho^\Sigma_Z(G_K))$ ($\subseteq \phi_l(G)$) is finite. This is a contradiction; see [8, Lemma 4.2 (iv)]. Thus, we conclude that $\phi_{n,l}(H_1) = \{1\}$, hence that $\phi_{n,l}(G_1)$ is finite. Therefore, since $\phi_{n,l}(G_1) \subseteq \phi_{n,l}(G)$ is a finite normal subgroup, and moreover, $\phi_{n,l}(G)$ is slim by Lemma 4.2 (iii), we get $\phi_{n,l}(G_1) = \{1\}$; see Remark 3.2 (i). This completes the proof of Claim 1.1.A.

Write $\chi_\Sigma\colon \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}}) \to (\hat{\mathbb{Z}}^\Sigma)^\times$ for the pro-Σ cyclotomic character, which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}}$. Then it follows from Claim 1.1.A that

$$\chi_\Sigma(G_1) = \{1\}.$$

For each $p \in \Sigma$, write

$$\phi_{(p)'}\colon \mathrm{Out}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}}) \twoheadrightarrow \mathrm{Out}(\Pi^{\Sigma\setminus\{p\}}_{Z_{\overline{\mathbb{Q}}}})$$

for the natural surjection.

Next, we verify the following assertion.

Claim 1.1.B. *There exists a finite subset $S \subseteq \Sigma$ such that, for each $p \in \Sigma \setminus S$, $\phi_{(p)'}(G_1) = \{1\}$.*

Indeed, let $\mathfrak{p}$ be a maximal ideal of the ring of integers of K such that

- the characteristic, which we denote by p, of the residue field at $\mathfrak{p}$ is contained in Σ, and
- Z has good reduction at $\mathfrak{p}$.

Let $F \in G_K$ be a lifting of the Frobenius element at $\mathfrak{p}$. We shall write,

- for each $i = 1, 2$, $\mathrm{pr}_i\colon G \twoheadrightarrow G_i$ for the natural projection,
- $J \subseteq G_K$ for the closed subgroup topologically generated by F, where we note that J is isomorphic to $\hat{\mathbb{Z}}$,
- $I \stackrel{\mathrm{def}}{=} \rho^\Sigma_Z(J) \subseteq G^{|\mathrm{C}|}$,
- $I_1 \stackrel{\mathrm{def}}{=} \mathrm{pr}_1(I) \times \{1\} \subseteq G_1 \times G_2^{|\mathrm{C}|} = G^{|\mathrm{C}|}$, $I_2 \stackrel{\mathrm{def}}{=} \{1\} \times \mathrm{pr}_2(I) \subseteq G_1 \times G_2^{|\mathrm{C}|} = G^{|\mathrm{C}|}$.

Then, since I is abelian,

$$I \subseteq I_1 \times I_2 \subseteq Z_{G^{|\mathrm{C}|}}(I).$$

Then it follows from Lemma 4.4 (ii), together with the theory of specialization isomorphism, that our assumption that $\chi_\Sigma(I_1) \subseteq \chi_\Sigma(G_1) = \{1\}$ implies that

$$\phi_{(p)'}(I_1) = \{1\}.$$

In particular, $\phi_{(p)'}(I) \subseteq \phi_{(p)'}(I_2)$. Thus, since $\chi_\Sigma(G_1) = \{1\}$, and $G_1 \subseteq Z_G(I_2)$, we conclude from Lemma 4.4 (ii), together with the theory of specialization isomorphism, that $\phi_{(p)'}(G_1) = \{1\}$. Observe that there exists a finite subset $S \subseteq \Sigma$ such that Z has good reduction at any maximal ideal of the ring of integers of K that lies over a prime number in $\Sigma \setminus S$. Therefore, we obtain the desired conclusion. This completes the proof of Claim 1.1.B.

Write $\Sigma' \stackrel{\text{def}}{=} \mathbf{P} \setminus \Sigma$. Note that, by Lemma 3.4 and the finiteness of $|\Sigma' \cup S|$, the composite

$$\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to \prod_{p \in \mathbf{P}\setminus(\Sigma' \cup S)} \mathrm{Out}((\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})^{(p)'}) \xrightarrow{\sim} \prod_{p \in \Sigma \setminus S} \mathrm{Out}(\Pi^{\Sigma\setminus\{p\}}_{Z_{\overline{\mathbb{Q}}}})$$

is injective. Thus, we conclude from Claim 1.1.B that $G_1 = \{1\}$, hence that G is indecomposable. This completes the proof of Theorem 1.1. □

Remark 5.3. It is natural to pose the following question.

Question. In the notation of Theorem 1.1, can the assumption that $|\Sigma| = 1$ or $|\mathbf{P} \setminus \Sigma|$ is finite be dropped?

However, at the time of writing, the authors do not know whether the answer to this question is affirmative or not.

Corollary 5.4. *In the notation of Theorem 1.1, suppose G is slim. Then $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G$ is strongly indecomposable.*

Proof. First, since $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ is center-free by [12, Proposition 1.4], we have an exact sequence of profinite groups

$$1 \to \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} \to \Pi_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G \to G \to 1.$$

Thus, since G is infinite, we conclude from Theorem 1.1, together with [8, Proposition 1.8 (i)] and [12, Propositions 1.4, 3.2], that $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G$ is strongly indecomposable. This completes the proof of Corollary 5.4. □

Lemma 5.5. *Let $m \geq 2$ be an integer, $\Sigma \subseteq \mathbf{P}$ a nonempty subset, and F_m a free profinite group of rank m. Then $\mathrm{Out}(F^{\Sigma}_m)$ is slim.*

Proof. First, we consider the case where $m = 2$. In this case, we claim the following assertion.

Claim 5.5.A. *There exists a closed subgroup*

$$H_1 \subseteq \mathrm{Out}(F_m^{\Sigma}) \quad (\textit{respectively, } H_2 \subseteq \mathrm{Out}(F_m^{\Sigma}))$$

such that, for every open subgroup $H_1' \subseteq H_1$ (respectively, $H_2' \subseteq H_2$),

$$Z_{\mathrm{Out}(F_m^{\Sigma})}(H_1') \cong \mathfrak{S}_3 \quad (\textit{respectively, } Z_{\mathrm{Out}(F_m^{\Sigma})}(H_2') \cong \mathbb{Z}/2\mathbb{Z}),$$

where we write $\mathfrak{S}_3$ for the symmetric group on 3 letters.

Indeed, write C_1 for the projective line minus $\{0, 1, \infty\}$ over a number field $K_1 \subseteq \overline{\mathbb{Q}}$. Let $(E, \{o\})$ be an elliptic curve, where o is the origin of E, over a number field $K_2 \subseteq \overline{\mathbb{Q}}$ such that the j-invariant of $(E, \{o\})$ is not equal to 0 or 1728. Write C_2 for the hyperbolic curve over K_2 obtained by removing $\{o\}$ from E. In particular, as is well known, for any finite extension $K_1 \subseteq L_1$ $(\subseteq \overline{\mathbb{Q}})$ (respectively, $K_2 \subseteq L_2$ $(\subseteq \overline{\mathbb{Q}})$), we have

$$\mathrm{Aut}_{L_1}((C_1)_{L_1}) \cong \mathfrak{S}_3 \quad (\text{respectively, } \mathrm{Aut}_{L_2}((C_2)_{L_2}) \cong \mathbb{Z}/2\mathbb{Z}).$$

Thus, since, by applying the Grothendieck Conjecture for hyperbolic curves over number fields [9, Theorem A], we have a natural isomorphism

$$\mathrm{Aut}_{L_1}((C_1)_{L_1}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^{\Sigma}_{(C_1)_{\overline{\mathbb{Q}}}})}(\rho^{\Sigma}_{C_1}(G_{L_1}))$$
$$(\text{respectively, } \mathrm{Aut}_{L_2}((C_2)_{L_2}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^{\Sigma}_{(C_2)_{\overline{\mathbb{Q}}}})}(\rho^{\Sigma}_{C_2}(G_{L_2}))),$$

we conclude that the image of $\rho^{\Sigma}_{C_1}(G_{K_1})$ (respectively, $\rho^{\Sigma}_{C_2}(G_{K_2})$) via any isomorphism

$$\mathrm{Out}(\Pi^{\Sigma}_{(C_1)_{\overline{\mathbb{Q}}}}) \xrightarrow{\sim} \mathrm{Out}(F_m^{\Sigma}) \quad (\text{respectively, } \mathrm{Out}(\Pi^{\Sigma}_{(C_2)_{\overline{\mathbb{Q}}}}) \xrightarrow{\sim} \mathrm{Out}(F_m^{\Sigma}))$$

determines the desired subgroup. This completes the proof of Claim 5.5.A.

Let us recall that, to verify Lemma 5.5, it suffices to show that, for every normal open subgroup $N \subseteq \mathrm{Out}(F_m^{\Sigma})$, we have $Z_{\mathrm{Out}(F_m^{\Sigma})}(N) = \{1\}$. Now suppose that $Z_{\mathrm{Out}(F_m^{\Sigma})}(N) \neq \{1\}$. Then, since

$$\begin{aligned} Z_{\mathrm{Out}(F_m^{\Sigma})}(N) &\subseteq Z_{\mathrm{Out}(F_m^{\Sigma})}(H_1 \cap N) \cap Z_{\mathrm{Out}(F_m^{\Sigma})}(H_2 \cap N) \\ &\subseteq Z_{\mathrm{Out}(F_m^{\Sigma})}(H_2 \cap N) \cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

by Claim 5.5.A, we have

$$\begin{aligned} Z_{\mathrm{Out}(F_m^{\Sigma})}(N) &= Z_{\mathrm{Out}(F_m^{\Sigma})}(H_1 \cap N) \cap Z_{\mathrm{Out}(F_m^{\Sigma})}(H_2 \cap N) \\ &= Z_{\mathrm{Out}(F_m^{\Sigma})}(H_2 \cap N) \cong \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Thus, since $Z_{\mathrm{Out}(F_m^\Sigma)}(N)$ is normal in $\mathrm{Out}(F_m^\Sigma)$ by our assumption that N is normal in $\mathrm{Out}(F_m^\Sigma)$, we conclude that $Z_{\mathrm{Out}(F_m^\Sigma)}(H_1 \cap N)$, which is isomorphic to $\mathfrak{S}_3$ by Claim 5.5.A, admits a normal subgroup of order 2, a contradiction. Therefore, we have $Z_{\mathrm{Out}(F_m^\Sigma)}(N) = \{1\}$. This completes the proof of Lemma 5.5 in the case where $m = 2$.

Next, we consider the case where $m = 3$. In this case, we claim the following assertion.

Claim 5.5.B. *There exists a closed subgroup*

$$H_1 \subseteq \mathrm{Out}(F_m^\Sigma) \quad (\textit{respectively, } H_2 \subseteq \mathrm{Out}(F_m^\Sigma))$$

satisfying the following conditions.

- *For every open subgroup $H_1' \subseteq H_1$ (respectively, $H_2' \subseteq H_2$),*

$$Z_{\mathrm{Out}(F_m^\Sigma)}(H_1') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$$(\textit{respectively, } Z_{\mathrm{Out}(F_m^\Sigma)}(H_2') \cong \mathbb{Z}/2\mathbb{Z}).$$

- *For any $l \in \Sigma$, every nontrivial element $\alpha_1 \in Z_{\mathrm{Out}(F_m^\Sigma)}(H_1')$ (respectively, the unique nontrivial element $\alpha_2 \in Z_{\mathrm{Out}(F_m^\Sigma)}(H_2')$) induces a $\mathbb{Z}_l$-automorphism $\overline{\alpha}_1$ (respectively, $\overline{\alpha}_2$) of $(F_m^l)^{\mathrm{ab}}$ such that the rank of the $\mathbb{Z}_l$-submodule of $(F_m^l)^{\mathrm{ab}}$ consisting of elements in $(F_m^l)^{\mathrm{ab}}$ fixed by $\overline{\alpha}_1$ (respectively, $\overline{\alpha}_2$) is 1 (respectively, 0).*

Indeed, write C_1 for the projective line minus $\{0, 1, 3, \infty\}$ over a number field $K_1 \subseteq \overline{\mathbb{Q}}$. Let $(E, \{o\})$ be an elliptic curve over a number field $K_2 \subseteq \overline{\mathbb{Q}}$ such that the j-invariant of $(E, \{o\})$ is not equal to 0 or 1728, and that E has a non-4-torsion K_2-rational point x. Write C_2 for the hyperbolic curve over K_2 obtained by removing $\{x, -x\}$ from E. Observe that the $\overline{\mathbb{Q}}$-automorphism "$[-1]$" of $E_{\overline{\mathbb{Q}}}$ (of order 2) preserves the set $\{x, -x\}$, and that if a nontrivial element $f \in \mathrm{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}})$ satisfies $f(x) = x$, then f coincides with the map

$$z \mapsto 2x - z$$

on the set of $\overline{\mathbb{Q}}$-rational points of $E_{\overline{\mathbb{Q}}}$, so $f(-x) \neq -x$. In particular, as is easily verified, for any finite extension $K_1 \subseteq L_1$ $(\subseteq \overline{\mathbb{Q}})$ (respectively, $K_2 \subseteq L_2$ $(\subseteq \overline{\mathbb{Q}})$), we have

$$\mathrm{Aut}_{L_1}((C_1)_{L_1}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad (\text{respectively, } \mathrm{Aut}_{L_2}((C_2)_{L_2}) \cong \mathbb{Z}/2\mathbb{Z}).$$

Let α_1 be a nontrivial element in $\mathrm{Aut}_{L_1}((C_1)_{L_1})$ that induces a permutation s on the set of cusps $\{0, 1, 3, \infty\}$. Here, we note that s may be written as the product of the transposition of two distinct elements in $\{0, 1, 3, \infty\}$ and the transposition of the other two distinct elements in $\{0, 1, 3, \infty\}$. Write α_2 for the unique nontrivial element in $\mathrm{Aut}_{L_2}((C_2)_{L_2})$. Then it follows from the well known structure of the étale fundamental group of a smooth curve over an algebraically closed field of characteristic 0 that if we identify $(\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$ (respectively, $(\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$) with the free $\mathbb{Z}_l$-module

$$(\langle a_0, a_1, a_3, a_\infty \mid a_0 + a_1 + a_3 + a_\infty = 0\rangle^l)^{\mathrm{ab}}$$

$$(\text{respectively, } (\langle b, c, d_x, d_{-x} \mid d_x + d_{-x} = 0\rangle^l)^{\mathrm{ab}}),$$

where $a_\circ$ (respectively, $d_\circ$) corresponds to the cusp $\circ \in \{0, 1, 3, \infty\}$ (respectively, $\circ \in \{x, -x\}$), b corresponds to a meridian of the "complex torus $E(\mathbb{C})$", and c corresponds to a longitude of $E(\mathbb{C})$, then α_1 (respectively, α_2) induces the $\mathbb{Z}_l$-automorphism

$$\overline{\alpha}_1 \colon (\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}} \xrightarrow{\sim} (\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}},$$
$$a_0 \mapsto a_{s(0)},\ a_1 \mapsto a_{s(1)},\ a_3 \mapsto a_{s(3)},\ a_\infty \mapsto a_{s(\infty)}$$
$$(\text{respectively, } \overline{\alpha}_2 \colon (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}} \xrightarrow{\sim} (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}},$$
$$b \mapsto -b,\ c \mapsto -c,\ d_x \mapsto d_{-x},\ d_{-x} \mapsto d_x).$$

Now one verifies easily that the rank of the $\mathbb{Z}_l$-submodule of

$$(\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}} \text{ (respectively, } (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}})$$

consisting of elements in

$$(\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}} \text{ (respectively, } (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}})$$

fixed by $\overline{\alpha}_1$ (respectively, $\overline{\alpha}_2$) is 1 (respectively, 0). Thus, Claim 5.5.B follows from a similar argument to the argument applied in the final portion of the proof of Claim 5.5.A.

In light of Claim 5.5.B, for every open subgroup $N \subseteq \mathrm{Out}(F_m^\Sigma)$, we have

$$Z_{\mathrm{Out}(F_m^\Sigma)}(N) \subseteq Z_{\mathrm{Out}(F_m^\Sigma)}(H_1 \cap N) \cap Z_{\mathrm{Out}(F_m^\Sigma)}(H_2 \cap N) = \{1\}.$$

This completes the proof of Lemma 5.5 in the case where $m = 3$.

Finally, we consider the case where $m \geq 4$. In this case, by considering the hyperbolic curve C over a number field $K \subseteq \overline{\mathbb{Q}}$ obtained by removing $m+1$ distinct $\mathbb{Q}$-rational points from $\mathbb{P}^1_K$ such that $\mathrm{Aut}_{\overline{\mathbb{Q}}}(C_{\overline{\mathbb{Q}}}) = \{1\}$, the next claim follows from a similar argument to the argument applied in the final portion of the proof of Claim 5.5.A.

Claim 5.5.C. *There exists a closed subgroup $H \subseteq \mathrm{Out}(F_m^\Sigma)$ such that, for every open subgroup $H' \subseteq H$, $Z_{\mathrm{Out}(F_m^\Sigma)}(H') = \{1\}$.*

It is easy to see from Claim 5.5.C that $\mathrm{Out}(F_m^{\Sigma})$ (where $m \geq 4$) is slim. This completes the proof of Lemma 5.5. □

Remark 5.6. Lemma 5.5 in the case where $m \geq 4$ also follows from [6, Theorem A].

Finally, we prove Corollary 1.2.

Proof of Corollary 1.2. The strong indecomposability of $\mathrm{Out}(F_m^{\Sigma})$ follows from Theorem 1.1 and Lemma 5.5. The strong indecomposability of $\mathrm{Aut}(F_m^{\Sigma})$ follows from Corollary 5.4 and Lemma 5.5. This completes the proof of Corollary 1.2. □

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