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# On the kernel of actions on asymptotic cones

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**Abstract.** Any finitely generated group $G$ acts on its asymptotic cones in natural ways. The purpose of this paper is to calculate the kernel of such actions. First, we show that, when $G$ is acylindrically hyperbolic, the kernel of the natural action on every asymptotic cone coincides with the unique maximal finite normal subgroup $K(G)$ of $G$. Secondly, we use this equivalence to interpret the kernel of the actions on asymptotic cones as the kernel of the actions on many spaces at "infinity". For instance, if $G \curvearrowright M$ is a non-elementary convergence group, then we show that the kernel of actions on the limit set $L(G)$ coincides with the kernel of the action on asymptotic cones. Similar results can also be established for the non-trivial Floyd boundary and the CAT(0) groups with the visual boundary, contracting boundary, and sublinearly Morse boundary. Additionally, the results are extended to another action on asymptotic cones, called Paulin's construction. In the last section, we calculate the kernel on asymptotic cones for various groups, and as an application, we show that the cardinality of the kernel can determine whether the group admits a non-elementary action under some mild assumptions.

## 1 Introduction

In geometric group theory, group action has proven to be a very powerful tool to study various properties of groups. Abundant examples of "nice" actions of groups on "good" spaces have been constructed and studied by many authors. Some of the classical examples include hyperbolic group actions on their Gromov boundary, Kleinian group actions on limit sets, $\mathrm{Out}(F_n)$ actions on outer spaces or free-factor complexes, mapping class group actions on Teichmüller spaces, and the list goes on and on.

There are rather general constructions of spaces on which groups act. One of the most famous and well-studied objects is the Cayley graph which can be defined for any group. While this graph depends on the choice of a generating set, it is well-defined up to quasi-isometry whenever its generating set is finite. In this case, the Cayley graph is connected and locally finite, hence also proper. Moreover, by construction, a group acts canonically on its Cayley graph, and this action is known

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to be isometric, cocompact, and properly discontinuous. Consequently, the Cayley graph plays a central role in the study of finitely generated groups.

Another such example is asymptotic cones. Roughly speaking, an asymptotic cone is a view of the metric space from infinity through our ultrafilter-glasses. For any finitely generated group $G$, we can construct an asymptotic cone of $G$ together with a non-principal ultrafilter $\omega$ and a suitable real sequence $d_n$, by considering $G$ as a metric space. We refer readers to Section 2.2 for precise definitions and further details of asymptotic cones. We also recommend M. Sapir's note [68], R. Young's note [76], [31], [67, Chapter 7], etc.

Asymptotic cones of a group reflect many properties of the group. For example, a group is of polynomial growth if and only if all of its asymptotic cones are locally compact (for a brief explanation, see [12, Chapter 4] and references therein). It is also known that a finitely generated group is hyperbolic if and only if all of its asymptotic cones are real trees [41]. Furthermore, if we restrict to finitely presented groups, then $G$ is hyperbolic if and only if one of its asymptotic cones is a real tree [55, Appendix]. Nevertheless, because of the unexpected and pathological phenomena of asymptotic cones, e.g., the existence of a group with uncountably many non-homeomorphic asymptotic cones, actions on asymptotic cones are less known than the action on the Cayley graph.

In this paper, we study the kernel of the actions of $G$ on its asymptotic cone. To the best of our knowledge, this is the first attempt to describe the kernel. Our first main theorem is that if $G$ is finitely generated acylindrically hyperbolic, then the kernel of the natural action on every asymptotic cone is the same as the unique maximal finite normal subgroup of $G$, denoted by $K(G)$.

**Theorem 1.1** (Acylindrically hyperbolic group, Theorem 4.3). *Let $G$ be a finitely generated acylindrically hyperbolic group. Then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = K(G)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

Combining with prior results obtained by Osin [58], Minasyan–Osin [53], and Dahmani–Guirardel–Osin [27] (for details, see Lemma 3.7), it turns out that these two subgroups also coincide with subgroups $\ker(G \curvearrowright \partial X)$, $\mathrm{FC}(G)$, and $\mathcal{A}(G)$.

**Corollary 1.2** (Corollary 4.4). *If $G$ is a finitely generated acylindrically hyperbolic group, then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$$

*for any ultrafilter $\omega$, sequence $d_n$, and $\delta$-hyperbolic space $X$ on which $G$ acts non-elementarily and acylindrically.*

For precise definitions and more details of these subgroups, see Section 3. We note that a $\delta$-hyperbolic space $X$ on which $G$ acts is not uniquely determined. Indeed, we can always choose $X$ to be a quasi-tree. See [5, 10]. Furthermore, it is known that some acylindrically hyperbolic groups have various asymptotic cones. Our proof does not depend on the choice of the ultrafilter $\omega$ and the sequence $d_n$, so it directly implies the following.

**Corollary 1.3** (Corollary 4.5). *Let $G$ be a finitely generated acylindrically hyperbolic group. Then the kernel* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is invariant under changes to the ultrafilter $\omega$ and sequence $d_n$.*

Next, we establish a connection between asymptotic cones and other spaces at "infinity" through the kernel. More specifically, we show that the kernel of the natural actions on asymptotic cones is equivalent to the kernel of the canonical actions on various spaces at 'infinity". First, we obtain the convergence group version corollary.

**Corollary 1.4** (Convergence group action, Corollary 5.4). *Let $G$ be a finitely generated group. Suppose that $G$ admits a convergence group action $G \curvearrowright M$ with $|L(G)| > 2$. Then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker\big(G \curvearrowright L(G)\big) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

Recall that a non-elementary convergence group is an acylindrically hyperbolic group [71] and the limit set of a convergence group can be considered as the space at "infinity". We note that if $G$ is a non-elementary convergence group, then in general, $\partial X \neq L(G)$. Corollary 5.4 can be obtained from the fact that the kernel of the action on its limit set is the same as $K(G)$.

According to [46], if $G$ has a non-trivial Floyd boundary $\partial_F G$, then the action of $G$ on $\partial_F G$ is a convergence group action. All the terminology and notions about the Floyd boundary can be found in Section 5.2. So Corollary 5.4 directly implies the following.

**Corollary 1.5** (Floyd boundary, Corollary 5.7).*Let $G$ be a finitely generated group. Suppose that $G$ has non-trivial Floyd boundary $\partial_F G$. Then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial_F G) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

It is a well-known fact that a CAT(0) group $G$ with a rank-one isometry is acylindrically hyperbolic, provided that $G$ is not virtually cyclic [70, 77]; see

Proposition 5.8 for the details. Motivated by this result, we obtain the following result.

**Corollary 1.6** (CAT(0) group, Corollary 5.17 and Corollary 5.18). *Suppose that a finitely generated group $G$ acts geometrically on proper* CAT(0) *space $X$. Also, assume that $G$ contains a rank-one isometry and $G$ is not virtually cyclic. All subgroups in the following formula are the same for any ultrafilter $\omega$, sequence $d_n$, and sublinear function $\kappa$:*

$$\begin{aligned}\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) &= \ker(G \curvearrowright \partial_v X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)\\ &= \ker(G \curvearrowright \partial_\kappa X) = \ker(G \curvearrowright \partial_* X).\end{aligned}$$

*Here, $\partial_v X$, $\partial_* X$, and $\partial_\kappa X$ are the visual, contracting, and sublinearly Morse boundary of $X$, respectively.*

So far, we have concentrated only on the canonical action on asymptotic cones. Now we will introduce another action on asymptotic cones known as Paulin's construction. Briefly speaking, Paulin's construction is a "twisted" action. This action is constructed by infinitely many (outer) automorphisms, so we need the condition that the outer automorphism group $\mathrm{Out}(G)$ should be infinite. Presumably, the main difference between the natural action and this new action is the existence of a global fixed point. It immediately follows that the natural action must have a global fixed point, but we eliminate global fixed points via the "twisted" method. Our second main result says that two kernels of these two actions on asymptotic cones coincide, so the kernel of Paulin's construction can also be characterized as in Corollary 4.4.

**Theorem 1.7** (Theorem 6.4). *Let $G$ be a finitely generated acylindrically hyperbolic group whose outer automorphism group* $\mathrm{Out}(G)$ *is infinite. For any sequence $[\phi_1], [\phi_2], \ldots$ in* $\mathrm{Out}(G)$ *with $[\phi_i] \neq [\phi_j]$, there exist automorphism representatives $\phi_1, \phi_2, \ldots \in \mathrm{Aut}(G)$ such that the kernels of two actions on an asymptotic cone are the same, that is,*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker\big(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G)\big)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

In the other direction, we concentrate on the kernel of various (non-acylindrically hyperbolic) groups and prove that if $G$ is a finitely generated infinite virtually nilpotent group, then the kernel of $G$ on its asymptotic cone is infinite. Using this fact, we can show that the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ determines whether the actions are non-elementary or not, as follows.

**Corollary 1.8** (Corollay 7.3). *Let $G$ be an infinite hyperbolic group. Then $G$ is non-elementary if and only if the kernel* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is finite.*

*Furthermore, suppose that a finitely generated group $G$ acts acylindrically on some $\delta$-hyperbolic space $X$ and contains a loxodromic element. Then the action of $G$ on $X$ is non-elementary (so $G$ is acylindrically hyperbolic) if and only if* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is finite.*

Then we also study the equalities in $(*)$ for various groups. For this purpose, we make the following definition.

**Definition 1.9** (Definition 7.1). If $G$ is a finitely generated group for which $K(G)$ exists and satisfies

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G) \qquad (*)$$

for some $\delta$-hyperbolic space $X$, then we say that $G$ satisfies condition $(*)$.

We explore condition $(*)$, and in particular, we show that the braided Thompson group BV satisfies condition $(*)$ but is not acylindrically hyperbolic.

The paper is organized as follows. In Section 2, we briefly recall the definition and properties of asymptotic cones, acylindrically hyperbolic groups, and related topics. In Section 3, we review some remarks about subgroups in Corollary 4.4, mainly $K(G)$, $\mathcal{A}(G)$, and $\mathrm{FC}(G)$. We also include some short proofs of easily obtained inclusions and equalities in this section. We prove the main theorem, Theorem 4.3, in Section 4. As an application, we prove Corollary 5.4 (a non-elementary convergence group), Corollary 5.7 (a group with a non-trivial Floyd boundary), and Corollary 1.6 (a CAT(0) group with a rank-one isometry including a Coxeter group) in Section 5. In Section 6, we characterize the kernel of Paulin's construction, and in Section 7, we consider general groups and find some algebraic conditions to disturb the equivalence. In this section, we record the kernels of some interesting groups acting on their asymptotic cones.

## 2 Preliminaries

### 2.1 Cayley graphs and $\delta$-hyperbolic spaces

In this subsection, we briefly recall the definitions of Cayley graphs and $\delta$-hyperbolic spaces. We also introduce our notation.

**Definition 2.1.** Let $G$ be a group with a fixed generating set $S$. The *Cayley graph* for $G$ with respect to $S$, denoted by $\Gamma(G, S)$, is the graph with vertex set $G$ and edge set given by $\{(g, gs) : g \in G,\ s \in S\}$.

Then it is an easy fact that $\Gamma(G, S)$ is connected, and locally finite whenever $S$ is a finite set. If $S$ and $T$ are finite generating sets for a group $G$, then usually $\Gamma(G, S)$ is different from the graph $\Gamma(G, T)$, but they are quasi-isometric when both $S$ and $T$ are finite.

Clearly, a group $G$ acts on the Cayley graph $\Gamma(G, S)$ by $g \cdot x = gx$. With the metric $d_S(g, h) = \|g^{-1}h\|_S$, where $\|x\|_S$ means the metric between $x \in G$ and the identity $e$ in the graph $\Gamma(G, S)$, this action is isometric and cocompact. When we write the word metric, we sometimes use $\|x\|$ when there is no confusion about the generating set, or when the generating set is not important.

Now we recall $\delta$-hyperbolic spaces and related notions.

**Definition 2.2.** A metric space $X$ is *$\delta$-hyperbolic* if $X$ is geodesic and there exists some $L \geq 0$ such that, for any geodesic triangle $(\gamma_1, \gamma_2, \gamma_3)$ in $X$, the geodesic $\gamma_i$ is contained in an $L$-neighborhood of the other two for all $i = 1, 2, 3$. In particular, we say that $X$ is a real tree if it is $\delta$-hyperbolic with $L = 0$.

A $\delta$-hyperbolic space $X$ has a typical and well-studied boundary called the *Gromov boundary*, denoted by $\partial X$. To define the Gromov boundary, we first recall the *Gromov product*. For a triple $x, y, z$ in $X$, the value

$$(x|y)_z := \frac{1}{2}\big(d(x,z) + d(y,z) - d(x,y)\big) \in \mathbb{R}$$

is called the Gromov product of three points $x, y, z$.

Now we define the Gromov boundary using the Gromov product. First, we say that a sequence $[x_n]$ in $X$ converges at infinity if

$$\lim_{(m,n)\to\infty} (x_m|x_n)_p = \infty$$

for fixed $p \in X$. Let $\mathcal{S}$ be a set of sequences that converge at infinity. We declare $[x_n] \approx [y_n]$ if and only if

$$\lim_{n\to\infty} (x_n|x_n)_p = \infty.$$

Then the relation $\approx$ is an equivalence relation. The Gromov boundary of $X$, $\partial X$, is defined by $\partial X := \mathcal{S}/\approx$.

It is a fact that $[x_n] \not\approx [y_n]$ if and only if $\sup_{m,n}(x_m|y_n)_p < \infty$, and the Gromov boundary does not depend on the choice of the basepoint $p$.

Let us review the terminology of isometry on a $\delta$-hyperbolic space $X$. An isometry $f$ on $X$ is called *elliptic* if the orbit of $f$ is bounded for some, equivalently any, orbit. It is called *parabolic* if $f$ has exactly one fixed point in the boundary $\partial X$. Lastly, we say that $f$ is *loxodromic* if it has exactly two fixed points in the boundary. Recall that $f$ is loxodromic if and only if the map $\mathbb{Z} \to X$ given by

$n \mapsto f^n \cdot x$ is a quasi-isometric embedding for any basepoint $x$ in $X$. We say that two loxodromic elements $f, g$ are *independent* if their fixed point sets are disjoint.

Suppose that a group $G$ acts isometrically on a $\delta$-hyperbolic group $X$. Then there exists a canonical $G$-action on $\partial X$. For $g \in G$ and $[x_n] \in \partial X$, define

$$g \cdot [x_n] := [g \cdot x_n].$$

Then this action is well-defined.

## 2.2 Asymptotic cones

Asymptotic cones are the main ingredient of this paper, so we briefly recall their definition. Since asymptotic cones are a special kind of ultralimit of metric spaces, we start with the definition of ultrafilters, which is a useful concept to define ultralimits.

**Definition 2.3.** An *ultrafilter* $\omega$ on $\mathbb{N}$ is a non-empty set of subsets in $\mathbb{N}$ satisfying

- $\emptyset \notin \omega$.
- If $A, B \in \omega$, then $A \cap B \in \omega$.
- If $A \in \omega$ and $A \subset B \subset \mathbb{N}$, then $B \in \omega$.
- For any $A \subset \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} - A \in \omega$.

Also, we say that an ultrafilter $\omega$ is *non-principal* if $F \notin \omega$ for all finite subsets $F$ in $\mathbb{N}$.

Recall that, from the first and second conditions, it cannot happen that both $A \in \omega$ and $\mathbb{N} - A \in \omega$. Now we will define the ultralimit of a sequence in a metric space. Using this, we can define an ultralimit of metric spaces.

**Definition 2.4.** Let $\omega$ be a non-principal ultrafilter on $\mathbb{N}$. When $\{x_n\}$ is a sequence of points in a metric space $(X, d)$, a point $x \in X$ is called the *ultralimit* of $\{x_n\}$, denoted by $x := \lim_\omega x_n$, if, for every $\epsilon > 0$, $\{n \in \mathbb{N} : d(x_n, x) \leq \epsilon\} \in \omega$.

Similarly to the usual limit, if an ultralimit exists, then it is unique, but in general, the ultralimit may not exist. However, when a metric space $X$ is compact, it is known that the ultralimit always exists. This implies that the ultralimit of any bounded sequence in $\mathbb{R}$ always exists.

Suppose that a sequence $\{x_n\}$ converges to $x$ in the usual limit sense. Then, for any non-principal ultrafilter $\omega$, $\lim_\omega x_n = x$. So we can think of an ultralimit as a generalization of the usual limits, and this explains why we use non-principal ultrafilters. One of the difficulties of an ultralimit is that an ultralimit depends on the choice of an ultrafilter. Now we define ultralimits of metric spaces.

**Definition 2.5.** Let $(X_n, d_n)$ be a sequence of metric spaces. Choose $p_n \in X_n$ for each $n \in \mathbb{N}$. We call the sequence $\{p_n\}$ the *observation sequence*. Suppose that $\omega$ is a non-principal ultrafilter on $\mathbb{N}$.

A sequence $\{x_n\}$ (here $x_n \in X_n$) is *admissible* if the sequence $\{d_n(x_n, p_n)\}$ is bounded. Let $\mathcal{A}$ be the set of all admissible sequences. We define $X_\infty := \mathcal{A}/\sim$, where $\{x_n\} \sim \{y_n\}$ if and only if $\lim_\omega d_n(x_n, y_n) = 0$. Then the space $X_\infty$ is a metric space with the metric defined by

$$d_\infty(\{x_n\}, \{y_n\}) := \lim_\omega d_n(x_n, y_n).$$

We say that the metric space $(X_\infty, d_\infty)$ is the *ultralimit* of $\{(X_n, d_n)\}$, denoted by $\lim_\omega(X_n, d_n, p_n) := (X_\infty, d_\infty)$.

Clearly, the ultralimit of metric spaces depends not only on $\{(X_n, d_n)\}$ but also the choice of the observation sequence $\{p_n\}$ and ultrafilter $\omega$.

**Definition 2.6.** Let $(X, d)$ be a metric space, $\omega$ a non-principal ultrafilter on $\mathbb{N}$, and $\{p_n\}$ a sequence in $X$. Suppose that a sequence $\{d_n\}$ is an unbounded non-decreasing sequence of positive real numbers. Then the ultralimit $\lim_\omega(X, \frac{1}{d_n}d, p_n)$ is called the *asymptotic cone* and denoted by $\mathrm{Cone}_\omega(X, d_n, p_n)$.

Let $X$ be a $\delta$-hyperbolic space. Recall that both elements of the Gromov boundary of $X$ and of an asymptotic cone of $X$ can be considered as sequences in $X$. To avoid confusion, we denote an element of the Gromov boundary by $[x_n] \in \partial X$ and an element of an asymptotic cone by $\{x_n\} \in \mathrm{Cone}_\omega(X, d_n, p_n)$.

We define an asymptotic cone of $G$ by an asymptotic cone of its Cayley graph.

**Definition 2.7.** Let $G$ be a finitely generated group with a finite generating set $S$. An *asymptotic cone* of $G$ is $\mathrm{Cone}_\omega(G, d_n, g_n) := \mathrm{Cone}_\omega(\Gamma(G, S), d_n, g_n)$ for some unbounded non-decreasing sequence $\{d_n\}$ and observation sequence $\{g_n\}$.

Since asymptotic cones are quasi-isometry invariant up to bi-Lipschitz, this definition is well-defined and we can omit the finite generating set $S$. Also, we can remove the observation sequence $g_n$ from the notation. Recall that a metric space $X$ is quasi-homogeneous if there exist a homogeneous space $Y \subset X$ and a constant $C > 0$ such that, for any $x \in X$, $d(x, Y) < C$. It is known that $\mathrm{Cone}(X, d_n, p_n)$ does not depend on the choice of observation sequence when $X$ is quasi-homogeneous. Thus any asymptotic cone of a group does not depend on the choice of observation sequence $\{g_n\}$, so we can simply write $\mathrm{Cone}_\omega(G, d_n)$.

However, an asymptotic cone of a group still depends on the choice of a real sequence $d_n$ and ultrafilters. First, S. Thomas and B. Velickovic constructed a group with two distinct (non-homeomorphic) asymptotic cones [73]. Their group is only finitely generated, not finitely presented. Later, a finitely presented group with two non-homeomorphic asymptotic cones was constructed [56]. This group has a simply connected space (not a real tree) and a non-simply connected space as an asymptotic cone.

However, it is known that any asymptotic cone of hyperbolic groups is unique and it is a real tree. In general, the converse is not true (recall that a group is called *lacunary hyperbolic* if one of its asymptotic cones is a real tree), but when a group $G$ is finitely presented, then $G$ is hyperbolic if and only if one of its asymptotic cones is a real tree. Furthermore, A. Sisto completely classified asymptotic cones of hyperbolic groups as follows. If a real tree $X$ is the asymptotic cone of a group, then it is a point, a line, or the $2^{\aleph_0}$ universal tree. We refer to [69, Corollary 5.9] for further details.

We end this subsection by introducing the natural action of $G$ on its asymptotic cone. Let $G$ be a group and consider one of its asymptotic cones $\mathrm{Cone}_\omega(G, d_n)$. Note that an element in the asymptotic cone can be considered as an (admissible) sequence of $G$, so let $\{x_n\} \in \mathrm{Cone}_\omega(G, d_n)$. Define a group action by

$$g \cdot \{x_n\} := \{gx_n\}.$$

It is straightforward that the group action is well-defined and isometric.

### 2.3 Acylindrically hyperbolic groups

In this subsection, we discuss acylindrically hyperbolic groups. Denis Osin first proposed this notion in [58]. It can be seen as a generalization of relatively hyperbolic groups. However, it still includes other significant groups in geometric group theory, e.g., the mapping class groups $\mathrm{Mod}(S)$, where $S$ is not an exceptional case and has no boundary cases, and the outer automorphism groups of the rank $n$ free group $\mathrm{Out}(F_n)$. First of all, we introduce the definition of an acylindrically hyperbolic group.

**Definition 2.8.** Let $G$ be a group and $X$ a metric space. An isometric action of $G$ on $X$ is called *acylindrical* if, for every $\epsilon > 0$, there exist $R, N > 0$ such that, for any two points $x, y \in X$ such that $d(x, y) \geq R$, then

$$|\{g \in G : d(x, gx) \leq \epsilon,\ d(y, gy) \leq \epsilon\}| \leq N.$$

Recall that an isometric action of $G$ on a $\delta$-hyperbolic space $X$ is called *elementary* if the limit set of $G$ on the Gromov boundary $\partial X$ contains at most 2 points.

**Definition 2.9.** We say that a group $G$ is *acylindrically hyperbolic* if it admits a non-elementary and acylindrical action on some $\delta$-hyperbolic space $X$.

Indeed, there are equivalent definitions for acylindrical hyperbolicity. The following theorem is well known and can be found in [58].

**Theorem 2.10** ([58, Theorem 1.2]). *For any group $G$, the following conditions are equivalent.*

(1) *$G$ admits a non-elementary acylindrical action on a $\delta$-hyperbolic space $X_1$, i.e., $G$ is acylindrical hyperbolic.*

(2) *There exists a generating set $X$ of $G$ such that the corresponding Cayley graph $\Gamma(G, X)$ is $\delta$-hyperbolic, $|\partial\Gamma(G, X)| > 2$, and the natural action of $G$ on $\Gamma(G, X)$ is acylindrical.*

(3) *$G$ contains a proper infinite hyperbolically embedded subgroup.*

(4) *$G$ is not virtually cyclic and admits an action on a $\delta$-hyperbolic space $X_2$ such that at least one element of $G$ is loxodromic and satisfies the WPD condition.*

**Convention 2.11.** Throughout this article, for a group property (P), we say that $G$ is *virtually* (P) if $G$ is infinite and $G$ has a finite index subgroup satisfying property (P). So $G$ is *virtually cyclic* if $G$ contains a finite index subgroup isomorphic to $\mathbb{Z}$. Note that any virtually (P) group cannot be finite in this paper.

We note that $\delta$-hyperbolic spaces $X_1$ and $X_2$ in the aforementioned theorem may be different. The terminology WPD stands for "Weakly Properly Discontinuous" and was first introduced by Bestvina and Fujiwara [11]. We briefly recall its definition.

**Definition 2.12.** Let $G$ be a group acting on a metric space $(X, d)$. We say that an element $g \in G$ is *WPD* if it satisfies all of the following conditions.

- The order of $g$ is infinite.
- For some $x \in X$, the map $\mathbb{Z} \to X$ given by $n \mapsto g^n \cdot x$ is a quasi-isometric embedding.
- For each $\epsilon > 0$ and each $x \in X$, there exists an integer $m > 0$ such that

$$|\{a \in G : d(x, a \cdot x) < \epsilon,\ d(g^m \cdot x, ag^m \cdot x) < \epsilon\}| < \infty.$$

By the definition, if $g$ is WPD for some action $G$ on a $\delta$-hyperbolic space $X$, then $g$ must be a loxodromic element. Also, if $G$ acts acylindrically on $X$, then every loxodromic element is WPD. Recall that an acylindrically hyperbolic group contains infinitely many independent loxodromic elements [58]. For more details about the equivalence definitions, Theorem 2.10, we refer to [27]. For readers who are interested in WPD elements, we refer to [11, 64].

We say that a subgroup $H$ of $G$ is *s-normal* if the intersection $g^{-1}Hg \cap H$ is infinite for every $g \in G$. It is known that the class of acylindrically hyperbolic groups is closed under taking s-normal subgroups [58, Lemma 7.2]. By the definition of s-normal subgroups, every infinite normal subgroup is s-normal, so we obtain the following lemma.

**Lemma 2.13.** *Let $G$ be an acylindrically hyperbolic group and $H$ an infinite normal subgroup of $G$. Then $H$ is also acylindrically hyperbolic.*

Note that the space does not change in Lemma 2.13. In other words, if $G \curvearrowright X$ is non-elementary and acylindrical, then for any infinite normal subgroup $H < G$, the induced action $H \curvearrowright X$ is also non-elementary and acylindrical.

## 3 On the subgroups that appeared in condition ($*$)

In this section, we focus on the subgroups that appeared in the equalities in condition ($*$). We will explain these subgroups more precisely and discuss some properties and remarks. Generally, the equalities among these subgroups fail, but we will prove $K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$ when $G$ is acylindrically hyperbolic from already-known results.

### 3.1 The unique maximal finite normal subgroup $K(G)$ and the amenable radical $\mathcal{A}(G)$

First, we will discuss the unique maximal finite normal subgroup $K(G)$, beginning with its definition.

**Definition 3.1.** Let $G$ be a group. *The unique maximal finite normal subgroup* is the finite normal subgroup of $G$ that contains all finite normal subgroups of $G$.

Note that if a maximal finite normal subgroup exists, then it must be unique. Hence "the" unique maximal finite normal subgroup in the definition makes sense.

Perhaps one of the natural questions is the existence of $K(G)$. Of course, $K(G)$ may not exist even if $G$ is finitely generated. For a concrete example, note that there is a finitely generated group $G$ whose center $Z(G)$ is isomorphic to the quotient group $\mathbb{Q}/\mathbb{Z}$ due to Abderezak Ould Houcine [60]. Recall that the center $Z(G)$ is a characteristic subgroup of $G$, and the subgroups $\langle 1/p \rangle < \mathbb{Q}/\mathbb{Z}$ are also characteristic subgroups (this follows from the fact that $\langle 1/p \rangle$ is the unique subgroup with $p$ elements). Thus we conclude that the subgroups $\langle 1/p \rangle$ are finite normal subgroups of $G$. Since $G$ has subgroups $\langle 1/p \rangle$ for every prime $p$, this implies that $G$ does not have the unique maximal finite normal subgroup.

However, it is known that, for any acylindrically hyperbolic group $G$, $K(G)$ exists. We refer to [27, Theorem 2.24].

The *amenable radical* $\mathcal{A}(G)$ is the unique maximal amenable normal subgroup. For more details about amenability, we refer to [31, Section 18.3] and [50, Chapter 9]. In contrast to $K(G)$, it is known that every group has an amenable radical. When $G$ is acylindrically hyperbolic, its amenable radical is finite [58, Corollary 7.3], which directly implies $K(G) = \mathcal{A}(G)$.

## 3.2 Description of the kernel of actions on asymptotic cones

Next, we will describe the kernel of group actions on asymptotic cones. Let $G$ be a group, and we fix an ultrafilter $\omega$ and sequence $d_n$. The value $d_\infty(\{x_n\}, \{gx_n\})$ is a metric between $\{x_n\}$ and $\{gx_n\}$ in the asymptotic cone. Note that this value is exactly

$$\lim_\omega \frac{\|x_n^{-1} g x_n\|_S}{d_n}.$$

Here, $S$ is a finite generating set for $G$. Since this value measures how far an element $g$ moves $\{x_n\}$, the kernel can be expressed as follows:

$$\begin{aligned}&\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big)\\ &\qquad = \left\{ g \in G : \lim_\omega \frac{\|x_n^{-1} g x_n\|_S}{d_n} = 0 \text{ for all } x_n \text{ with } \lim_\omega \frac{x_n}{d_n} < \infty \right\}.\end{aligned}$$

## 3.3 Finite conjugacy classes subgroup

Now, we give the definition of $\mathrm{FC}(G)$. Recall that FC stands for "Finite Conjugacy classes".

**Definition 3.2.** Let $G$ be a group. We define the *finite conjugacy classes subgroup* $\mathrm{FC}(G)$, also called the FC-*center*, to be the set of all elements having finitely many conjugacy classes, that is,

$$\mathrm{FC}(G) := \big\{ g \in G : |\{x^{-1} g x : x \in G\}| < \infty \big\}.$$

Then it is straightforward that $\mathrm{FC}(G)$ is a normal subgroup of $G$.

### 3.4 Easily obtained inclusions and equalities

Now we will prove some easily obtained equalities and inclusions.

**Lemma 3.3.** *If a finitely generated group $G$ has a unique maximal finite normal subgroup $K(G)$, then*

$$K(G) \subset \mathrm{FC}(G) \subset \ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

*Proof.* The first inclusion follows directly. Now choose $a \in \mathrm{FC}(G)$. Then the set $\{\|g^{-1}ag\|_S : g \in G\} \subset \mathbb{R}$ is bounded. Thus, for any unbounded sequence $d_n$ of $\mathbb{R}$ and any sequence $x_n$ in $G$, we have

$$\lim_{n\to\infty} \frac{x_n^{-1}ax_n}{d_n} = 0,$$

which means that $a$ is an element of the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$. □

However, these subgroups in Lemma 3.3 may not coincide in general. For the first case, just consider $G = \mathbb{Z}$. In order to find a counterexample for the second case, consider the three-dimensional Heisenberg group $H$. Let

$$x = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

be two generators of $H$, and let $z = [x, y] := x^{-1}y^{-1}xy$. It is known that the commutator subgroup of $H$ is an infinite cyclic group generated by $z$, and $z$ is a distortion element. Here, we say that $g \in G$ is a *distortion element* if $g$ has an infinite order and

$$\lim_{k\to\infty} \frac{\|g^k\|}{k} = 0$$

for some word metric $\|\cdot\|$ with respect to some finite generating set. The distortedness can be obtained from $[x^l, y^m] = z^{lm}$.

Now we will give an example of $\mathrm{FC}(G) \subsetneq \ker(G \to \mathrm{Isom}(\mathrm{Cone}_\omega(G, d_n)))$. Let $G = H$, the Heisenberg group, and pick $g = x$, one of its generators. Then $g \notin \mathrm{FC}(G)$ since

$$y^{-m}gy^m = \begin{bmatrix} 1 & 1 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Our goal is to show that $g$ is in $\ker(G \to \mathrm{Isom}(\mathrm{Cone}_\omega(G, d_n)))$. First note that any element in $H$ can be expressed as

$$\begin{bmatrix} 1 & a_k & c_k \\ 0 & 1 & b_k \\ 0 & 0 & 1 \end{bmatrix} = y^{b_k} z^{c_k} x^{a_k}.$$

Then we have

$$[g, y^{b_k} z^{c_k} x^{a_k}] = \begin{bmatrix} 1 & 0 & b_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now assume $g \notin \ker(G \to \mathrm{Isom}(\mathrm{Cone}_\omega(G, d_n)))$. Then there exists a sequence $\{x_n\}$ in $H$ such that $\lim_\omega \frac{\|x_n\|_S}{d_n} = L > 0$ and $\{x_n\} \neq g \cdot \{x_n\}$, equivalently,

$$\lim_\omega \frac{\|x_n^{-1} g x_n\|_S}{d_n} = d > 0.$$

Since $g$ is fixed and $\lim_\omega \frac{d_n}{\|x_n\|_S} = 1/L > 0$, we have

$$\lim_\omega \frac{\|g^{-1} x_n^{-1} g x_n\|_S}{\|x_n\|} = \lim_\omega \frac{\|g^{-1} x_n^{-1} g x_n\|_S}{d_n} \frac{d_n}{\|x_n\|_S} = \frac{d}{L} > 0,$$

which means that

$$\lim_\omega \frac{\|[g, x_n]\|_S}{\|x_n\|_S} > 0.$$

Let $x_n = y^{b_n} z^{c_n} x^{a_n}$, so

$$[g, x_n] = \begin{bmatrix} 1 & 0 & b_n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = z^{b_n}.$$

Since $z$ is distortion,

$$\lim_{n \to \infty} \frac{\|z^{b_n}\|_S}{b_n} = 0,$$

and this implies

$$\lim_\omega \frac{\|z^{b_n}\|_S}{b_n} = 0.$$

However, $\|x_n\|_S \geq b_n$, a contradiction. Therefore,

$$g \in \ker\big(G \to \mathrm{Isom}(\mathrm{Cone}_\omega(G, d_n))\big) - \mathrm{FC}(G),$$

which means $\mathrm{FC}(G) \subsetneq \ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$.

Also, the following inclusion is known to be satisfied for a countable group. Note that we concentrate only on finitely generated groups, so this inclusion holds in our setting.

**Lemma 3.4.** *Let G be a countable group. Then we have* $\mathrm{FC}(G) \subset \mathcal{A}(G)$.

This inclusion follows immediately using the AC-center. This notion was first suggested in [74], but we use the definition introduced in [47].

**Definition 3.5.** Let $G$ be a countable group. The AC-*center* of $G$, denoted by $\mathrm{AC}(G)$, is defined by

$$\mathrm{AC}(G) := \{g \in G : \text{the quotient group } G/C_G(\langle\langle g\rangle\rangle) \text{ is amenable}\}.$$

Here, $\langle\langle S\rangle\rangle$ is the normal closure of $S$, meaning the minimal normal subgroup containing $S$.

*Proof of Lemma* 3.4. By definition, the AC-center contains $\mathrm{FC}(G)$ since any finite group is amenable. By [47, Lemma 3.1], $\mathrm{AC}(G)$ is an amenable normal subgroup of $G$, so by the definition, $\mathrm{AC}(G)$ is contained in the amenable radical $\mathcal{A}(G)$. □

Like the previous inclusions, the two subgroups $\mathrm{FC}(G)$ and $\mathcal{A}(G)$ need not be the same generally. The Heisenberg group $H$ also provides a counterexample. Recall that $g := x \notin \mathrm{FC}(H)$ but $\mathcal{A}(H) = H$ since $H$ is nilpotent, so $\mathrm{FC}(H) \subsetneq \mathcal{A}(H)$.

The only remaining inclusion relation concerns

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) \quad \text{and} \quad \mathcal{A}(G).$$

In Section 7, we give an example for $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)) \subsetneq \mathcal{A}(G)$ (see Lemma 7.5). However, we have not found a group satisfying the reverse inclusion.

**Question 3.6.** For a finitely generated group $G$, does the inclusion

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) \subseteq \mathcal{A}(G)$$

always hold? Or does there exist a finitely generated group $G$ such that

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) \supsetneq \mathcal{A}(G)?$$

The referee pointed out to the authors that [48] provides examples of finitely generated groups with a trivial amenable radical that are not $C^*$-simple. Furthermore, [14] characterizes $C^*$ simplicity (for any discrete group) as the existence of a topologically free boundary action. This suggests that the examples in [14, 48] could be promising candidates for a negative answer to Question 3.6. Thus it would be interesting to compute the kernel of the natural action on the asymptotic cone for these examples.

Recall that the equality among $K(G)$, $\mathrm{FC}(G)$, and $\mathcal{A}(G)$ does not hold in general, but they are the same when $G$ is acylindrically hyperbolic.

**Lemma 3.7.** *Let $G$ be an acylindrically hyperbolic group and $X$ a $\delta$-hyperbolic space on which $G$ acts non-elementarily and acylindrically. Then*

$$\ker(G \curvearrowright \partial X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G).$$

*Proof.* As mentioned before, we already have $K(G) = \mathcal{A}(G)$ since the amenable radical of $G$ is finite [58, Corollary 7.3]. To verify $K(G) = \mathrm{FC}(G)$, consider the quotient $G/K(G)$. By construction, $K(G/K(G)) = 1$ and it is again acylindrically hyperbolic [53, Lemma 3.9]. By [27, Theorem 2.35], it follows that

$$\mathrm{FC}\big(G/K(G)\big) = 1.$$

This means $\mathrm{FC}(G) \subset K(G)$ and the opposite inclusion follows directly. Hence $K(G) = \mathrm{FC}(G)$.

The only remaining part is $\ker(G \curvearrowright \partial X) = K(G)$. Recall that any finite normal subgroup of $G$ acts trivially on $\partial X$, so $\ker(G \curvearrowright \partial X) \supset K(G)$. The reverse inclusion is obtained from Lemma 2.13. If $N$ is an infinite normal subgroup of $G$, then the induced action $N \curvearrowright X$ is also non-elementary. So $N$ cannot be the kernel $\ker(G \curvearrowright \partial X)$ and we have $\ker(G \curvearrowright \partial X) \subset K(G)$. □

## 4 The kernel of acylindrically hyperbolic groups acting on asymptotic cones

In this section, we will characterize the kernel of group actions on asymptotic cones. We start this section with the following lemma. We use it to construct a quasi-isometric embedding.

**Lemma 4.1** ([72, Lemma 3.2]). *Let $X$ be a $\delta$-hyperbolic space and, for $1 \le i \le k$, let $g_i \in \mathrm{Isom}(X)$ be isometries of $X$ such that, for some $x_0 \in X$, we have*

$$d(x_0, g_i \cdot x_0) \ge 2(g_j^{\pm 1} \cdot x_0 | g_l^{\pm 1} \cdot x_0)_{x_0} + 18\delta + 1$$

*for all $1 \le i, j, l \le k$ except when $j = l$ and the exponent on the $g_j$ and $g_l$ are the same. Then the orbit map $\langle g_1, \ldots, g_k \rangle \to X$ given by $g \mapsto g \cdot x_0$ is a quasi-isometric embedding.*

We slightly modify the statement as follows. The following version is more suitable for our purpose.

**Lemma 4.2.** *Let $G$ be a group and suppose $G$ acts isometrically on a $\delta$-hyperbolic space $X$. Let $x, y$ be two independent loxodromic elements. Then there exists $M \in \mathbb{N}$ such that the orbit map $\langle x^M, y^M \rangle \to X$ defined by $g \mapsto g \cdot o$ is a quasi-isometric embedding for any $o \in X$.*

*Proof.* First, think of the previous lemma with $k = 2$. In this case, the required conditions are the following inequalities:

$$\begin{aligned}
d(x_0, g_1 \cdot x_0) &\geq 2(g_1^{\pm 1} \cdot x_0 | g_2^{\pm 1} \cdot x_0)_{x_0} + 18\delta + 1,\\
d(x_0, g_1 \cdot x_0) &\geq 2(g_1 \cdot x_0 | g_1^{-1} \cdot x_0)_{x_0} + 18\delta + 1,\\
d(x_0, g_1 \cdot x_0) &\geq 2(g_2 \cdot x_0 | g_2^{-1} \cdot x_0)_{x_0} + 18\delta + 1,\\
d(x_0, g_2 \cdot x_0) &\geq 2(g_1^{\pm 1} \cdot x_0 | g_2^{\pm 1} \cdot x_0)_{x_0} + 18\delta + 1,\\
d(x_0, g_2 \cdot x_0) &\geq 2(g_1 \cdot x_0 | g_1^{-1} \cdot x_0)_{x_0} + 18\delta + 1,\\
d(x_0, g_2 \cdot x_0) &\geq 2(g_2 \cdot x_0 | g_2^{-1} \cdot x_0)_{x_0} + 18\delta + 1.
\end{aligned}$$

Here, the inequalities containing $\pm 1$ must hold for any choice of signs $\pm$. Put $x = g_1$, $y = g_2 \in G$, and $p = x_0 \in X$. By the definition of the Gromov product, it suffices to check the following three types of inequalities for some $p \in X$ and $M > 0$:

$$\begin{aligned}
d(x^{2M} \cdot p, p) &\geq d(x^M \cdot p, p) + 18\delta + 1,\\
d(y^{2M} \cdot p, p) &\geq d(y^M \cdot p, p) + 18\delta + 1,
\end{aligned} \tag{†}$$

$$\begin{aligned}
d(y^{2M} \cdot p, p) + d(x^M \cdot p, p) &\geq 2d(y^M \cdot p, p) + 18\delta + 1,\\
d(x^{2M} \cdot p, p) + d(y^M \cdot p, p) &\geq 2d(x^M \cdot p, p) + 18\delta + 1,
\end{aligned} \tag{††}$$

$$\begin{aligned}
d(x^{\pm M} y^{\pm M} \cdot p, p) &\geq d(y^{\pm M} \cdot p, p) + 18\delta + 1,\\
d(y^{\pm M} x^{\pm M} \cdot p, p) &\geq d(x^{\pm M} \cdot p, p) + 18\delta + 1.
\end{aligned} \tag{†††}$$

The inequalities in (†) and (††) follow from the stable translation length. For an isometry $g$ on $X$, the stable translation length is defined by

$$\tau(g) := \lim_{k \to \infty} \frac{d(p, g^k \cdot p)}{k}.$$

It is known that $\tau(g)$ is well-defined. Also, it does not depend on the choice of the basepoint $p \in X$, and when $X$ is $\delta$-hyperbolic, then $g$ is loxodromic if and only if $\tau(g) > 0$. For each $m > 0$, $\tau(g^m) = m\tau(g)$.

From these facts, we can obtain the first inequalities. Suppose

$$d(x^{2n} \cdot p, p) < d(x^n \cdot p, p) + 18\delta + 1$$

for all $n$. Then we have

$$\frac{d(x^{2n} \cdot p, p)}{n} < \frac{d(x^n \cdot p, p)}{n} + \frac{18\delta + 1}{n}.$$

By taking $n \to \infty$, we have $\tau(x^2) < \tau(x)$ but $\tau(x) > 0$. This is a contradiction. Indeed, this contradiction implies that the inequalities hold for sufficiently large $n$.

Similarly, we can show that all inequalities in (††) are satisfied for sufficiently large $M$.

The inequalities in (†††) follow from the assumption that $x, y$ are independent. Since $x, y$ are independent, we get

$$\lim_{n\to\infty} (x^{\pm n} \cdot p | y^{\pm n} \cdot p)_p < \infty$$

for any choice of signs $\pm$. This implies that, for all $n$,

$$d(x^{\pm n} \cdot p, p) + d(y^{\pm n} \cdot p, p) - d(x^{\pm n} \cdot p, y^{\pm n} \cdot p) < B$$

for some $M$. Since $d(x^{\pm n} \cdot p, p) \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} d(x^{\pm n} \cdot p, y^{\pm n} \cdot p) - d(y^{\pm n} \cdot p, p) = \infty.$$

This means that there exists some $M_1 > 0$ such that

$$d(x^{\pm N} y^{\pm N} \cdot p, p) - d(y^{\pm N} \cdot p, p) \geq 18\delta + 1$$

whenever $N > M_1$. Similarly, there exists some $M_2 > 0$ such that

$$d(y^{\pm N} x^{\pm N} \cdot p, p) - d(x^{\pm N} \cdot p, p) \geq 18\delta + 1$$

whenever $N > M_2$.

Therefore, we see that all the inequalities hold for sufficiently large $M$. So the subgroup $\langle x^M, y^M \rangle$ satisfies all the conditions in [72, Lemma 3.2]. The result now follows. □

Now we prove the main result.

**Theorem 4.3.** *Let $G$ be a finitely generated acylindrically hyperbolic group. Then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = K(G)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

*Proof.* By Lemma 3.3 and Lemma 3.7, it suffices to show

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) \subset \ker(G \curvearrowright \partial X)$$

for some $\delta$-hyperbolic space $X$. We choose $X$ such that the action $G \curvearrowright X$ is cobounded ([58, Theorem 1.2] or [5, Theorem 1.7]). In this case, the limit set $L(G)$ is the same as the Gromov boundary $\partial X$.

In order to obtain the inclusion, we will show that $g \notin \ker(G \curvearrowright \partial X)$ implies $g \notin (G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$, so choose $g$ in the complement of $\ker(G \curvearrowright \partial X)$. Since the set of loxodromic fixed points is dense in $L(G)$ ([28, Corollary 7.4.3]) and $L(G) = \partial X$, there exists a loxodromic element $h \in G$ such that

$$[h^n] \neq g^{-1} \cdot [h^n] \quad \text{and} \quad \mathrm{Fix}(h) \cap \mathrm{Fix}(g^{-1}hg) = \emptyset.$$

Let $x := h$ and $y := g^{-1}hg$. Then, using the ping-pong lemma, we can prove that $H := \langle x^k, y^k \rangle$ is the free group of rank 2 for sufficiently large $k$. By Lemma 4.2, we may assume that the orbit map $\langle x^k, y^k \rangle \to X$ is a quasi-isometric embedding. Let

$$z_n := g^{-1}h^{-nk}gh^{nk} = y^{-nk}x^{nk} \in H.$$

Note that the length of $z_n$ in $H$ is $2n$ with respect to the generating set $\{x^k, y^k\}$. We have

$$2n \leq d(o, z_n \cdot o) \leq C\|z_n\|_S$$

for $C := \sup_{s\in S} d(o, s \cdot o)$, where $S$ is a finite generating set for $G$. Note that the lower bound is obtained from [72, Lemma 3.2]. This implies

$$\|z_n\|_S \geq \frac{2}{C}n$$

and we thus get

$$\lim_{n\to\infty} \frac{\|h^{-nk}gh^{nk}\|_S}{n} = \frac{2}{C} > 0.$$

Now suppose that we are given an unbounded non-decreasing sequence $d_n$ and any ultrafilter $\omega$. Then $\{h^{\lfloor d_n \rfloor}\}$ is an admissible sequence, that is,

$$\{h^{\lfloor d_n \rfloor}\} \in \mathrm{Cone}_\omega(G, d_n).$$

Then

$$\lim_{n\to\infty} \frac{\|h^{-k\lfloor d_n \rfloor}gh^{k\lfloor d_n \rfloor}\|_S}{d_n} = \frac{2}{C} > 0$$

and this means $\{h^{\lfloor d_n \rfloor}\} \neq \{gh^{\lfloor d_n \rfloor}\}$ in the asymptotic cone $\mathrm{Cone}_\omega(G, d_n)$ for any ultrafilter $\omega$. □

Combining with Lemma 3.7, we can conclude that the following subgroups coincide when $G$ is finitely generated acylindrically hyperbolic.

**Corollary 4.4.** *If $G$ is a finitely generated acylindrically hyperbolic group, then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$$

*for any ultrafilter $\omega$, sequence $d_n$, and $\delta$-hyperbolic space $X$ on which $G$ acts non-elementarily and acylindrically.*

Recall that, for a finitely generated group $G$, an asymptotic cone of $G$ depends on the choice of ultrafilter $\omega$ and non-decreasing sequence $d_n$. Examples of such cases can be found in [56, 73]. In particular, an acylindrically hyperbolic group may have various asymptotic cones and this is guaranteed by the fact that, for any finitely presented group $G$, the free product $G * F_n$ is acylindrically hyperbolic for some $n$, due to Osin [57]. We also note that $G$ has a lot of asymptotic cones for some relatively hyperbolic group $G$, and it is proved that if its peripheral subgroups have a unique asymptotic cone, then it also has the unique asymptotic cone [59].

According to the characterization, we prove that the kernel of the action of an acylindrically hyperbolic group $G$ on its asymptotic cone is invariant even though $G$ may have a lot of asymptotic cones.

**Corollary 4.5.** *Let $G$ be a finitely generated acylindrically hyperbolic group. Then the kernel* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is invariant under changes to the ultrafilter $\omega$ and sequence $d_n$.*

Our main result can be applied to inner amenability, von Neumann algebras, and $C^*$-algebras of groups. It is known that many algebraic properties are related to inner amenability and such algebras of groups. For more details, we refer to [27, Theorem 8.14] and the references therein. Applying our main theorem, we can extend [27, Theorem 8.14] to obtain the following corollary.

**Corollary 4.6.** *Let $G$ be a finitely generated acylindrically hyperbolic group. Then the following conditions are equivalent.*

(1) *$G$ has no non-trivial finite normal subgroups.*

(2) *$G$ contains a proper infinite cyclic hyperbolically embedded subgroup.*

(3) *$G$ is ICC. In other words,* $\mathrm{FC}(G) = 1$.

(4) *$G$ is not inner amenable.*

(5) *The amenable radical of $G$ is trivial.*

(6) *The natural action of $G$ on its asymptotic cone* $\mathrm{Cone}_\omega(G, d_n)$ *is faithful for any ultrafilter $\omega$ and a sequence $d_n$.*

(7) *The natural action of $G$ on its asymptotic cone* $\mathrm{Cone}_\omega(G, d_n)$ *is faithful for some ultrafilter $\omega$ and a sequence $d_n$.*

(8) *The induced action of $G$ on the Gromov boundary $\partial X$ is faithful, where $X$ is any $\delta$-hyperbolic space on which $G$ acts non-elementarily and acylindrically.*

(9) *The reduced $C^*$-algebra of $G$ is simple.*

(10) *The reduced $C^*$-algebra of $G$ has a unique normalized trace.*

The equivalence of (1)–(4), (9), and (10) is proved in [27, Theorem 8.14]. Our contribution in the list is the equivalence of (1), (6), and (7), and the equivalence of (1), (3), (5), and (8) is a consequence of Lemma 3.7.

## 5 Applications to other spaces at "infinity"

We devote this section to connecting the natural action on an asymptotic cone and the canonical action on other spaces at "infinity". The main results in this section demonstrate that the kernels of the canonical group action on many spaces at "infinity" are the same as the kernel of the natural action on asymptotic cones. This section covers the limit set of non-elementary convergence group actions, non-trivial Floyd boundary, and various interesting boundaries of a $\mathrm{CAT}(0)$ group. We prove them using acylindrical hyperbolicity and comparing both kernels with the subgroups in Lemma 3.7.

### 5.1 Non-elementary convergence groups case

First, we will consider non-elementary convergence group cases. As we mentioned in the introduction, non-elementary convergence groups are acylindrically hyperbolic [71]. In this subsection, we will relate the kernel of the action on its asymptotic cone to the kernel of a group acting on its limit set. We will briefly recall the notion of convergence group.

The convergence group action was first suggested by Gehring and Martin [37]. They considered only actions on the closed $n$-ball or the $(n-1)$-dimensional sphere, and later, Tukia extended this to actions on general compact Hausdorff spaces [75]. This can be seen as a generalization of the action of the Kleinian group by Möbius transformations, as well as a generalization of the action of non-elementary hyperbolic groups on their Gromov boundary and the action of non-elementary relatively hyperbolic groups on their Bowditch boundary.

First, we define the convergence group action. Our definition adopts the stronger condition, namely, we will consider only group actions on a compact metrizable space.

**Definition 5.1.** Let $M$ be a compact metrizable space and suppose that a group $G$ acts on $M$ by homeomorphisms. This action is called a *convergence action* if, for every infinite distinct sequence of elements $g_n \in G$, there exist a subsequence $g_{n_k}$ and points $x, y \in M$ (not necessarily distinct) satisfying the following:

- for every open set $U$ containing $y$ and every compact set $K \subset M - \{x\}$, there exists $N > 0$ such that, for every $k > N$, $g_{n_k}(K) \subset U$.

In this case, we also say that $G$ is a convergence group or $G$ acts on $M$ as a convergence group, etc.

When $G$ acts on $M$ as a convergence group, there are two subsets of $M$, *the domain of discontinuity* $\Omega(G)$ (also called the *ordinary set*) and the *limit set* $L(G)$. We define the domain of discontinuity, $\Omega(G)$ to be the set of all points in $M$ where $G$ acts properly discontinuously and the limit set is the complement, that is, $L(G) := M - \Omega(G)$. Recall that a group action of $G$ on a topological space $M$ is *properly discontinuous* if, for every compact subset $K \subset M$, the set $\{g \in G : g \cdot K \cap K \neq \emptyset\}$ is finite.

We note that any finitely generated group can be considered as a convergence group with $|L(G)| = 1$ ([36, Example 1.2]). Also, for any group $G$ and discrete topological space $M$ with $|M| = 2$, the trivial action of $G$ on $M$ is a convergence group action. To avoid such trivial cases, we only consider non-elementary convergence groups.

**Definition 5.2.** Let $G$ be a convergence group acting on $M$. If $|L(G)| \geq 3$, then $G$ is said to be *non-elementary*. Otherwise, we say that it is *elementary*.

The following fact is well known.

**Fact 5.3.** Let $G$ be a group acting on a compact metrizable space $M$ with $|M| > 2$, by homeomorphisms, and let $\mathrm{Trip}(M) := \{(x, y, z) : x, y, z \in M, |\{x, y, z\}| = 3\}$ be the set of distinct triples in $M$. Then $G$ is a convergence group if and only if the action of $G$ on $\mathrm{Trip}(M)$ is properly discontinuous.

This fact implies that if a subgroup $H$ of $G$ fixes three distinct points on $M$, then $H$ must be finite. Thus the kernel of the convergence group action is finite. Conversely, any finite normal subgroup $N$ of $G$ acts trivially on the limit set $L(G)$ when $G$ is non-elementary (see [36,52]), so it immediately follows that the kernel of the convergence group action coincides with $K(G)$. Since a non-elementary convergence group is acylindrically hyperbolic [71], applying Corollary 4.4, we have the following corollary.

**Corollary 5.4.** *Let $G$ be a finitely generated group. Suppose that $G$ admits a convergence group action $G \curvearrowright M$ with $|L(G)| > 2$. Then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker\big(G \curvearrowright L(G)\big) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

### 5.2 Groups with non-trivial Floyd boundary case

In this subsection, we investigate another well-understood boundary, the Floyd boundary. This boundary was first suggested by Floyd in [34]. First, we recall the definition. We will use the definition from [40, 49].

**Definition 5.5.** Let $G$ be a finitely generated group and $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ a function, called the *Floyd function*, satisfying the following conditions.

- $f(n) > 0$ for all $n \in \mathbb{N}$ and $f(0) = f(1)$.
- There exists $K \geq 1$ such that
$$1 \leq \frac{f(n)}{f(n+1)} \leq K$$
for all $n$.
- $f$ is summable, that is,
$$\sum_{n=1}^{\infty} f(n) < \infty.$$

Fix a finite generating set $S$ for $G$. Now we construct a new graph $\Gamma_f(G, S)$. Combinatorially, $\Gamma_f(G, S)$ is the same as the Cayley graph $\Gamma(G, S)$. We assign a length of an edge $l$ between two vertices $v, w$ to $f(n)$, where $n$ is the distance between the identity $e$ and the edge $l$ in the Cayley graph. With this new length, we can give a path metric on $\Gamma_f(G, S)$. Namely, for any two vertices $a, b$, we define the Floyd distance by

$$d_f(a, b) = \inf\left\{\sum_{i=1}^{n} f\big(\min\{d(e, p_{i-1}), d(e, p_i)\}\big)\right\},$$

where the infimum is taken over all paths $a = p_0, p_1, \ldots, p_n = b$. The metric completion $\overline{G_f(G, S)}$ is called the *Floyd completion*, and the subspace

$$\partial_F(G, S, f) := \overline{G_f(G, S)} - G_f(G, S)$$

is the *Floyd boundary* of $G$.

We say that $G$ has a *non-trivial Floyd boundary* if $|\partial_F(G, S, f)| > 2$.

**Remark 5.6.** A Floyd boundary depends on the choice of Floyd function $f$, but we will omit $f$ in the notation when there is no confusion. Similarly, we will omit $S$, so we write simply $\partial_F G$.

The Floyd boundary is related to many well-known boundaries. For example, Gromov showed that if $G$ is hyperbolic, then $\partial_F G$ is homeomorphic to the Gromov boundary [41], and Gerasimov proved that if $G$ is relatively hyperbolic, then there is a continuous equivariant map from $\partial_F G$ to $\partial_B G$ (see [39]). They obtained the results by putting $f(n) = \lambda^n$ for some $0 < \lambda < 1$ as a Floyd function. In Floyd's original paper [34], he also showed that, for a given geometrically finite discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, there is a continuous equivariant map from $\partial_F G$ to the limit set $L(G)$, with a Floyd function $f(n) = \frac{1}{n^2+1}$.

It is known that the induced action of $G$ on its non-trivial Floyd boundary is a convergence group action, thanks to A. Karlsson [46]. Combining the fact that the limit set is exactly $\partial_F G$, we have the following.

**Corollary 5.7.** *Suppose that a finitely generated group $G$ has non-trivial Floyd boundary $\partial_F G$. Then*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial_F G) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$$

*for any ultrafilter $\omega$ and sequence $d_n$. Moreover, the kernel* $\ker(G \curvearrowright \partial_F G)$ *is not dependent on the choice of Floyd function.*

### 5.3 CAT(0) groups with rank-one isometries case

In this subsection, we apply our main theorem to CAT(0) spaces. Namely, we will relate the kernel of the action on some boundaries of CAT(0) spaces to the kernel of the action on asymptotic cones. CAT(0) spaces have many classical boundaries including the visual boundary $\partial_v X$ and Morse boundary $\partial_* X$. The definition of the visual boundary is similar to the Gromov boundary of a proper $\delta$-hyperbolic space, but it lacks the desirable properties of the Gromov boundary. Of particular note is that this is not a quasi-isometry invariant [26]. Furthermore, it is proved by Cashen [24] that the Morse boundary is also not a quasi-isometry invariant. This stops the boundary relating to a group. Namely, if $G$ acts geometrically on two CAT(0) spaces $X_1$ and $X_2$, then the visual boundaries $\partial_v X_1$ and $\partial_v X_2$ may be different.

On the other hand, the sublinearly Morse boundary is known to be quasi-isometry invariant, so the sublinearly Morse boundary of a group is well-defined whenever $G$ acts geometrically on a CAT(0) space [65]. When $G$ acts isometrically, properly discontinuously, and cocompactly on $X$, we say that the action is *geometric*.

Suppose that a group $G$ acts geometrically on a proper CAT(0) space $X$ and $X$ has a non-empty sublinearly Morse boundary. Then $G$ contains a rank-one isometry [77, Corollary 3.14]. Also, it is known that if a group $G$ acts properly on a proper CAT(0) space $X$ and $G$ contains rank-one isometry $g$, then $g$ is contained in a virtually cyclic hyperbolically embedded subgroup [70, Definition 1.2, Proposition 3.14 and Theorem 4.7]. Note that this hyperbolically embedded subgroup may not be a proper subgroup, so we can summarize the above as follows.

**Proposition 5.8** ([70, 77]). *Let $G$ be a group acting geometrically on a proper* CAT(0) *space $X$. If $G$ contains a rank-one isometry and $G$ is not virtually cyclic, then $G$ is acylindrically hyperbolic.*

Therefore, based on the assumption above, we can directly use Corollary 4.4. Our first application is to investigate the kernel of actions on the visual boundary. First of all, we briefly recall elementary CAT(0) geometry, an important isometry on CAT(0) spaces called rank-one isometry, and many boundaries of a CAT(0) space including the sublinearly Morse boundary.

**Definition 5.9.** Suppose that $X$ is a geodesic space and $\triangle(p,q,r)$ is a geodesic triangle in $X$ with three vertices $p,q,r \in X$. Let $\overline{\triangle}$ be the (unique) triangle in the Euclidean plane $\mathbb{R}^2$ with the same side lengths. We call it the comparison triangle for $\triangle$.

We say that $\triangle(p,q,r)$ in $X$ satisfies the CAT(0) *inequality* if, for any points $x, y \in \triangle(p,q,r)$ with $x$ and $y$ on the geodesics $[p,q]$ and $[p,r]$, respectively, and if we choose two points $\overline{x}, \overline{y}$ in $\overline{\triangle}$ such that

$$d_X(p,x) = d_{\mathbb{R}^2}(\overline{p},\overline{x}) \quad \text{and} \quad d_X(p,y) = d_{\mathbb{R}^2}(\overline{p},\overline{y}),$$

then we have $d_X(x,y) \le d_{\mathbb{R}^2}(\overline{x},\overline{y})$. We say that $X$ is a CAT(0) *space* if all geodesic triangles in $X$ satisfy the CAT(0) inequality.

In this subsection, we assume that a CAT(0) space is proper. Recall that a metric space $X$ is *proper* if closed balls in $X$ are compact. With properness, a CAT(0) space has more properties, in particular, nearest-point projections. We explain this more precisely. Let $l$ be a geodesic line and $x$ a point in $X$. Then there exists a unique point $p$ on $l$ such that $d_X(x,l) = d_X(x,p)$. We call the point $p$ the *nearest-point projection* from $x$ to $l$, and we denote it by $\pi_l(x)$. Next, we record the definition of the CAT(0) group.

**Definition 5.10.** A group $G$ is said to be a CAT(0) *group* if it acts discretely, cocompactly, and isometrically on some CAT(0) space.

By the definition, we allow a CAT(0) space to be flat, so a free abelian group $\mathbb{Z}^n$ is a CAT(0) group. This is an immediate counterexample for the generalization of the main theorem to general CAT(0) groups. However, we will prove that the generalization holds if a CAT(0) group contains a rank-one isometry. We now give the definition of rank-one isometry.

**Definition 5.11.** Let $X$ be a CAT(0) space.

- An isometry $g$ on $X$ is called *hyperbolic* (or *axial*) if there exists a geodesic line $l: \mathbb{R} \to X$ which is translated non-trivially by $g$, that is, $g(l(t)) = l(t+a)$ for some $a$. The geodesic line $l$ is called an *axis* of $g$.
- A *flat half-plane* means a totally geodesic embedded isometric copy of a Euclidean half-plane in $X$.
- We say that an isometry $g \in \mathrm{Isom}(X)$ is *rank-one* if it is hyperbolic and some (equivalently, every) axis of $g$ does not bound a flat half-plane.

Now we describe various boundaries of CAT(0) space $X$. First, we consider the visual boundary. As mentioned before, its construction is similar to the Gromov boundary of a proper $\delta$-hyperbolic space, namely, we use geodesic lines to define the visual boundary. After establishing the visual boundary, we give a topology on the boundary, called the cone topology. Although there is another well-studied topology on the boundary, that given by the Tits metric, we will not use it, so we will omit any introductory explanation about the Tits metric. Instead, we refer to [15] for readers who are interested in the metric.

**Definition 5.12.** Let $X$ be a complete CAT(0) space. Two geodesic rays

$$c, c': [0, \infty) \to X$$

are asymptotic if $d_X(c(t), c'(t)) < K$ for some $K$ and all $t \geq 0$. Let $\mathcal{R}(X)$ be the set of all geodesic rays and declare two geodesic rays $c, c'$ are equivalent if and only if two geodesics are asymptotic. Then the quotient space $\mathcal{R}(X)/\sim$ is called the *visual boundary* of $X$. We denote the visual boundary of $X$ by $\partial_v X$.

Now suppose that the geodesic ray $c$ and two positive numbers $t, r$ are given. Consider the following subset of $\mathcal{R}(X)$:

$$B(c, t, r) := \{c' : c'(0) = c(0),\ d_X\big(c(t), c'(t)\big) < r\}.$$

Then it follows that $B(c,t,r)$ forms a basis for a topology on $\partial_v X$. We call this topology the *cone topology*.

Next, we introduce the Morse boundary. One of the weaknesses of the visual boundary is that it is not a quasi-isometry invariant. As a result, there have been many attempts to construct a boundary of a CAT(0) space which is quasi-isometry invariant, for several years. In 2015, Ruth Charney and Harold Sultan suggested a new boundary called the Morse boundary (also known as the contracting boundary) [25]. In the paper, we will denote it by $\partial_* X$. The Morse boundary is the subset of the visual boundary $\partial_v X$, consisting of Morse geodesic rays. Recall that a geodesic $c$ is called a *Morse* if, for any constants $A \geq 1$ and $B \geq 0$, there is a constant $M(A,B)$ only depending on $A, B$ such that, for every $(A,B)$-quasi-geodesic $q$ with endpoints on $c$, we have $q$ is in the $M$ neighborhood of $c$. Also, we say that a geodesic $c$ is *$D$-contracting* if, for any points $x, y \in X$,

$$d_X(x,y) < d_X\big(x, \pi_c(x)\big) \quad \text{implies} \quad d_X\big(\pi_c(x), \pi_c(y)\big) < D.$$

Recall that $\pi_c(x)$ is the nearest-point projection. We simply say that a geodesic $c$ is contracting if $c$ is $D$-contracting for some $D$. It is known that a geodesic $c$ is contracting if and only if it is Morse, so we can make a boundary using contracting geodesics. This explains why people use the Morse boundary and the contracting boundary interchangeably.

Maybe the most natural topology on $\partial_* X$ is the subspace topology, but they suggested another topology, using a direct limit to topologize. For more details, we refer to their paper [25].

From the direction of finding the quasi-isometry invariant boundary, Yulan Qing and Kasra Rafi generalize the notion of the Morse geodesic and construct a new boundary called the sublinearly Morse boundary [65]. As the name suggests, they generalize this by allowing the error term to be sublinear. The precise definition is the following.

**Definition 5.13.** A map $\kappa\colon \mathbb{R} \to \mathbb{R}$ is called *sublinear* if $\lim_{t\to\infty} \frac{\kappa(t)}{t} = 0$.

Now fix a function $\kappa\colon [0,\infty) \to [1,\infty)$ that is monotone increasing, concave, and sublinear. For a geodesic line $c$ and a constant $a$, we define the $(\kappa, a)$-neighborhood of $c$ by

$$\mathcal{N}_\kappa(c,a) := \big\{x \in X : d_X(x,c) \leq a\kappa\big(d_X(c(0),x)\big)\big\}.$$

We say that two quasi-geodesics $c_1$ and $c_2$ *$\kappa$-fellow travel* each other if

$$c_1 \in \mathcal{N}_\kappa(c_2,\epsilon), \quad c_2 \in \mathcal{N}_\kappa(c_1,\epsilon)$$

for some $\epsilon$. Then it is known that $\kappa$-fellow traveling is an equivalence relation. Using a sublinear function $\kappa$ and the notion of $\kappa$-fellow traveling, we define a $\kappa$-Morse quasi-geodesic and a $\kappa$-contracting quasi-geodesic as follows.

**Definition 5.14** ([65, Definition 3.6]). Let $\kappa$ be as before. A quasi-geodesic $c$ is called a *(weakly) $\kappa$-Morse* if there is a function $m_c\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ such that if $\alpha\colon [s,t] \to X$ is a $(q, Q)$-quasi-geodesic with endpoints on $c$, then

$$\alpha([s,t]) \subset \mathcal{N}_\kappa\big(c, m_c(q, Q)\big).$$

We remark that the authors of [65] also defined the notion of strongly $\kappa$-Morse quasi-geodesic and $\kappa$-contracting quasi-geodesic [65, Definitions 3.7 and 3.9]. But we omit these definitions since they are all equivalent [65, Theorem 3.10]. Now, we define a sublinearly Morse boundary of $X$.

**Definition 5.15** ([65, Definition 4.1]). Let $X$ be a proper CAT(0) space and let $\kappa$ be as before. The *$\kappa$-Morse boundary* of $X$, denoted by $\partial_\kappa X$, is the set of $\kappa$-fellow traveling classes of $\kappa$-Morse quasi-geodesic rays in $X$.

We can use [65, Theorem 3.10] to define the $\kappa$-Morse boundary of $X$ as the set of $\kappa$-contracting geodesic rays in $X$, up to $\kappa$-fellow traveling. Since distinct geodesic rays do not $\kappa$-fellow travel each other [65, Lemma 3.5], this means that $\partial_\kappa X \subset \partial_v X$.

In order to relate the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ to other boundaries via the kernel of the actions, we use the following remarkable relation between CAT(0) spaces and $\delta$-hyperbolic spaces. In [64], for a given CAT(0) space $X$, they define a new metric $D$ such that the metric space $(X, D)$ is $\delta$-hyperbolic. In addition, they prove many interesting results, for instance, $g$ is rank-one for the action on CAT(0) space if and only if g acts loxodromically on $X_D$ (see [64, Theorem C]). The space $X_D$ is called the *curtain model* of the CAT(0) space $X$ and the construction is quite technical, so we refer to [64] for details of the construction and the astonishing results therein. To obtain the main result of this subsection, we use the following lemma.

**Lemma 5.16.** *Let $G$ be a finitely generated group which acts geometrically on a proper* CAT(0) *space $X$. If $G$ contains a rank-one isometry and $G$ is not virtually cyclic, then* $\ker(G \curvearrowright \partial X_D)$ *is finite.*

*Proof.* By Proposition 5.8, $G$ is acylindrically hyperbolic. Let $N$ be an arbitrary infinite normal subgroup of $G$. We will show that $N$ cannot be $\ker(G \curvearrowright \partial X_D)$, so the lemma follows. By Lemma 2.13, $N$ is also acylindrically hyperbolic. Then, by

Theorem 2.10, there exists a $\delta$-hyperbolic space $Y$ such that $N$ contains at least one WPD (loxodromic) element for the action on $Y$.

It is proved that if $G$ is a CAT(0) group acting on a $\delta$-hyperbolic space $H$ and $g \in G$ is a WPD element for the action on $H$, then $g$ is also WPD for the action on $X_D$ (see [64, Theorem F]). From this result, $N$ contains a WPD element for the action on $X_D$. Recall that a WPD element is loxodromic. To conclude $N \neq \ker(G \curvearrowright \partial X_D)$, we need to show $|\partial X_D| \geq 3$.

Note that, in our setting, $|\partial_v X| = \infty$ by [61, Lemma 16]. Also, $\partial X_D$ is known as a dense subset of $\partial_v X$ (see [64, Theorem L]). Therefore, $\partial X_D$ is infinite and the lemma follows. □

We are now ready to characterize the kernel of CAT(0) groups acting on their asymptotic cones. First, we will show that the kernel is the same as that of the natural action on the visual boundary.

**Corollary 5.17.** *Suppose that a finitely generated group $G$ acts geometrically on a proper* CAT(0) *space $X$. If $G$ contains a rank-one isometry and $G$ is not virtually cyclic, then we have*

$$\ker\bigl(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\bigr) = \ker(G \curvearrowright \partial_v X) = \mathrm{FC}(G) = K(G) = \mathcal{A}(G)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

*Proof.* By Proposition 5.8, $G$ is acylindrically hyperbolic, so we already have

$$\ker\bigl(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\bigr) = \mathrm{FC}(G) = K(G) = \mathcal{A}(G).$$

To conclude the result, it suffices to show that

$$\mathrm{FC}(G) \subset \ker(G \curvearrowright \partial_v X) \quad \text{and} \quad \ker(G \curvearrowright \partial_v X) \subset \ker(G \curvearrowright \partial X_D) \subset K(G),$$

where $X_D$ is the curtain model of $X$.

We show the first inclusion. Choose $g \in \mathrm{FC}(G)$ and $[\gamma] \in \partial_v X$. Since $[\gamma]$ is a geodesic ray and $G$ acts cocompactly on $X$, there exist a compact set $K$ and a sequence $g_n$ in $G$ such that

$$\gamma \subset \bigcup_{k=1}^{\infty} g_k \cdot K.$$

We assume that we choose $g_k$ as minimal, that is, $g_k \cdot K \cap \gamma \neq \emptyset$ for all $k$. Also, we can assume that $\gamma(Ck) \in g_k \cdot K$ for some $C > 0$. Then $g \cdot \gamma(Ck) \in gg_k \cdot K$, so for any $k$, we have

$$\begin{aligned} d_X\bigl(\gamma(Ck), g \cdot \gamma(Ck)\bigr) &\leq \mathrm{diam}(g_k \cdot K \cup gg_k \cdot K) \\ &= \mathrm{diam}(K \cup g_k^{-1} gg_k \cdot K) < ML, \end{aligned}$$

where

$$\operatorname{diam}(K) < M \quad \text{and} \quad L := \max\{\|x^{-1}gx\| : x \in G\}.$$

Recall that $L$ exists since $g \in \mathrm{FC}(G)$. Thus $[\gamma] = g \cdot [\gamma]$ and $g \in \ker(G \curvearrowright \partial_v X)$.

Now consider the second inclusion. By Lemma 5.16, the only remaining thing to prove is

$$\ker(G \curvearrowright \partial_v X) \subset \ker(G \curvearrowright \partial X_D).$$

Choose $g \notin \ker(G \curvearrowright \partial X_D)$; then there exists $[\gamma] \in \partial X_D$ such that $[\gamma] \neq g \cdot [\gamma]$. From [64, Theorem L], there exists an embedding $f : \partial X_D \to \partial_v X$ and this directly implies $g \notin \ker(G \curvearrowright \partial_v X)$. This completes the proof. □

Now we show that the kernel of the action on the other boundaries, sublinearly Morse boundary and Morse boundary, are the same as the kernel of the action on the visual boundary.

**Corollary 5.18.** *Suppose that a finitely generated group $G$ acts geometrically on a proper* CAT(0) *space $X$. Assume that $G$ contains a rank-one isometry and $G$ is not virtually cyclic. The following three kernels are the same for any sublinear function $\kappa$:*

$$\ker(G \curvearrowright \partial_v X) = \ker(G \curvearrowright \partial_\kappa X) = \ker(G \curvearrowright \partial_* X).$$

*Proof.* First, the inclusions

$$\partial_* X \subset \partial_\kappa X \subset \partial_v X \tag{$\star$}$$

follow from [65, Proposition 4.10] and [66, Lemma 2.13].

Now consider the Morse boundary $\partial_* X$ with the cone topology. Murray proved that the Morse boundary is dense in the visual boundary with respect to the cone topology whenever $G$ is not virtually cyclic and $X$ admits a proper cocompact group action [54]. Our setting satisfies all the conditions, so the Morse boundary is a dense subset in the visual boundary. Density ensures that the two kernels

$$\ker(G \curvearrowright \partial_v X), \quad \ker(G \curvearrowright \partial_* X)$$

are the same.

Then the kernel $\ker(G \curvearrowright \partial_\kappa X)$ must be the same as the other two kernels. The result follows from the inclusion ($\star$). □

Recall that the sublinearly Morse boundary depends on the choice of a sublinear function $\kappa$, but Corollary 5.17 tells us that the kernel is independent of the choice of $\kappa$.

**Corollary 5.19.** *Suppose that a finitely generated group $G$ acts geometrically on a proper* CAT(0) *space $X$. Also, assume that $G$ contains a rank-one isometry and that $G$ is not virtually cyclic. Then the kernel* $\ker(G \curvearrowright \partial_\kappa X)$ *is independent of the choice of sublinear function $\kappa$.*

**Remark 5.20.** One might wonder if the following holds: if a CAT(0) group $G$ satisfies

$$\begin{aligned}\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) &= \ker(G \curvearrowright \partial_v X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G),\\ &= \ker(G \curvearrowright \partial_\kappa X) = \ker(G \curvearrowright \partial_* X),\end{aligned}$$

then $G$ contains a rank-one isometry. This is false and counterexamples are easily established using the direct product. In particular, let $G = F_2 \times F_2$; then the assumption is satisfied (all these subgroups are trivial), but $G$ does not contain a rank-one isometry.

As an application, we deduce the result for Coxeter groups. It is worth noting that all Coxeter groups are CAT(0) groups because of the Davis complex [29]. Any finitely generated Coxeter group $W$ can always be decomposed as

$$W = W_F \times W_A \times W_{L_1} \times \cdots \times W_{L_n},$$

where $W_F$ is a finite Coxeter group, $W_A$ is a virtually abelian group, and $W_{L_i}$ are irreducible non-spherical and non-affine Coxeter groups. In the decomposition, each component may be trivial (that is, we allow $W_A = 1$ or $W_{L_1} = 1$ with $n = 1$). Furthermore, it is known which Coxeter groups contain a rank-one isometry, so we can prove the following Coxeter group application.

**Corollary 5.21.** *Let $W$ be a finitely generated Coxeter group and $\Sigma$ the Davis complex of $W$. Decompose $W$ as*

$$W = W_F \times W_A \times W_{L_1} \times \cdots \times W_{L_n}.$$

*If $W_A$ is trivial and $n = 1$, then the following subgroups are all the same for any sublinear function $\kappa$:*

$$\begin{aligned}\ker\big(W \curvearrowright \mathrm{Cone}_\omega(W, d_n)\big) &= \ker(W \curvearrowright \partial_v \Sigma) = K(W)\\ &= \mathrm{FC}(W) = \mathcal{A}(W) = W_F\\ &= \ker(W \curvearrowright \partial_\kappa \Sigma) = \ker(W \curvearrowright \partial_* \Sigma).\end{aligned}$$

*Moreover, we can replace $\Sigma$ with any proper* CAT(0) *space $X$ on which $W$ acts geometrically.*

*Proof.* Recall that $W$ acts geometrically on its Davis complex $\Sigma$. For simplicity, put $W_{L_1} = W_L$, so by assumption, we have $W = W_F \times W_L$. Then $W_L$ does not contain a finite-index subgroup that splits as a direct product of two infinite subgroups ([62, Theorem 4.1 (2)] and that paper uses the terminology *strongly indecomposable*). Since $W_F$ is finite, $W$ also does not contain a finite-index subgroup that splits as a direct product of two infinite subgroups.

From [44, Theorem 1.1], every irreducible Coxeter group is either finite, affine, or contains a rank-one isometry. Thus $W_L$ contains a rank-one isometry. Since $W_L$ is not virtually cyclic, $W_L$ is acylindrically hyperbolic and so is $W$ by [53, Lemma 3.9]. Since $K(W_L) = 1$ (see [62, Proposition 4.3, Assertion 2]), we have $K(W) = W_F$; hence we have completed the proof for the case of $\Sigma$, the Davis complex.

For a general proper CAT(0) space $X$, the result follows immediately from the fact that $W \curvearrowright \Sigma$ contains a rank-one isometry if and only if $W \curvearrowright X$ contains a rank-one isometry [44, Theorem 1.1]. □

## 6 The kernel of Paulin's construction

In this section, we concentrate on another group action on its asymptotic cone. This action was first suggested by Paulin, so nowadays this action is called *Paulin's construction* [63]. One of the advantages of using Paulin's construction is to obtain a group action without a global fixed point. Recall that the natural action $G \curvearrowright \mathrm{Cone}_\omega(G, d_n)$ has a global fixed point $\{e\}$, where $e$ is the identity of $G$. But Paulin's construction has no global fixed point. We denote it by $G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G)$ and we remove $d_n$ from the notation since we cannot choose the sequence $d_n$ freely. The main idea is to use infinitely many automorphisms of $G$, so the condition $|\mathrm{Out}(G)| = \infty$ is required. With this condition, we define a new action of $G$ on an asymptotic cone by $g \mapsto (\{x_n\} \mapsto \{\phi_n(g)x_n\})$ with a sequence $d_n$ defined by

$$d_n := \inf_{g \in G} \max_{s \in S} d_S\big(g, \phi_n(s)g\big).$$

Recall that, when $\phi_n(g) = g$, this is the canonical action on an asymptotic cone. Paulin proved that if $|\mathrm{Out}(G)| = \infty$, then we can choose infinitely many automorphisms $\phi_n \in \mathrm{Aut}(G)$ such that the new action does not have a global fixed point. In fact, Paulin's construction has other advantages. For more details, we refer to Paulin's paper [63].

When Paulin designed the new action, he assumed that $G$ was hyperbolic. The reason is probably to construct a group action on a real tree without a global fixed point (more precisely, a small action). However, the main idea can be extended to general groups with the assumption $|\mathrm{Out}(G)| = \infty$. Indeed, A. Genevois confirmed that, for any group $G$ with $|\mathrm{Out}(G)| = \infty$, $G$ can act on $\mathrm{Cone}_\omega(G, d_n)$ without a global fixed point [38, Section 5.3].

The main purpose of this section is to show that if $G$ is finitely generated acylindrically hyperbolic, then the kernel of Paulin's construction, $\ker(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G))$, is the same as the kernel of the natural action on asymptotic cone, hence all the subgroups that appeared in Corollary 4.4. We start with the following elementary observations.

**Lemma 6.1.** *The kernel* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is a characteristic subgroup.*

*Proof.* Suppose that an automorphism $\phi$ of $G$, a finite generating set $S$ for $G$, and a sequence $d_n$ are given. Then $\phi(S) = \{\phi(s) : s \in S\}$ is also a finite generating set for $G$. Thus there exist $C, D > 0$ satisfying

$$C < \frac{\|\phi(g)\|_S}{\|g\|_S} < D$$

for every $g \in G$. Let $g \in \ker(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G, d_n))$. Then, for any sequence $\{x_k\}$ in $G$ such that $\frac{\|x_k\|_S}{d_k} < \infty$, we have

$$\lim_\omega \frac{\|x_k^{-1} g x_k\|_S}{d_k} = 0.$$

This implies that

$$\lim_\omega \frac{\|\phi(x_k)^{-1}\phi(g)\phi(x_k)\|_S}{d_k} = \lim_\omega \frac{\|\phi(x_k^{-1} g x_k)\|_S}{d_k} = 0.$$

Thus we have $\phi(g) \in \ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$. This completes the proof. □

**Lemma 6.2.** *Let $G$ be a group and suppose that $G$ admits an acylindrical and non-elementary action on a $\delta$-hyperbolic space $X$. Then, for any two independent loxodromic elements $h_1, h_2$, the pointwise stabilizer subgroup* $\mathrm{Stab}([h_1^{\pm n}], [h_2^{\pm n}])$ *is finite.*

*Proof.* Let $N := \mathrm{Stab}([h_1^{\pm n}], [h_2^{\pm n}])$. Then $N$ is a subgroup of a virtually cyclic group $S := \mathrm{Stab}([h_1^{\pm n}])$. Thus there exists a finite index infinite cyclic normal subgroup $M < S$.

Consider $N \cap M$. This intersection is a finite index normal subgroup of $N$ and either $N \cap M$ is trivial or isomorphic to $\mathbb{Z}$. If $N \cap M = 1$, then $N$ is finite, so to deduce a contradiction, assume that $N \cap M$ is isomorphic to $\mathbb{Z}$. Let $g$ be a generator for $N \cap M$. Since $M$ is a finite index subgroup of $S$, $g^{n_1} \in S$ for some $n_1$. Recall that $h_1 \in S$, so $\langle h_1 \rangle$ is a finite index infinite cyclic subgroup of $S$. Thus

$(g^{n_1})^{n_2} \in \langle h_1 \rangle$ for some $n_2$. Thus $g^n = h_1^m$ for some $n, m$, and this means that $g$ is a loxodromic element.

Hence $g \in N$ and $g$ is loxodromic. Since any loxodromic element has exactly two fixed points in $\partial X$, $N$ cannot contain a loxodromic element. This is a contradiction. Therefore, $N$ is a finite subgroup. □

**Lemma 6.3.** *Let $G$ be a group and suppose that $G$ admits an acylindrical and non-elementary action on a $\delta$-hyperbolic space $X$. Then there exist finitely many loxodromic elements $l_1, \dots, l_p \in G$ satisfying* $\ker(G \curvearrowright \partial X) = \mathrm{Stab}([l_1^n], \dots, [l_p^n])$.

*Proof.* The inclusion $\subset$ is trivial. By Lemma 6.2, it follows that there exist four points $[h_1^{\pm n}], [h_2^{\pm n}] \in \partial X$ such that their pointwise stabilizer subgroup, say $N$, is finite. If $N = \ker(G \curvearrowright \partial X)$, we are done. Otherwise, there exist only finitely many elements $g_1, \dots, g_m \in N - \ker(G \curvearrowright \partial X)$. Recall that the set of loxodromic fixed points is dense in $L(G)$ ([28, Corollary 7.4.3]), so for each $1 \le i \le m$, there exists a loxodromic element $k_i$ such that $[k_i^n] \neq g_i[k_i^n]$ in $\partial X$. By the construction of $h_1, h_2$, and $k_1, \dots, k_m$, if $g \notin \ker(G \curvearrowright \partial X)$, then $[l^n] \neq g \cdot [l^n] \in \partial X$ for some $l \in \{h_1^{\pm}, h_2^{\pm}, k_1, \dots, k_m\}$. By taking $\{l_1, \dots, l_p\} := \{h_1, h_2, k_1, \dots, k_m\}$, we complete the proof. □

Using these lemmas, we deduce that if $G$ is acylindrically hyperbolic, then the kernels of these two group actions are the same whenever Paulin's construction is well-defined and we choose suitable representatives.

**Theorem 6.4.** *Let $G$ be a finitely generated acylindrically hyperbolic group whose outer automorphism group* $\mathrm{Out}(G)$ *is infinite. For any sequence* $[\phi_1], [\phi_2], \dots$ *in* $\mathrm{Out}(G)$ *with* $[\phi_i] \neq [\phi_j]$*, there exist automorphism representatives*

$$\phi_1, \phi_2, \dots \in \mathrm{Aut}(G)$$

*such that the kernels of two actions on an asymptotic cone are the same, that is,*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker\big(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G)\big)$$

*for any ultrafilter $\omega$ and sequence $d_n$.*

*Proof.* For simplicity, put

$$N_1 := \ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) \quad \text{and} \quad N_2 := \ker\big(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G)\big).$$

First, we assume

$$d_n = \inf_{g \in G} \max_{s \in S} d_S\big(g, \phi_n(s)g\big);$$

in other words, $d_n$ is the sequence in Paulin's construction. Choose $g \in N_1$. Then, as the kernel $N_1$ is characteristic (Lemma 6.1), $\phi(g)$ is in $N_1$ for any $\phi \in \mathrm{Aut}(G)$.

So, in Paulin's construction, $\phi_n(g) \in N_1$ for all $n$, so we get $g \in N_2$. The inclusion $N_1 \subset N_2$ follows immediately since $N_1$ is finite.

In order to show the opposite inclusion, it suffices to show that $g \notin N_1$ implies $g \notin N_2$. Fix a finite generating set $S$ for $G$ and choose $g \notin N_1$. Then $\phi_i(g) \notin N_1$. From Corollary 4.4, this means that $\phi_i(g) \notin \ker(G \curvearrowright \partial X)$. Here, $X$ is any $\delta$-hyperbolic space on which $G$ acts acylindrically and non-elementarily, so we take $X := \Gamma(G, Y)$ as in [58, Theorems 1.2 and 5.4]. In particular, we may assume $Y$ contains $S$ by [27, Corollary 4.27].

By Lemma 6.3, choose finitely many loxodromic elements $l_1, \ldots, l_k \in G$. Then there is some $j \in \{1, 2, \ldots, k\}$ such that $\{i \in \mathbb{N} : [l_j^n] \neq \phi_i(g) \cdot [h_j^n]\} \in \omega$. For simplicity, we will write $l := l_j$, so $l$ is a loxodromic element and

$$\{i \in \mathbb{N} : [l^n] \neq \phi_i(g) \cdot [l^n]\} \in \omega.$$

Choose constants $C > 0$ and $M \geq 0$ such that $d_X(e, l^k) \geq Ck - M$ for all $k > 0$. From $[l^n] \neq \phi_i(g) \cdot [l^n]$ for $\omega$-almost $i$, we have

$$\lim_{k\to\infty} \big(d_X(\phi_i(g)l^k, e) + d_X(l^k, e) - d_X(\phi_i(g)l^k, l^k)\big) \not\to \infty.$$

This implies that

$$d_X\big(\phi_i(g)l^k, l^k\big) \geq d_X(l^k, e) \geq Ck - M.$$

Since $S \subset Y$, we obtain $\|l^{-k}\phi_i(g)l^k\|_S \geq \|l^{-k}\phi_i(g)l^k\|_X := d_X(l^k, \phi_i(g)l^k)$. Therefore, we get $\|l^{-k}\phi_i(g)l^k\|_S \geq Ck - M$. This means that

$$\lim_\omega \frac{\|l^{-\lfloor d_n \rfloor}\phi_n(g)l^{\lfloor d_n \rfloor}\|_S}{d_n} \geq C.$$

It is obvious that $l^{\lfloor d_n \rfloor}$ is an admissible sequence, that is, $\{l^{\lfloor d_n \rfloor}\} \in \mathrm{Cone}_\omega(G, d_n)$.

Now the only remaining thing to prove is that the ultra limit

$$\lim_\omega \frac{\|l^{-\lfloor d_n \rfloor}\phi_n(g)l^{\lfloor d_n \rfloor}\|_S}{d_n}$$

is finite. To do this, we choose a suitable representative $\phi_n(g)$. Recall that

$$d_n = \inf_{g\in G} \max_{s\in S} d_S\big(g, \phi_n(s)g\big),$$

so for each $n$, there is $x_n \in G$ such that $d_n \leq \max_{s\in S} d_S(x_n, \phi_n(s)x_n) \leq d_n + 1$. In other words,

$$d_n \leq \max_{s\in S} \|x_n^{-1}\phi_n(s)x_n\|_S \leq d_n + 1.$$

Taking $\tau_n(s) := x_n^{-1}\phi_n(s)x_n$, we have

$$d_n \leq \max_{s\in S} \|\tau_n(s)\|_S \leq d_n + 1$$

and $[\phi_n] = [\tau_n]$ in $\mathrm{Out}(G)$. Since $\|\tau_n(s)\|_S \le d_n + 1$ for all $s \in S$, we obtain

$$\begin{aligned} \lim_\omega \frac{\|l^{-\lfloor d_n \rfloor}\tau_n(g) l^{\lfloor d_n \rfloor}\|_S}{d_n} &\le \lim_\omega \frac{2\|l^{\lfloor d_n \rfloor}\|_S}{d_n} + \lim_\omega \frac{\|\tau_n(g)\|_S}{d_n} \\ &\le 2L + \|g\|_S < \infty, \end{aligned}$$

which means that $g \notin N_2$. Thus we conclude that the two kernels coincide when $d_n = \inf_{g\in G} \max_{s\in S} d_S(g, \phi_n(s)g)$.

The general case follows from the fact that the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ is independent of the choice of $d_n$. □

Note that the condition $|\mathrm{Out}(G)| = \infty$ is essential since, for some acylindrically hyperbolic group $G$, $\mathrm{Out}(G)$ is finite. For example, the mapping class group $\mathrm{Mod}(S)$ has a finite outer automorphism group; we refer to [3,45].

We end this section by establishing an example where two kernels are different. We first note that Paulin's construction is also valid for any abelian group $G$ if $G$ satisfies the original conditions, that is, $G$ is finitely generated and $|\mathrm{Out}(G)| = \infty$. From this observation, we can establish an example where

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) \ne \ker\big(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G)\big).$$

For a concrete example, take $G = \mathbb{Z}^2$. Since $G$ is abelian,

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = G$$

for any $\omega$ and $d_n$. Recall that $G$ is finitely generated and $\mathrm{Out}(G) = \mathrm{GL}(2, \mathbb{Z})$, so Paulin's construction for $G = \mathbb{Z}^2$ is well-defined. Since Paulin's construction does not have global fixed points, this means that $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)) \ne G$. More precisely, choose an infinite sequence

$$M_n := \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in \mathrm{Out}(G).$$

Since the inner automorphism group $\mathrm{Inn}(G)$ is trivial, we can consider $M_n$ as a sequence of automorphisms. Let $S := \{e_1 = (1, 0),\ e_2 = (0, 1)\}$ be the canonical generating set for $G$, then $M_n \cdot e_1 = (1, 0)$, $M_n \cdot e_2 = (n, 1)$. From Paulin's construction, we have

$$\begin{aligned} d_n &:= \inf_{g\in G} \max_{s\in S} d_S\big(g, M_n(s)g\big) \\ &= \inf_{g\in G} \max_{s\in S} d_S\big(e, M_n(s)\big) \quad \text{(because } G \text{ is abelian)} \\ &= n + 1 \end{aligned}$$

Thus Paulin's construction gives the action $\{e\} = \{(0,0)\} \mapsto \{M_n(e_2)e\} = \{(n,1)\}$. Since

$$\frac{d_S(e, M_n(e_2)e)}{d_n} = 1,$$

this implies that $\{e\} \neq \{M_n(e_2)e\}$ in $\mathrm{Cone}_\omega(G, d_n)$. This means that $e_2$ is not in $\ker(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G))$, so we obtain

$$e_2 \notin \ker\big(G \overset{p}{\curvearrowright} \mathrm{Cone}_\omega(G)\big) \neq \ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = G.$$

## 7 The kernel of general group actions on their asymptotic cones

We have proved that the kernel must be finite if $G$ is acylindrically hyperbolic. This section is dedicated to the kernel of general groups acting on their asymptotic cones. We study the kernel of various groups and relate the kernel of group actions on asymptotic cones to the elementariness of the group action. It is well known that if $G$ is hyperbolic, then $G$ is either finite, virtually cyclic, or non-elementary. Also, many properties, including both algebraic and geometric ones, are actually equivalent to elementariness. For example, assuming that $G$ is hyperbolic, the following are equivalent.

- $G$ is elementary hyperbolic.
- $G$ is amenable.
- $G$ is of polynomial growth.
- $G$ satisfies a non-trivial law [32].
- $G$ is boundedly generated [33].

The first purpose is to give another equivalent condition for elementariness. To do this, we suggest the following definition and consider the virtually nilpotent group case.

**Definition 7.1.** If $G$ is a finitely generated group for which $K(G)$ exists and satisfies

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G) \qquad (*)$$

for some $\delta$-hyperbolic space $X$, then we say that $G$ satisfies condition (*).

The following proposition implies that we can use the kernel

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big)$$

to classify whether $G$ is a non-elementary hyperbolic group or not.

**Proposition 7.2.** *Let $G$ be a finitely generated virtually nilpotent group. Then* $\mathrm{FC}(G)$ *is infinite, so the kernel* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is also infinite.*

*Proof.* Let $H$ be a nilpotent group with $[G : H] < \infty$. Recall that $H$ is a finitely generated nilpotent group, so $H$ is noetherian. Letting

$$N := \bigcap_{g \in G} g^{-1} H g,$$

we obtain a finite index normal subgroup $N$ which is contained in $H$. Note that $N$ is also a finitely generated nilpotent group.

Consider the center of $N$, $Z(N)$. It is known that the center of a nilpotent group is a normality-large subgroup, that is, the intersection of the center with any non-trivial normal subgroup is non-trivial. From this and the fact that every finitely generated nilpotent group is virtually torsion-free [31, Proposition 13.75 and Corollary 13.81], we can deduce that $Z(N)$ is infinite. More precisely, since $N$ is a finitely generated nilpotent group, $N$ is virtually torsion-free, so we choose a finite index torsion-free subgroup $F$ of $N$. Let

$$S := \bigcap_{g \in G} g^{-1} F g$$

be a normal core of $F$; then $S$ is a finite index torsion-free normal subgroup of $N$. By the normality-large subgroup property, the intersection

$$Z(N) \cap S \neq \{e\}.$$

Since $S$ is torsion-free, the intersection must be infinite; hence $Z(N)$ is infinite.

Choose $a \in Z(N)$. Then $nan^{-1} = a$ for all $n \in N$. Since $N$ is a finite index normal subgroup of $G$, we have that the set $\{gag^{-1} : g \in G\}$ is finite. This means $a \in \mathrm{FC}(G)$. Therefore, $\mathrm{FC}(G)$ is infinite as $Z(N) \subset \mathrm{FC}(G)$. Also, it follows that the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ is infinite from Lemma 3.3. □

Therefore, if $G$ is virtually nilpotent, condition $(*)$ cannot be satisfied regardless of the existence of $K(G)$. If it exists, then by the definition, $K(G)$ is a finite subgroup, so $K(G) \neq \ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$. Thus, in this sense, condition $(*)$ detects properties of the negative curvature of the group.

Recall that an infinite hyperbolic group is either virtually cyclic or non-elementary. Similarly, if an infinite group $G$ acts acylindrically on some $\delta$-hyperbolic space $X$, then either $G$ has a bounded orbit, is virtually cyclic, or is non-elementary [58, Theorem 1.1]. From these facts, we obtain the following results.

**Corollary 7.3.** *Let $G$ be an infinite hyperbolic group. Then $G$ is non-elementary if and only if the kernel* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is finite.*

*Furthermore, suppose that a finitely generated group $G$ acts acylindrically on some $\delta$-hyperbolic space $X$ and contains a loxodromic element. Then the action of $G$ on $X$ is non-elementary (so $G$ is acylindrically hyperbolic) if and only if* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ *is finite.*

Motivated by Proposition 7.2, one might wonder whether a solvable group can have trivial kernel. In the class of solvable groups, there is an example satisfying condition ($*$) excluding only the amenable radical. One such example is the Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n > 1$. Recall that this group has group presentation $\mathrm{BS}(1,n) = \langle a, t : tat^{-1} = a^n \rangle$.

First, it is well known that $\mathrm{BS}(1,n)$ is solvable (actually, $\mathrm{BS}(1,n) = \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$) and torsion-free, so we have $K(G) = 1$ for $G = \mathrm{BS}(1,n)$. Next, we show that $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)) = 1$. This directly implies that all three subgroups, $K(G)$, $\mathrm{FC}(G)$, and the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ are trivial.

In order to show this, we need an asymptotic metric on $\mathrm{BS}(1,n)$. Since every element $x \in \mathrm{BS}(1,n)$ can be uniquely expressed as $x = t^{-l} a^N t^m$, where $l, m \geq 0$ and $n \nmid N$ if $l, m > 0$, the following lemma gives full asymptotic metric information.

**Lemma 7.4** ([19, Proposition 2.1]). *There exist constants $C_1, C_2, D_1, D_2 > 0$ such that, for every element $x = t^{-l} a^N t^m$ of* $\mathrm{BS}(1,n)$*, $N \neq 0$, we have*

$$C_1(l + m + \log|N|) - D_1 \leq \|x\| \leq C_2(l + m + \log|N|) + D_2,$$

*where $\|x\|$ is the word metric with respect to the generators $a, t$.*

**Lemma 7.5.** *Let $G =$* $\mathrm{BS}(1,n)$*. Then the kernel* $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_k))$ *is trivial.*

*Proof.* Choose a non-trivial element $g \in \mathrm{BS}(1,n)$. Using the normal form, we can write $g = t^{-l} a^N t^m$ for some $l, m, N$, and let $x_k = t^{-l_k} a^{N_k} t^{m_k}$. Then we have

$$\begin{aligned} x_k^{-1} g x_k &= (t^{-m_k} a^{-N_k} t^{l_k}) t^{-l} a^N t^m (t^{-l_k} a^{N_k} t^{m_k}) \\ &= t^{-m_k} a^{-N_k} t^{l_k - l} a^N t^{m - l_k} a^{N_k} t^{m_k}. \end{aligned}$$

Obviously, the rightmost part is not the normal form, so we need to compute more. By using the relation $ta = a^n t$, we have $t^A a^B = a^{n^A B} t^A$, so this formula gives us the following equalities:

$$\begin{aligned} t^{-m_k} a^{-N_k} t^{l_k - l} a^N t^{m - l_k} a^{N_k} t^{m_k} &= t^{-m_k} a^{-N_k} (t^{l_k - l} a^N) t^{m - l_k} a^{N_k} t^{m_k} \\ &= t^{-m_k} a^{-N_k} (a^{n^{l_k - l} N} t^{l_k - l}) t^{m - l_k} a^{N_k} t^{m_k} \end{aligned}$$

$$
\begin{aligned}
&= t^{-m_k} a^{n^{l_k - l} N - N_k} t^{m-l} a^{N_k} t^{m_k} \\
&= t^{-m_k} a^{n^{l_k - l} N - N_k} (t^{m-l} a^{N_k}) t^{m_k} \\
&= t^{-m_k} a^{n^{l_k - l} N - N_k} (a^{n^{m-l} N_k} t^{m-l}) t^{m_k} \\
&= t^{-m_k} a^{n^{l_k - l} N + n^{m-l} N_k - N_k} t^{m_k + m - l}.
\end{aligned}
$$

The last expression

$$
t^{-m_k} a^{n^{l_k - l} N + n^{m-l} N_k - N_k} t^{m_k + m - l} \tag{§}
$$

looks like the normal form. Putting $l_k = l$ and $N_k = N$, the expression (§) would be $t^{-m_k} a^{n^{m-l} N} t^{m_k + m - l}$. Note that this is not a normal form. But since

$$
t^{m-1} a^N = a^{n^{m-1} N} t^{m-1},
$$

we have

$$
\begin{aligned}
t^{-m_k} a^{n^{m-l} N} t^{m_k + m - l} &= t^{-m_k} (a^{n^{m-l} N} t^{m-1}) t^{m_k - l + 1} \\
&= t^{-m_k} (t^{m-1} a^N) t^{m_k - l + 1} \\
&= t^{-m_k + m - 1} a^N t^{m_k - l + 1}.
\end{aligned}
$$

Now take $m_k = \lfloor d_k \rfloor$. Then we can easily check that the expressions

$$
x_k = t^{-l} a^N t^{\lfloor d_k \rfloor} \quad \text{and} \quad x_k^{-1} g x_k = t^{-\lfloor d_k \rfloor + m - 1} a^N t^{\lfloor d_k \rfloor - l + 1}
$$

are the normal forms, at least for sufficiently large $d_k$. Thus, by Lemma 7.4, we have

$$
\lim_{k \to \infty} \frac{\|x_k\|}{d_k} = \alpha,
$$

$$
\lim_{k \to \infty} \frac{\|x_k^{-1} g x_k\|}{d_k} = \lim_{k \to \infty} \frac{\|t^{-\lfloor d_k \rfloor + m - 1} a^N t^{\lfloor d_k \rfloor - l + 1}\|}{d_k} = \beta
$$

and $C_1 \le \alpha \le C_2$ and $2C_1 \le \beta \le 2C_2$ (here, $C_1, C_2$ are constants in Lemma 7.4). This means that $\{x_k\}$ is admissible, that is, $\{x_k\} \in \mathrm{Cone}_\omega(G, d_k)$ and

$$
g \notin \ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_k)\big).
$$

Since $g \neq 1$ is arbitrary, the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_k))$ is trivial. □

Now we concentrate on group actions of $\mathrm{BS}(1, n)$ on $\delta$-hyperbolic spaces. Recall that $\mathrm{BS}(1, n)$ is finitely generated solvable but not virtually nilpotent, so it has a finite index subgroup which admits a focal action on some $\delta$-hyperbolic space

[6, Proposition 3.7]. Indeed, all the cobounded actions of $\mathrm{BS}(1,n)$ on $\delta$-hyperbolic space are known in [1]. Among these actions, we consider the action of $\mathrm{BS}(1,n)$ on $\mathbb{H}^2$ via the representation $\mathrm{BS}(1,n) \to \mathrm{PSL}(2,\mathbb{R})$ given by

$$a \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad t \mapsto \begin{bmatrix} \sqrt{n} & 0 \\ 0 & \frac{1}{\sqrt{n}} \end{bmatrix}.$$

Then it follows that $\ker(G \curvearrowright \partial X)$ is trivial, letting $G = \mathrm{BS}(1,n)$ and $X = \mathbb{H}^2$. Combining this with Lemma 7.5, we obtain the following result.

**Proposition 7.6.** *There exists a finitely generated solvable group $G$ such that*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial X) = K(G) = \mathrm{FC}(G) = 1$$

*for any ultrafilter $\omega$ and sequence $d_n$, where $X$ is some $\delta$-hyperbolic space and the action $G \curvearrowright X$ is cobounded.*

The following example shows that, ruling out the condition of actions of a group $G$ on a $\delta$-hyperbolic space $X$, there exists a group satisfying condition ($*$) excluding the kernel of the action on $\partial X$.

**Proposition 7.7.** $G = \mathrm{SL}(3,\mathbb{Z})$ *satisfies the following equalities:*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G) = 1.$$

*Proof.* Margulis' normal subgroup theorem tells us that, for any normal subgroup $N$ of $G = \mathrm{SL}(3,\mathbb{Z})$, either $N$ is trivial, or $N$ is of finite index. Obviously, we have $K(G) = 1$ and $\mathcal{A}(G) = 1$ since $G$ is not amenable ($G$ contains a rank 2 free group) and any virtually amenable group is again amenable.

Since $\mathrm{FC}(G) \subset \ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$, in order to complete the proof, it suffices to prove that the kernel $\ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$ is not a finite index subgroup of $G$.

For simplicity, put $N := \ker(G \curvearrowright \mathrm{Cone}_\omega(G, d_n))$, and we shall prove that $N$ cannot be of finite index. For $i \neq j$, let $e_{i,j}$ be the matrix in $G = \mathrm{SL}(3,\mathbb{Z})$ which has 1 on the main diagonal and $(i,j)$ component and 0 otherwise. Then the following statements are well known.

- For distinct $1 \leq i, j, k \leq 3$, $[e_{i,j}^M, e_{j,k}^N] = e_{i,k}^{MN}$.
- There exists a constant $C > 0$ such that $\frac{1}{C}\log(M) \leq \|e_{1,2}^M\| \leq C\log(M)$ (see [51, (2.14)]).

Thus there exists some $D > 0$ such that

$$\frac{1}{D}\log(M) \le \|e_{i,j}^{M}\| \le D\log(M)$$

for every $i \neq j$. Now put $g = e_{1,2}$ and $x_n = e_{2,3}^n$. Then, for $k \in \mathbb{Z} - \{0\}$, we have $\|[g^k, x_n]\| = \|e_{1,3}^{kn}\|$, so we have

$$\frac{1}{D}\log(kn) \le \|[g^k, x_n]\| \le D\log(kn).$$

However, it also holds that $\frac{1}{D}\log(n) \le \|x_n\| \le D\log(n)$. Therefore, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{\|x_n^{-1}g^k x_n\|}{\|x_n\|} &= \lim_{n\to\infty}\frac{\|g^{-k}x_n^{-1}g^k x_n\|}{\|x_n\|} = \lim_{n\to\infty}\frac{\|[g^k, x_n]\|}{\|x_n\|}\\ &\ge \lim_{n\to\infty}\frac{1}{D^2}\frac{\log(k)+\log(n)}{\log(n)} = \frac{1}{D^2} > 0.\end{aligned}$$

The above means that $g^k \notin N$ for every $k \in \mathbb{Z}$, unless $k = 0$. But if $N$ is a finite index subgroup of $G$, then for every $a \in G$, $a^M \in N$ for some $M > 0$. This implies that $N$ cannot be a finite index subgroup. Now the result $N = 1$ follows from Margulis' normal subgroup theorem. □

We note that $\mathrm{SL}(3,\mathbb{Z})$ does not admit any interesting actions on a $\delta$-hyperbolic space $X$ (see [4, 43]). If Property (NL) denotes No Loxodromics, as in [6], then $\mathrm{SL}(3,\mathbb{Z})$ satisfies Property (NL). The Thompson group $T$ is another example; $T$ also satisfies Property (NL) [6].

**Proposition 7.8.** *Thompson group $T$ satisfies the following equalities*

$$\ker\big(T \curvearrowright \mathrm{Cone}_\omega(T, d_n)\big) = K(T) = \mathrm{FC}(T) = \mathcal{A}(T) = 1.$$

*Proof.* Recall that $T$ is simple [23, Theorem 5.8], so in order to show the equalities hold, we just check that these normal subgroups are not the whole group $T$. First, the simpleness and infiniteness of $T$ imply that $K(T) = 1$. In [42], we have $\mathrm{FC}(T) = 1$. Since $T$ is not amenable [42], we have $\mathcal{A}(T) = 1$, so it only remains to show that the kernel $\ker(T \curvearrowright \mathrm{Cone}_\omega(T, d_n))$ is trivial.

It is well known that $T$ has a normal form [23, Theorem 5.7] and its word metric property [18, Theorem 5.1]. Let $x_0, x_1 \in T$ be two generators for $T$. Then it follows directly from [18, Theorem 5.1] that there is some $C > 0$ such that

$$\frac{n}{C} \le \|x_0^{-n}\| \le Cn \quad \text{and} \quad \frac{2n+1}{C} \le \|x_0^n x_1 x_0^{-n}\| \le (2n+1)C$$

(recall that $x_0^{-n}$ and $x_0^n x_1 x_0^{-n}$ are already normal forms). Thus

$$0 < \frac{2n+1}{C^2 n} \leq \frac{\|x_0^n x_1 x_0^{-n}\|}{\|x_0^{-n}\|} \leq \frac{C^2(2n+1)}{n}.$$

This means that $\{x_0^{-n}\} \neq x_1 \cdot \{x_0^{-n}\}$ in the asymptotic cone $\mathrm{Cone}_\omega(T, n)$, so for a general sequence $d_n$, replacing $x_0^{-n}$ with $x_0^{-\lfloor d_n \rfloor}$, we can conclude that $x_1$ is not in the kernel $\ker(T \curvearrowright \mathrm{Cone}_\omega(T, d_n))$. Since $T$ is simple, the kernel

$$\ker\bigl(T \curvearrowright \mathrm{Cone}_\omega(T, d_n)\bigr)$$

is trivial. □

We conclude this section with another interesting example and a question that arises from the example. There exists a group $G$ that acts on $\delta$-hyperbolic space and satisfies all our equalities, condition $(*)$, but $G$ itself is not acylindrically hyperbolic. The braided Thompson group BV serves as an example. This group was first described by Brin [16] and Dehornoy [30], independently. We refer to these two articles for more details. In this paper, we use the "tree-braid-tree" form for describing elements in BV. Let $\mathcal{I}$ be the set of triples $(T_1, \sigma, T_2)$, where $T_1$ and $T_2$ are binary trees with $n$ leaves and $\sigma$ is a braid with $n$ strings. In this paper, we only consider a *binary tree* that is rooted, finite, and planar. We say that a vertex $v$ in a binary tree is a *leaf* if its degree is 1. In other words, $v$ is a leaf if and only if $v$ is connected to only one vertex via an edge.

We will define an equivalence relation called expansion in [35]. Recall that there exists a surjective group homomorphism $\pi_n \colon B_n \to S_n$ from the braid group to the symmetric group, for all $n > 1$. We call the binary tree with 3 vertices and 2 leaves the caret. Let $(T_1, \sigma, T_2) \in \mathcal{I}$ and suppose that $T_1$ and $T_2$ have $n$ leaves. For $1 \leq k \leq n$, the $k$-th expansion of the triple $(T_1, \sigma, T_2)$ is $(T_1', \sigma', T_2')$, where $T_1'$ is a new binary tree obtained from $T_1$ by adding the caret to the $k$-th leaf of $T_1$. Similarly, $T_2'$ is a binary tree obtained from $T_2$ by adding the caret to the $\pi_n(k)$-th leaf of $T_2$, and $\sigma'$ is a braid in $B_{n+1}$ obtained from $\sigma \in B_n$ by bifurcating the $k$-th string into two parallel strings. Now we say that two triples $(T_1, \sigma_1, T_2), (T_1', \sigma_2, T_2')$ in $\mathcal{I}$ are equivalent if one is the $k$-th expansion of the other for some $k$. Rigorously, it is not an equivalence relation. It is neither reflexive nor transitive. But by considering the equivalence relation generated by this binary relation, we obtain the equivalence relation, and as a set, BV is the set of all equivalence classes. To describe an element of BV, we abuse the notation, namely, we denote it by a triple $(T_1, \sigma, T_2)$.

Now we define a group operation on BV. For two given elements $(T_1, \sigma_1, T_2)$, $(T_1', \sigma_2, T_2')$, let $U$ be a binary tree formed from finitely many consecutive expansions of both $T_2$ and $T_1'$. Then, up to the equivalence class, these two elements are expressed as $(\tilde{T}_1, \tilde{\sigma}_1, U), (U, \tilde{\sigma}_2, \tilde{T}_2')$. Define

$$(T_1, \sigma_1, T_2) \circ (T_1', \sigma_2, T_2') = (\tilde{T}_1, \tilde{\sigma}_1, U) \circ (U, \tilde{\sigma}_2, \tilde{T}_2') := (\tilde{T}_1, \tilde{\sigma}_1\tilde{\sigma}_2, \tilde{T}_2').$$

Then the binary operation $\circ$ is well-defined and BV with the binary operation $\circ$ is a group.

Also, it is known that there exists a surjection $\pi\colon \mathrm{BV} \twoheadrightarrow V$. This group homomorphism is induced by the group homomorphisms $\pi_n\colon B_n \to S_n$ and the kernel is denoted by $bP$ [35], $PBV$ [13], and $P_{br}$ [78]. However, we denote it by $\mathrm{PB}_\infty := \ker(\pi)$ since it is a well-known fact that the kernel is isomorphic to the pure braid group with infinite strings $\mathrm{PB}_\infty$. Finally, we remark that $\mathrm{PB}_\infty$ is the set of all triples $(T, \rho, T)$ where $\rho$ is a pure braid.

Now we will establish that BV is a counterexample by proving the following series of lemmas.

**Lemma 7.9.** *The braided Thompson group* BV *is not acylindrically hyperbolic.*

*Proof.* Recall that BV contains the pure braid group with infinite strings $\mathrm{PB}_\infty$ as a normal subgroup [2, Remark 10.6] and [20, Corollary 3.2]. It suffices to check that $\mathrm{PB}_\infty$ is not acylindrically hyperbolic.

Recall that if $G$ is acylindrically hyperbolic and $g \in G$ is loxodromic, then the centralizer $C(g)$ is virtually cyclic. Hence, by the definition of acylindrical hyperbolicity, whenever $G$ is acylindrically hyperbolic, there exists some $g \in G$ whose centralizer $C(g)$ is virtually cyclic. To show that $\mathrm{PB}_\infty$ is not acylindrically hyperbolic, we will show that, for any $x \in \mathrm{PB}_\infty$, the centralizer $C(x)$ cannot be virtually cyclic.

Choose $x \in \mathrm{PB}_\infty$. Then $x \in \mathrm{PB}_k$ for some $k \in \mathbb{N}$. Let $\{\sigma_n : n \in \mathbb{N}\}$ be the natural generator for $B_\infty$. Then $\sigma_l^2 \in \mathrm{PB}_\infty$ and $\sigma_l^2 \in C(x)$ for all $l > k$. This implies that $C(x)$ is not virtually cyclic. Further, $C(x)$ contains $\mathbb{Z}^n$ for all $n \in \mathbb{N}$, and actually, $C(x)$ contains a subgroup isomorphic to $\mathrm{PB}_\infty$. Since $x$ is arbitrary, there is no $g \in \mathrm{PB}_\infty$ for which $C(g)$ is virtually cyclic, so $\mathrm{PB}_\infty$ is not acylindrically hyperbolic. □

The braided Thompson group BV acts on a $\delta$-hyperbolic space called the ray graph. The ray graph $R$ was first designed by Danny Calegari on his blog [21], and obviously, the mapping class group of $\mathbb{R}^2 - C$, denoted by $M$, acts on the ray graph, where $C$ stands for the Cantor set. He suggested this space as the analog of the complex of curves. Later J. Bavard ([7] and English translation [8]) showed that the ray graph has infinite diameter and is $\delta$-hyperbolic. Also, it is known that $M$ admits two independent loxodromic elements. It is known that BV is a subgroup of $M$ (see [35]), so clearly, BV also acts on the ray graph $R$ which is $\delta$-hyperbolic.

Our next lemma shows that the mapping class group $M$ acts faithfully on the Gromov boundary of $R$. The Gromov boundary of $R$ is described in [9], and for the sake of completeness, we briefly introduce their results and notation. By one-point compactification, we can consider $\mathbb{R}^2 - C$ as $S := S^2 - (C \cup \{\infty\})$. A *short ray* is a simple arc embedded in $S$ connecting $\infty$ to a point on $C$, up to isotopy. The *ray graph* is the graph whose vertices are short rays, with an edge between two short rays whenever they are disjoint up to isotopy.

To define the notion of long ray, they used a specific covering $\tilde{S}$ of $S$, called the conical cover. We refer to [9, Section 2.2] for details about this covering. Let $\pi\colon \tilde{S} \to S$ be the covering map. Let $\tilde{\gamma}$ be a geodesic ray from $\tilde{\infty}$ to $\partial\tilde{S}$, and $\gamma = \pi(\tilde{\gamma})$. We say that $\gamma$ is a *long ray* if $\gamma$ is simple but neither a short ray nor a loop. Here, a loop means an embedding $\alpha\colon [0,1] \to S$ with $\alpha(0) = \alpha(1) = \infty$. Like the ray graph $R$, they define the following "extended" ray graph as follows. Its vertex set consists of short rays, long rays, and loops. They are connected by an edge whenever they can be realized disjointly. This new graph is called the *completed ray graph*, and we denote it by $\mathcal{R}$.

**Definition 7.10** (High-filling rays [9, Definition 2.7.1]). Let $l$ be a long ray. We say that

- $l$ is *loop-filling* if it intersects every loop,
- $l$ is *ray-filling* if it intersects every short ray,
- $l$ is *$k$-filling* if there exists a short ray $l_0$ and long rays $l_1, \ldots, l_k = l$ such that $l_i$ is disjoint from $l_{i+1}$ for all $i$, and $k$ is minimal,
- $l$ is *high-filling* if it is ray-filling but not $k$-filling for any $k \in \mathbb{N}$.

A *clique* is simply a complete subgraph. The notable result in [9] is the following description of $\partial R$.

**Theorem 7.11** ([9, Theorem 5.4.3]). *Let $\mathcal{E}$ be the set of cliques of high-filling rays in $\mathcal{R}$. Then there exists a homeomorphism $F\colon \mathcal{E} \to \partial R$.*

Before proving the lemma, we introduce the final ingredient used for our proof, called the simple circle. The notion of the simple circle was introduced in [22], and this circle is related to the conical cover $\tilde{S}$ and the conical circle $S^1_C := \partial\tilde{S}$ which can be identified with the circle $S^1$. D. Calegari and L. Chen provided many interesting results including the relationship between the conical circle and the simple circle, as well as the rigidity theorem [22, Theorem 6.1]. For further details, we refer to [22]. In this paper, we record only the construction of the simple circle.

For the construction, we begin with short rays and long rays. Let $R_S$ and $R_L$ be the set of short rays and long rays in $S$, respectively. Then, from [22, Theorem 3.4], the union $R_S \cup R_L$ is a Cantor set in the conical circle $S^1_C$.

**Definition 7.12** (The simple circle [22, p. 8]). The simple circle $S^1_S$ is the quotient of the conical circle $S^1_C$ obtained by collapsing every complementary interval of $R_S \cup R_L$ to a point.

**Lemma 7.13.** *Let $R$ be the ray graph. Then the kernel* $\ker(\mathrm{BV} \curvearrowright \partial R)$ *is trivial.*

*Proof.* Recall that we can consider BV as a subgroup of the mapping class group of $\mathbb{R}^2 - C$, so we will show that $\ker(M \curvearrowright \partial R)$ is trivial. The result follows immediately from this.

Let $x \in M$ be an element fixing $\partial R$ pointwise. Then, by [9, Theorem 5.4.3], $x$ preserves all cliques of high-filling rays. In particular, there exist cliques that contain only one high-filling ray (see [9, Section 2.7.1]). Thus $x$ must fix all such high-filling rays.

Note that the set of all such high-filling rays is dense in the simple circle. This follows from the fact that $M$ acts minimally on the simple circle [22, Theorem 3.4]. Recall that such cliques are $M$-invariant and the simple circle contains all high-filling rays.

Therefore, $x$ acts trivially on the simple circle. Since the simple circle also contains all short rays, $x$ acts trivially on the ray graph $R$. Since the mapping class group $M$ acts faithfully on $R$, we conclude that $x$ is trivial. Therefore, $\ker(M \curvearrowright \partial R) = 1$. □

It is known that BV is finitely presented ([16] and [13, Theorem 3.1]), so its asymptotic cone is well-defined. The following lemma shows that the kernel of the action of BV on asymptotic cones is trivial.

**Lemma 7.14.** *The kernel* $\ker(\mathrm{BV} \curvearrowright \mathrm{Cone}_\omega(\mathrm{BV}, d_n))$ *is trivial for any ultrafilter $\omega$ and sequence $d_n$.*

*Proof.* For simplicity, put $N := \ker(\mathrm{BV} \curvearrowright \mathrm{Cone}_\omega(\mathrm{BV}, d_n))$ and fix an ultrafilter $\omega$ and a sequence $d_n$. The asymptotic word metric on BV is well known due to [17, Theorem 3.6].

First, recall that BV contains Thompson's group $F$. Every element in $F$ can be represented as two binary tree diagrams with the same number of leaves. By adding the trivial braid between two diagrams, every element in $F$ is also an element in BV. For example, two generators $A, B$ for $F$ in [23] can be expressed in the tree-braid-tree form depicted in Figure 1.

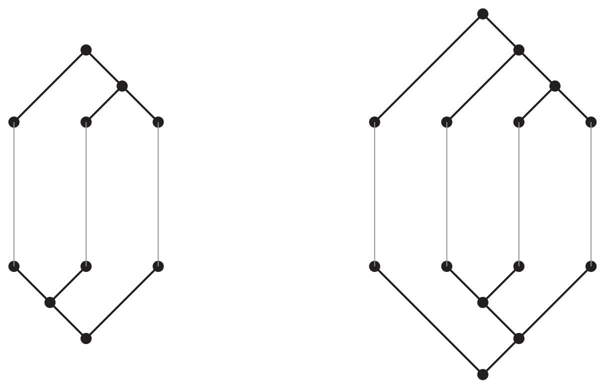

Figure 1. Two generators $A$, $B$ for $F$ which are contained in BV. The vertical lines in the middle (gray thin lines) represent the trivial braid.

Consider $X_n := A^{-(n-1)}BA^{n-1}$. Then it is obvious that $X_n$ has $n+3$ leaves and there is no crossing (in the tree-braid-tree form representing $X_n$, the braid part must be trivial). Also, $A^n$ has $n+2$ leaves and there is no crossing in the braid part. By [17, Theorem 3.6], we have

$$C_1 n \le \|A^n\| \le C_2 n + D_1, \quad C_1 n \le \|X_n\| \le C_2 n + D_2$$

for some $C_1, C_2, D_1, D_2 > 0$. Thus these two inequalities imply that

$$C_1 \le \lim_{\omega} \frac{\|X_{\lfloor d_n \rfloor}\|}{d_n} \le C_2.$$

Since $\{A^{\lfloor d_n \rfloor}\}$ is admissible (that is, $0 < C_1 \le \lim_{\omega} \|A^{\lfloor d_n \rfloor}\|/d_n \le C_2 < \infty$), we have $B \notin N$.

So the kernel $N$ is a proper subgroup of BV. By [78, Corollary 2.8], $N$ is contained in $\mathrm{PB}_\infty$ which is the kernel of the surjective homomorphism $\pi\colon \mathrm{BV} \twoheadrightarrow V$.

To complete the proof, we will prove that, for any $g \in \mathrm{PB}_\infty$ with $g \neq 1$, $g \notin N$. This means $N = 1$, so we are done. Choose $g \in \mathrm{PB}_\infty$. Then recall that its tree-braid-tree form is $(T, \rho, T)$, where $T$ is a binary tree diagram with $n$ leaves and $\rho$ is a pure braid with $n$ strings. Since $g \neq 1$, we may assume $\rho \neq 1$. We will use the technique given in [35, Section 2.2]. To describe this technique, we need to define the *right depth* ([35, Definition 2.9]), the homomorphism $\chi_1$ ([35, Definition 2.10]), and so on. We briefly introduce these concepts which are slightly modified for our purpose. More precisely, the only difference is the domain; originally, they defined these for $\widehat{bV}$, but we will define these for a more general case. We refer to [35, Section 2.2] for an explicit definition of $\widehat{bV}$ and for the original definitions.

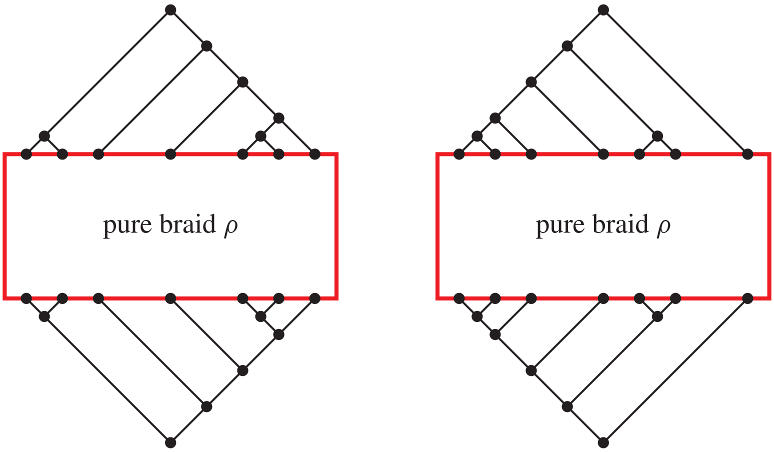


Figure 2. An example for Claim 1. The left diagram represents $x = (T, \rho, T)$, where $T$ has right depth 4. The right diagram is the tree-braid-tree form of $A^{-3}xA^3$, so $A^{-3}xA^3 \in K_1$.

Let $P$ be the subgroup of all elements of the form $(T_1, \beta, T_2)$, where $\beta$ is a pure braid. For a given binary tree $T$, the right depth of $T$ is the distance from its rightmost leaf to its root, by giving length 1 at each edge. Using right depth, we can define a group homomorphism $\chi\colon P \to \mathbb{Z}$ by letting $\chi((T_1, \beta, T_2))$ be the right depth of $T_1$ minus the right depth of $T_2$. It is easy to show that this is not well-defined if we consider $\chi$ as a homomorphism on the whole group BV, but by restriction to $P$, it is well-defined. Moreover, $\mathrm{PB}_\infty$ is contained in the intersection $P \cap \ker(\chi)$.

Let $K_1 \subset P$ be the subgroup of elements that can be represented in the form $(T_1, \beta, T_2)$, where both $T_1, T_2$ have right depth 1.

**Claim 1.** *For any $x \in \mathrm{PB}_\infty$, there exists some $p \geq 0$ such that $A^{-p}xA^p \in K_1$.*

*Proof of Claim 1.* The proof is actually the same as the proof of [35, Lemma 2.12]. Pick $x \in \mathrm{PB}_\infty$; then its tree-braid-tree form is $(T, \rho, T)$. If $T$ admits a representative with right depth 1, there is nothing to prove. Suppose that $T$ has right depth $r > 1$. Then $A^{-(r-1)}xA^{r-1} \in K_1$. We give an example in Figure 2.

Since $T$ has right depth $r$, the number of leaves in $T$ is at least $r+1$. Note that $A^{r-1}$ has the form $(T_1, 1, T_2)$, where the right depths of $T_1, T_2$ are $r$ and 1, respectively. Since $A^{-(r-1)}$ has the form $(T_2, 1, T_1)$, first we compute $A^{-(r-1)}x$. To do this, we find an expansion $E$ of both $T_1$ and $T$. Since the right depths of $T_1$ and $T$ are both $r$, we can choose an expansion $E$ whose right depth is also $r$.

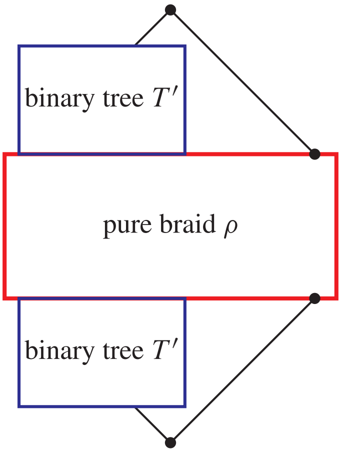


Figure 3. An element $x = (T, \rho, T) \in \mathrm{PB}_\infty \cap K_1$.

Indeed, we just put $E = T$; in other words, $T$ is an extension of $T_1$. Thus, using such an extension, the tree-braid-tree form of $A^{-(r-1)}x$ is $(T_2', \rho, T)$. We note that the right depth of $T_2'$ is 1. Similarly, the tree-braid-tree form of $A^{-(r-1)}xA^{r-1}$ is $(T_2', \rho, T_2')$, but $T_2'$ has right depth 1. With this, we are done. □

**Claim 2.** *For any $x \in \mathrm{PB}_\infty \cap K_1$ with $l$ leaves, the number of leaves of $A^{-n}xA^n$ is exactly $l + n$.*

*Proof of Claim 2.* We will prove that if $x \in \mathrm{PB}_\infty \cap K_1$ has $l$ leaves, then $A^{-1}xA$ is also in $\mathrm{PB}_\infty \cap K_1$ and has $l + 1$ leaves. Then the proof is completed by induction.

First, since $\mathrm{PB}_\infty$ is a normal subgroup of BV, $A^{-1}xA \in \mathrm{PB}_\infty$. The fact that $A^{-1}xA \in K_1$ and it has $l + 1$ leaves follows from the operation on BV. Choose $x \in \mathrm{PB}_\infty \cap K_1$. Then its tree-braid-tree form must be as in Figure 3.

Since $T$ has $l$ leaves, $T'$ has $l - 1$ leaves. Here, we assume that $T'$ is reducible, that is, $T'$ is not an extension of any binary tree $T''$ with less than $l - 1$ leaves. Then the element $A^{-1}xA$ can be expressed as in Figure 4.

Here, the pure braid $\tilde{\rho}$ can be constructed from $\rho$ as follows. First, recall that $\rho \in \mathrm{PB}_l$ is a pure braid. Let $s$ be the $l$-th string (the right-most string in the picture). Now we consider $s$ as two parallel strings, that is, we bifurcate the $l$-th string into two parallel strings. Then the resulting braid is an element of $\mathrm{PB}_{l+1}$. This braid is $\tilde{\rho}$. From its tree-braid-tree form, $A^{-1}xA \in K_1$ and it has $l + 1$ leaves. This completes the proof. □

Now we will prove that $N = 1$. Recall $g \in \mathrm{PB}_\infty$. Then, by Claim 1, we have $A^{-p}gA^p \in K_1$. Assume that $A^{-p}gA^p$ has $l$ leaves. Then, by Claim 2, it follows

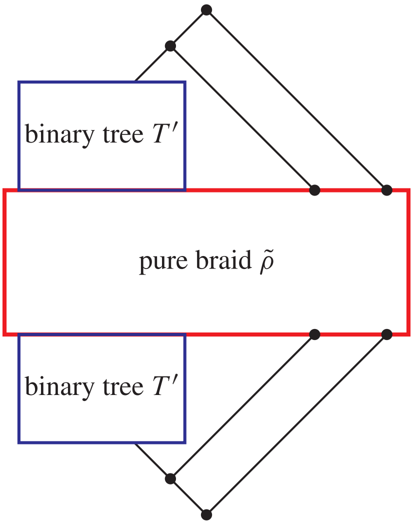


Figure 4. Tree-braid-tree form of $A^{-1}xA$.

that $A^{-n}(A^{-p}gA^{p})A^{n}$ has $l+n$ leaves. By [17, Theorem 3.6], we obtain

$$\|A^{-(n+p)}gA^{n+p}\| \geq C_1(l+n)$$

for some $C_1 > 0$. Also, recall that

$$C_2 n \leq \|A^n\| \leq C_3 n + C_4$$

for some $C_2, C_3, C_4 > 0$. Therefore, we get

$$0 < C_1 \leq \lim_{\omega} \frac{\|A^{-(\lfloor d_n \rfloor + p)} g A^{\lfloor d_n \rfloor + p}\|}{d_n} \leq \lim_{\omega} \frac{\|A^{\lfloor d_n \rfloor + p}\|}{d_n} \leq 2C_3 < \infty.$$

Since $\{A^{\lfloor d_n \rfloor + p}\}$ is admissible, we show that $g \notin N$. Since $g \neq 1$ is arbitrary, we get $N = 1$. □

**Lemma 7.15.** *The amenable radical of* BV*,* $\mathcal{A}(\mathrm{BV})$ *is trivial.*

*Proof.* Recall that $\mathrm{PB}_\infty$ is the kernel of the natural surjection $\pi: \mathrm{BV} \twoheadrightarrow V$. By [78, Corollary 2.8], $\mathcal{A}(G)$ is contained in $\mathrm{PB}_\infty$ since $\mathcal{A}(G) \neq G$. Since any amenable normal subgroup of $G$ is also an amenable normal subgroup of $\mathrm{PB}_\infty$, we deduce that $\mathcal{A}(G) \subset \mathcal{A}(\mathrm{PB}_\infty)$. Thus we will show that $\mathcal{A}(\mathrm{PB}_\infty)$ is trivial.

To justify this, we need to claim that the pure braid group with $n$ strings, $\mathcal{A}(\mathrm{PB}_n)$ is isomorphic to $\mathbb{Z}$ for all $n > 1$. We use mathematical induction. When

$n = 2$, the braid group $B_2$ is isomorphic to $\mathbb{Z}$ and so is $\mathrm{PB}_2$. Thus $\mathcal{A}(\mathrm{PB}_2) = \mathbb{Z}$. Now assume that $\mathcal{A}(\mathrm{PB}_{n-1})$ is isomorphic to $\mathbb{Z}$. Recall that there exists a short exact sequence

$$1 \to F_{n-1} \to \mathrm{PB}_n \xrightarrow{p} \mathrm{PB}_{n-1} \to 1,$$

where $F_{n-1}$ is the rank $n-1$ free group. This induces

$$1 \longrightarrow F_{n-1} \cap \mathcal{A}(\mathrm{PB}_n) \longrightarrow \mathcal{A}(\mathrm{PB}_n) \xrightarrow{p|_{\mathcal{A}(\mathrm{PB}_n)}} p(\mathcal{A}(\mathrm{PB}_n)) \longrightarrow 1.$$

Consider $F_{n-1} \cap \mathcal{A}(\mathrm{PB}_n)$. Recall that $F_{n-1}$ and $\mathcal{A}(\mathrm{PB}_n)$ are both normal in $\mathrm{PB}_n$. Thus $F_{n-1} \cap \mathcal{A}(\mathrm{PB}_n)$ is a normal subgroup of both $F_{n-1}$ and $\mathcal{A}(\mathrm{PB}_n)$. Thus the intersection $F_{n-1} \cap \mathcal{A}(\mathrm{PB}_n)$ is amenable and normal in $F_{n-1}$. This means that $F_{n-1} \cap \mathcal{A}(\mathrm{PB}_n)$ is trivial. Now we have

$$1 \longrightarrow \mathcal{A}(\mathrm{PB}_n) \xrightarrow{p|_{\mathcal{A}(\mathrm{PB}_n)}} p(\mathcal{A}(\mathrm{PB}_n)) \longrightarrow 1,$$

and equivalently, $\mathcal{A}(\mathrm{PB}_n)$ is isomorphic to $p(\mathcal{A}(\mathrm{PB}_n))$, but

$$p(\mathcal{A}(\mathrm{PB}_n)) \subset \mathcal{A}(\mathrm{PB}_{n-1}) = \mathbb{Z},$$

so $p(\mathcal{A}(\mathrm{PB}_n))$ is isomorphic to either 1 or $\mathbb{Z}$. But the center of $\mathrm{PB}_n$ is $\mathbb{Z}$ and the amenable radical must contain the center. Therefore, $\mathcal{A}(\mathrm{PB}_n)$ is isomorphic to $\mathbb{Z}$.

Now we will prove that $\mathcal{A}(\mathrm{PB}_\infty)$ is trivial. Suppose not, and choose

$$1 \neq x \in \mathcal{A}(\mathrm{PB}_\infty).$$

By definition, $x \in \mathrm{PB}_k$ for some $k$. We choose $k$ to be minimal, so we may assume that $x \notin \mathrm{PB}_l$ for all $l < k$ but $x \in \mathrm{PB}_l$ for all $l \geq k$. Note that, for any $n \in \mathbb{N}$, the intersection $\mathcal{A}(\mathrm{PB}_\infty) \cap \mathrm{PB}_n$ is an amenable normal subgroup of $\mathrm{PB}_n$. Then, for any $m \geq k$, the intersection $\mathcal{A}(\mathrm{PB}_\infty) \cap \mathrm{PB}_m$ is an amenable normal subgroup of $\mathrm{PB}_m$ containing $x$. By the claim above, the intersection is isomorphic to $\mathbb{Z}$ since the intersection contains $x \neq 1$.

Consider the intersection $\mathcal{A}(\mathrm{PB}_\infty) \cap \mathrm{PB}_{k+2}$. This contains $x$ and $x \in \mathrm{PB}_k$, so in braid representation, $x$'s $k+1$ and $k+2$ strings are just vertical lines. Since the intersection is isomorphic to $\mathbb{Z}$ and it contains the center $Z(\mathrm{PB}_{k+2})$, $x^N \in Z(\mathrm{PB}_{k+2})$ for some $N$. This is a contradiction. Therefore, $\mathcal{A}(\mathrm{PB}_\infty)$ is trivial and $\mathcal{A}(G)$ is also trivial. □

These lemmas can be summarized as follows.

**Proposition 7.16.** *Let $G := \mathrm{BV}$ be the braided Thompson group and $R$ the ray graph. Then $K(G)$ exists and* BV *satisfies the equalities*

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial R) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G) = 1$$

*for any ultrafilter $\omega$ and sequence $d_n$. However,* BV *is not acylindrically hyperbolic.*

It is still unknown whether the group action of BV on the ray graph $R$ contains two independent loxodromics. Motivated by the above result and question, we pose the following question.

**Question 7.17.** Let $G$ be a finitely generated group and suppose that $G$ acts isometrically on some $\delta$-hyperbolic space $X$ and $G$ contains two independent loxodromic elements. If $K(G)$ exists and the equalities

$$\ker\big(G \curvearrowright \mathrm{Cone}_\omega(G, d_n)\big) = \ker(G \curvearrowright \partial X) = K(G) = \mathrm{FC}(G) = \mathcal{A}(G)$$

hold for any ultrafilter $\omega$ and sequence $d_n$, then must $G$ be acylindrically hyperbolic?

If the action of BV on the ray graph admits two independent loxodromic elements, then the answer is automatically negative. If the question is answered affirmatively, then this gives another characterization for being acylindrically hyperbolic.

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