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Visual right-angled Artin subgroups of two-dimensional right-angled Coxeter groups

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Abstract. There is a procedure, due to Dani and Levcovitz, for taking a finite simplicial graph Γ and a subgraph Λ of its complement, checking some conditions, and, if satisfied, producing a graph Δ such that the right-angled Artin group with presentation graph Δ is a finite index subgroup of the right-angled Coxeter group with presentation graph Γ. They do not tell us how to find Λ, given Γ. We show, in the 2-dimensional case, that the existence of such a Λ is connected to the graph property of satellite-dismantlability of Γ, and we use this connection to give an algorithm for producing a suitable Λ or deciding that one does not exist.

1 Introduction

Every right-angled Artin group (RAAG) is a finite index subgroup of a right-angled Coxeter group (RACG), and given the presentation graph Δ of the RAAG A_Δ, there is a simple graph operation that turns it into a graph Γ that is the presentation graph of the RACG supergroup W_Γ (see [8]). The converse is not true; RACGs are more varied, and there are invariants such as divergence that show that some RACGs are not even quasiisometric to a RAAG. So what graph conditions on Γ imply that W_Γ is commensurable to a RAAG?

Consider the case that Γ is a square. Each pair of diagonal vertices generate an infinite dihedral group, and these two dihedral groups commute. Each of the dihedral groups has an index two, infinite cyclic subgroup, and these make an index four, $\mathbb{Z}^2$ subgroup of W_Γ. This is the basic example of a finite index *visual* RAAG subgroup; it is "visual" in the sense that we "see" the RAAG generators as pairs of non-adjacent vertices of Γ, and they commute when the vertex pairs from Γ make a square.

This situation generalizes as follows: let Λ be a subgraph of the complement graph Γ^c of Γ, that is, Γ^c has the same vertex set as Γ, and has an edge if and only if Γ does not. Edges of Λ give pairs of generators of W_Γ that generate an infinite

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dihedral subgroup. Let Δ be the graph with one vertex for each edge of Λ, and an edge between two vertices if the corresponding subgroups commute, which is the case exactly when they span a square in Γ. There is a homomorphism from A_Δ to W_Γ given by sending a generator of A_Δ to the product in W_Γ of the two endpoints of the corresponding edge of Λ. In general, however, this homomorphism is not injective, nor does it have finite index image. Based on initial results of LaForge [12], Dani and Levcovitz [5] give conditions on Λ that determine whether the natural homomorphism is injective when Λ has at most two connected components. In the 2-dimensional case, when Γ is triangle-free, visual RAAG subgroups only occur for Λ with at most two components [5, Lemma 4.7], and in this case, they give necessary and sufficient conditions on Λ for A_Δ to be a finite index subgroup of W_Γ.

We focus on the 2-dimensional case, and call a $\Lambda < \Gamma^c$ satisfying their conditions a *finite index Dani–Levcovitz* Λ *(FIDL-*Λ*)*. Checking that a given subgraph of Γ^c is a FIDL-Λ is algorithmic, and since Γ is finite, one can simply enumerate subgraphs of Γ^c and check them all. This is slow, even for rather small examples. We are interested in a more efficient algorithm for starting from Γ alone and either producing a FIDL-Λ or deciding that one does not exist.

We give such an algorithm as Global Search Algorithm 5.3. The key step, Theorem 4.7, is that a FIDL-Λ exists if and only if Γ admits a satellite-dismantling sequence that reduces it to a square and satisfies some additional conditions that can be checked only from Γ. We apply this algorithm to a large number of examples via computer computations in forthcoming work.

2 Preliminaries

2.1 Graphs

The *join* of A and B is denoted by $A * B$, that is, the complete bipartite graph with one part the elements of A and the other the elements of B. A graph is *complete* if, for every pair of vertices, there exists an edge between them. It is *incomplete* if there exist two vertices that are not joined by an edge. The empty graph and a graph consisting of a single vertex are complete. A *clique* is a complete subgraph. A graph Γ is *separated by a clique* if there is a clique C such that $\Gamma - C$ has more than one connected component. A disconnected graph is separated by the empty clique. The *link* $\mathrm{lk}(v)$ of a vertex v in a graph is the induced subgraph on its neighboring vertices. The *star* $\mathrm{st}(v)$ of a vertex v is $\{v\} * \mathrm{lk}(v)$. A *loop* is an edge path that starts and ends at the same vertex, and a *cycle* is a loop that has no repeated vertices. A set of vertices V of Γ, or the vertex set of a subgraph, are said to *span* or *induce* the maximal subgraph of Γ with V as its vertex set. A subgraph of Γ is *induced* or *full* if it spans itself. Given a cycle γ, an *n-chord*, or just *chord* when $n = 1$, is a path of length n between vertices x and y of γ such that both subsegments of γ between x and y have length greater than n.

Vertices $v \neq w$ are *twins* if $\mathrm{lk}(v) = \mathrm{lk}(w)$, and v is a *satellite of* w if $v \neq w$ and $\mathrm{lk}(v) \subset \mathrm{lk}(w)$, so v is a satellite of each of its twins, but not of itself. A vertex is a *satellite* if it is a satellite of some vertex. A graph Γ is *satellite-dismantlable to a square* if there exists a sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_n$ such that $\Gamma_i - \Gamma_{i+1}$ is a single satellite and Γ_n is a square. This is reminiscent of the more common graph-theoretic notion of a *dismantlable graph*, in which a vertex v is *dominated by* w if $\mathrm{st}(v) \subset \mathrm{st}(w)$, and a graph is dismantlable if it is possible to reduce it to a single vertex by removing one dominated vertex at a time.

2.2 RACGs and RAAGs

See [7] and [4, Section 2.6] for background on RACGs and RAAGs.

Definition 2.1. The *right-angled Coxeter group (RACG)* W_Γ defined by a finite, simplicial graph Γ is

$$W_\Gamma = \langle s \in \Gamma \mid s^2 = 1 \text{ for all } s \in \Gamma,\ st = ts \text{ if } (s,t) \in \mathrm{Edges}(\Gamma) \rangle.$$

The *right-angled Artin group (RAAG)* A_Δ defined by a finite, simplicial graph Δ is

$$A_\Delta = \langle m \in \Delta \mid mn = nm \text{ if } (m,n) \in \mathrm{Edges}(\Delta) \rangle.$$

The graphs are called the *presentation graphs*[1].

Definition 2.2. If W_Γ is a RACG and Υ is an induced subgraph of Γ, then the subgroup of W_Γ generated by vertices of Υ is called a *special subgroup*. It is a RACG with presentation graph Υ, and is denoted $W_{V(\Upsilon)}$ or W_Υ. The analogous statement and terminology also applies to RAAGs.

We will restrict to one-ended groups. A RACG is one-ended if and only if its presentation graph is incomplete and has no separating clique [7, Theorem 8.7.2]. A RAAG is one-ended if and only if its presentation graph is connected and has at least two vertices. To further simplify the set-up, we consider only RACGs and RAAGs whose Davis and Salvetti complexes, respectively, are two-dimensional. This is satisfied for both RACGs and RAAGs if the presentation graph is triangle-free, in addition to the above conditions for one-endedness.

[1] This is different from the conventions used to define the *Coxeter graph*, which is more commonly used for not-necessarily right-angled Coxeter groups.

There is a quasiisometry invariant known as *divergence*. In particular, if a group has polynomial divergence, then the degree of the polynomial is a quasiisometry invariant. RAAGs have at most quadratic divergence [1, Corollary 4.8], thus so does every group quasiisometric to a RAAG. By work of Dani and Thomas [6, Theorem 1.1], a one-ended, two-dimensional RACG has at most quadratic divergence if and only if its presentation graph has a property known as $\mathcal{CFS}$ (component of full support/constructed from squares). This was generalized to higher dimension in [2, Definition 1.3].

Definition 2.3. The *diagonal graph* $\boxslash(\Gamma)$ of Γ is the graph whose vertices are diagonals of induced squares in Γ, with $\{a,b\}$ and $\{c,d\}$ connected by an edge if $\{a,b\} * \{c,d\}$ is an induced square in Γ.

The *support* $\operatorname{supp}(\{a,b\})$ of a vertex $\{a,b\}$ of $\boxslash(\Gamma)$ is the pair of vertices $\{a,b\}$ in Γ. The support of a subset of $\boxslash(\Gamma)$ is the union of the supports of its vertices.

When Γ is triangle-free and has no separating cliques,[2] it is $\mathcal{CFS}$ if $\boxslash(\Gamma)$ has a connected component whose support is all of Γ.

The graph Γ is *strongly* $\mathcal{CFS}$ if it is $\mathcal{CFS}$ and $\boxslash(\Gamma)$ is connected.

Remark. The usual definition of $\mathcal{CFS}$ uses a graph $\square(\Gamma)$ whose vertices are induced squares of Γ, with an edge between two vertices if they intersect in a diagonal. The graphs $\square(\Gamma)$ and $\boxslash(\Gamma)$ carry the same information, but $\boxslash(\Gamma)$ is topologically simpler, since many squares intersecting in a common diagonal form a star in $\boxslash(\Gamma)$ but a clique in $\square(\Gamma)$. The diagonal graph is also more natural for our purposes because, when we have Γ with a FIDL-Λ, then the commuting graph Δ of Λ sits as a subgraph in $\boxslash(\Gamma)$; see Remark 2.6.

Theorem 2.4 ([6]). *If Γ is an incomplete, triangle-free graph without separating cliques such that W_Γ is quasiisometric to a RAAG, then Γ is $\mathcal{CFS}$.*

Lemma 2.5. *If Γ is incomplete, triangle-free, and $\mathcal{CFS}$, then it has no separating clique.*

Proof. Take $a, b \in \Gamma$. If a and b are the diagonal of some square, then they are not separated by a clique. Otherwise, there is a path $\{p_0, q_0\}, \ldots, \{p_n, q_n\}$ in the full support component of $\boxslash(\Gamma)$ with $n > 0$, $p_0 = a$, and $p_n = b$. This path corresponds to a chain of squares $\{p_i, q_i\} * \{p_{i+1}, q_{i+1}\}$ in Γ, with successive squares sharing a diagonal. The union of the squares is not separated by a clique. □

[2] In the general case, Γ is $\mathcal{CFS}$ if $\boxslash(\Gamma)$ has a connected component whose support is all non-cone vertices of Γ. If Γ is triangle-free and not a star, then it has no cone vertices.

2.3 Dani–Levcovitz conditions

Let $\Theta = \Theta(\Gamma, \Lambda)$ be the graph with vertex set Γ, with edges from both Γ and $\Lambda < \Gamma^c$. The edges coming from Γ are Γ-*edges*, and the edges coming from Λ are Λ-*edges*. Similarly, a path consisting only of Γ-edges is a Γ-path, etc. The Λ-hull, hull_Λ, of a subset of vertices of Θ is the vertex set of their convex hull in Λ. A set of vertices is Λ-convex if it is equal to its Λ-hull.

Dani and Levcovitz [5] give *subgroup conditions* $\mathcal{R}_1$–$\mathcal{R}_4$ to determine that the RAAG A_Δ on the commuting graph Δ associated to Λ is a visual RAAG subgroup of W_Γ. They give *index conditions* $\mathcal{F}_1$ and $\mathcal{F}_2$ to ensure that the visual RAAG subgroup is of finite index in W_Γ. They show in the 2-dimensional case that it always suffices to find Λ with two components, and for two-component Λ, their conditions are necessary and sufficient.

The conditions are as follows, simplified by specializing to the case that Γ is an incomplete, triangle-free graph without separating cliques. Let $\Lambda_{\mathtt{r}}$ (red) and $\Lambda_{\mathtt{b}}$ (blue) be disjoint, connected subgraphs of Γ^c, with $\Lambda = \Lambda_{\mathtt{r}} \sqcup \Lambda_{\mathtt{b}}$.

- $\mathcal{R}_1$: $\Lambda_{\mathtt{r}}$ and $\Lambda_{\mathtt{b}}$ are trees.
- $\mathcal{R}_2$: $\Lambda_{\mathtt{r}}$ and $\Lambda_{\mathtt{b}}$ are induced subgraphs of Θ.
- $\mathcal{F}_1$: Λ spans Γ.

These conditions are true if and only if Γ is bipartite, with a bicoloring $\mathtt{r}/\mathtt{b}$ (every vertex is colored either $\mathtt{r}$ or $\mathtt{b}$, and adjacent vertices have different colors), and $\Lambda_{\mathtt{r}}$ and $\Lambda_{\mathtt{b}}$ are trees in Γ^c spanning the $\mathtt{r}$ and $\mathtt{b}$ parts, respectively. We will not state $\mathcal{F}_2$. In our case, it is always satisfied if $\mathcal{R}_2$ and $\mathcal{F}_1$ are [5, Remark 4.3]. Assuming these conditions, we can state the remaining two conditions in simplified form.

- $\mathcal{R}_3$: If $\{a,b\} * \{c,d\}$ is a square in Γ, then $\text{hull}_\Lambda\{a,b\} * \text{hull}_\Lambda\{c,d\} \subset \Gamma$.
- $\mathcal{R}_4$: If $a \multimap b$ is an edge in a cycle γ of Γ, then there is a square $\{a,a'\} * \{b,b'\}$ with $a', b' \in \text{hull}_\Lambda(\gamma)$.

Remark 2.6. Notice that the assumption that Γ is incomplete with no separating clique implies W_Γ is 1-ended, so A_Δ is 1-ended, so Δ is connected and has more than one vertex. Thus, every edge of Λ is a diagonal of a square in Γ, since otherwise it would have nothing to commute with, so would give an isolated vertex in Δ. Thus, we may identify Δ with a subgraph of $\boxslash(\Gamma)$.

One nice application of these conditions in [5] is to connect them to conditions given by Nguyen and Tran [15] on deciding when a *planar* graph Γ defines a RACG that is quasiisometric to a RAAG. The conclusion is that, for planar Γ,

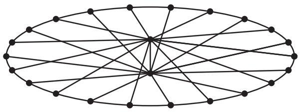
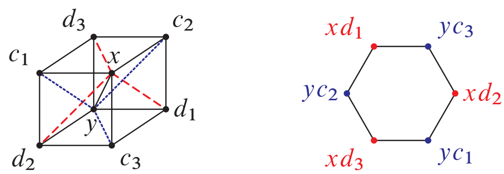


Figure 1. A bicycle wheel.



(a) $\Theta(\Gamma, \Lambda)$ with $\Lambda_{\mathtt{r}}$ = - - -, $\Lambda_{\mathtt{b}}$ = ········ (b) Δ

Figure 2. Example of Γ, a 2-component FIDL-Λ and the commuting graph Δ of its edges for Γ the 1-skeleton of a 3-cube with one space diagonal.

W_Γ being quasiisometric to a RAAG implies graph conditions that imply $\mathcal{R}_1$–$\mathcal{R}_4$ and $\mathcal{F}_1$ and $\mathcal{F}_2$, so W_Γ actually has a finite index visual RAAG subgroup, which happens always to be defined by a tree Δ. Dani and Levcovitz also give two families of non-planar graphs to which their conditions apply and yield Δ that are not trees. We mention one of these families here.

Example 2.7. A *bicycle wheel* is a graph consisting of adjacent vertices x and y, the "hub", a circle of even length $2n \geq 6$ given by $c_1, d_1, c_2, \ldots, d_n$, the "rim", and edges from each c_i to x, and from each d_i to y, the "spokes".

A bicycle wheel admits a 2-component FIDL-Λ consisting of the opposite of the spokes: one star consisting of an edge from x to each d_i and another consisting of an edge from y to each c_i. The commuting graph Δ is a circle of the same length as the rim.

In Figure 2, the case $n = 3$ is also recognizable as the 1-skeleton of a 3-cube with one space diagonal.

2.4 Splittings of RACGs

We have mentioned that W_Γ is one-ended when it is incomplete with no separating clique. This corresponds to not having a splitting as an amalgamated product over a finite group. The next simplest splittings are over two-ended, or virtually $\mathbb{Z}$, groups.

A *JSJ decomposition* of a finitely presented group is a certain maximal graph of groups decomposition (see [11] for the precise definition) with two-ended edge groups and vertex groups in three categories: two-ended, *hanging* or *rigid*. A hanging vertex group is essentially the fundamental group of a surface with boundary and a rigid vertex group does not split any further with respect to its incident edge groups. JSJ decompositions are not unique, but there is a way to encode all of them simultaneously in a *JSJ graph of cylinders*. It is the canonical representative for the deformation space of JSJ decompositions of the group and it can be used to deduce quasiisometry invariants, by [3]. The idea is that some of the two-ended edge and vertex groups in a JSJ decomposition may be commensurable, and these can be grouped together to form *cylinders*, and from this, a new decomposition is derived. For RACGs, all of this is visible in the presentation graph: Mihalik and Tschantz [14] show a one-ended, two-dimensional RACG W_Γ admits a splitting over a two-ended subgroup if and only if Γ has a *cut* $\binom{a}{b}$.

Definition 2.8. If Γ is an incomplete and triangle-free graph without separating cliques, a *cut* $\binom{a}{b}$ means one of the following, both of which have the property that the element $ab \in W_\Gamma$ generates an infinite cyclic subgroup that is finite index in $W_{\binom{a}{b}}$.

- A *cut pair* $\binom{a}{b} = \{a, b\}$: a pair of non-adjacent vertices such that $\Gamma - \{a, b\}$ is not connected.
- A 2-*path cut triple* $\binom{a}{b} = \{a, b, c\}$: a triple of vertices with $c \in \operatorname{lk}(a) \cap \operatorname{lk}(b)$ such that $\Gamma - \{a, b\}$ is connected but $\Gamma - \{a, b, c\}$ is not.

Definition 2.8 implies that every component of $\Gamma - \binom{a}{b}$ contains a neighbor of each vertex in $\binom{a}{b}$.

A cut pair $\{a, b\}$ is *crossed* by another, disjoint, cut pair $\{c, d\}$ if a and b lie in different connected components of $\Gamma - \{c, d\}$. A cut $\binom{a}{b} = \{a, b, c\}$ is *crossed* by a cut $\binom{d}{e} = \{d, e, f\}$ if c is equal to f and a and b lie in different connected components of $\Gamma - \{d, e, c\}$. Cut pairs and 2-path cut triples do not cross each other. To see this, suppose $a - c - b$ is a 2-path separating Γ and $\{x, y\}$ is a cut pair such that $\{a, b, c\}$ has vertices in different complementary components of $\{x, y\}$. Without loss of generality, $c = x$ and $y \notin \{a, b\}$. Let z be a vertex in a different complementary component of $\{a, b, c\}$ than y. There are no separating cliques, and $\{a, c\}$ is a clique, so $\{a, b, c\}$ separates z from y, but $\{a, c\}$ does not, so there is a shortest path γ'_b from z to y that goes through b and avoids a and c. Let γ_b be the initial segment of γ'_b that ends at b. Similarly, let γ'_a be a shortest path from z to y that goes through a and avoids b and c, and let γ_a be the initial segment ending at a. The concatenation of the inverse of γ_a with γ_b is a path from a to b

that avoids y and $c = x$, contradicting the supposition that a and b are in different complementary components of $\{x, y\}$.

A cut that is not crossed by any other cut is *uncrossed*. Crossing cuts are responsible for hanging vertices in the JSJ decomposition, but RAAGs do not have these [13], so they cannot appear in groups quasiisometric to RAAGs.

Theorem 2.9 ([9, Theorem 3.29]). *Let Γ be an incomplete, triangle-free graph without separating cliques. If the JSJ graph of cylinders of W_Γ has no hanging vertices, it consists of the following.*

- *For every pair $\{a, b\}$ such that there is an uncrossed cut $\binom{a}{b}$, there is a cylinder vertex with vertex group $W_{\{a,b\}\cup(\mathrm{lk}(a)\cap\mathrm{lk}(b))}$.*
- *For every set B of essential (valence at least 3) vertices in Γ satisfying the following conditions, there is a rigid vertex with vertex group W_B.*
  - (B1) *No cut separates B.*
  - (B2) *The set B is maximal among all sets satisfying (B1).*
  - (B3) $|B| \geq 4$.

Furthermore, a pair of vertices is connected by an edge if and only if the pair consists of a cylinder vertex and a rigid vertex whose vertex groups intersect in a subgroup containing the two-ended cut defining the cylinder. The edge group is the intersection of its vertex groups.

3 FIDL-Λ convexity

If Λ is a forest and x and y are in the same component, let $[x, y]_\Lambda$ denote the unique Λ-geodesic joining them.

Lemma 3.1. *Let Γ be an incomplete, triangle-free graph with no separating clique that admits a FIDL-Λ. Let $\binom{v}{v'}$ be a cut of Γ. Every component of $\Gamma - \binom{v}{v'}$ contains a common neighbor of v and v', that is, a single vertex that is adjacent to both v and v'.*

Proof. Pick a component of $\Gamma - \binom{v}{v'}$; let b be a vertex in that component, and let a be a vertex from a different component. By Theorem 2.4, we can choose a geodesic $\{p_0, q_0\}, \dots, \{p_n, q_n\}$ in $\boxslash(\Gamma)$ such that $a \in \{p_0, q_0\}$ and $b \in \{p_n, q_n\}$. Let i_0 be the least index such that $\{p_{i_0}, q_{i_0}\}$ contains a vertex, say p_{i_0}, in the same component of $\Gamma - \binom{v}{v'}$ as b.

If $i_0 = 0$, then $q_{i_0} = a$ and there is a square with one diagonal containing vertices in different components of the cut. This is only possible if the cut $\binom{v}{v'}$ is a cut pair $\{v, v'\}$ and the square is $\{a, p_0\} * \{v, v'\}$, in which case p_0 is a common neighbor of v and v' in the b-component.

If $i_0 > 0$, then $\{p_{i_0-1}, q_{i_0-1}\} * \{p_{i_0}, q_{i_0}\}$ is a square with p_{i_0} in the b-component and with p_{i_0-1} and q_{i_0-1} non-adjacent vertices that are both adjacent to p_{i_0}, but neither of which are in the same component of $\Gamma - \binom{v}{v'}$ as p_{i_0}. Then $\{p_{i_0-1}, q_{i_0-1}\} = \{v, v'\}$. □

Lemma 3.2. *Let* Γ *be an incomplete, triangle-free graph with no separating clique that admits a FIDL-*Λ. *Let* $\binom{v}{v'}$ *be a cut of* Γ. *Then* Λ *contains an edge between* v *and* v'.

Proof. By Lemma 3.1, it is possible to choose vertices a and b that are common neighbors of v and v' and contained in different components of $\Gamma - \binom{v}{v'}$. Every common neighbor of a and b must lie in the cut, so the triangle-free condition implies the only common neighbors of a and b are v and v'.

Now, $\{a, b\} * \{v, v'\}$ is a square, so $\mathcal{R}_3$ implies $\{a, b\} * \text{hull}_\Lambda\{v, v'\} \subset \Gamma$, so $\text{hull}_\Lambda\{v, v'\} = \{v, v'\}$, so there is a Λ-edge between v and v'. □

Lemma 3.3. *Let* Γ *be an incomplete, triangle-free graph with no separating clique that admits a FIDL-*Λ. *The link of every vertex is* Λ*-convex.*

Proof. Suppose not. Then there exist a, b, v with $a, b \in \text{lk}(v)$ such that

$$\text{lk}(v) \cap \text{hull}_\Lambda\{a, b\} = \{a, b\}$$

and a and b are not adjacent in Λ. Assume $v \in \Lambda_{\mathtt{r}}$ and $a, b \in \Lambda_{\mathtt{b}}$. Let

$$a = c_0, c_1, \ldots, c_n = b, \quad n \geq 2,$$

be the vertices of $[a, b]_\Lambda$. According to Remark 2.6, each pair $\{c_i, c_{i+1}\}$ is the diagonal of some square $\{c_i, c_{i+1}\} * \{d_i, d_i'\}$ of Γ. None of the d_i and d_i' equal v, since the c_i for $i \neq 0, n$ are not in $\text{lk}(v)$.

We build a cycle γ as follows: start with b, v, $a = c_0$, and d_0. Next add c_{j_0}, where $j_0 \geq 1$ is the largest index such that c_{j_0} is adjacent to d_0. Then add d_{j_0}. Continue, where, having most recently added d_i, we next add c_j such that j is the maximal index with d_i adjacent to c_j. If $j < n$, then add d_j and repeat. The point is that, while d_j is adjacent to c_{j+1}, the previous d_i that occur in γ are not adjacent to any c_k for $k > j$, so we guarantee that no d is repeated in γ.

Thus, γ is a cycle in Γ with $\text{hull}_{\Lambda_{\mathtt{b}}}(\gamma) = \text{hull}_\Lambda\{a, b\} = \{c_0, \ldots, c_n\}$. Condition $\mathcal{R}_4$ implies there exists a square $\{a, a'\} * \{v, v'\}$ with $a', v' \in \text{hull}_\Lambda(\gamma)$. However, a and b are the only vertices of $\text{lk}(v) \cap \text{hull}_\Lambda(\gamma)$, so $a' = b$. Condition $\mathcal{R}_3$ implies $\{v\} * \text{hull}_\Lambda\{a, b\} \subset \Gamma$, contradicting $\text{hull}_\Lambda\{a, b\} \not\subset \text{lk}(v)$. □

Corollary 3.4. *Let $\binom{v}{v'}$ be a cut of Γ, and let Γ' be a connected component of $\Gamma - \binom{v}{v'}$. Let $\bar{\Gamma}' := \Gamma' \cup \binom{v}{v'}$. The intersection of each component of Λ with $\bar{\Gamma}'$ is Λ-convex.*

Proof. For every pair of vertices $a, b \in \bar{\Gamma}'$, there exists an embedded Γ-path

$$c_0 = a, \dots, c_n = b$$

such that $c_i \in \Gamma'$ for all $i \neq 0, n$. Suppose a and b are in the same component of Λ. Then n is even, and for i odd, $\mathrm{lk}(c_i) \subset \bar{\Gamma}'$ is Λ-convex and contains c_{i-1} and c_{i+1}, so $[c_0, c_2]_\Lambda + \cdots + [c_{n-2}, c_n]_\Lambda$ is a Λ-path from a to b with vertices in $\bigcup_{i \text{ odd}} \mathrm{lk}(c_i) \subset \bar{\Gamma}'$. □

Corollary 3.5. *For any two vertices $a, b \in \Gamma$, $\mathrm{lk}(a) \cap \mathrm{lk}(b)$ is Λ-convex.*

Proof. For $c, d \in \mathrm{lk}(a) \cap \mathrm{lk}(b)$, Lemma 3.3 says $[c, d]_\Lambda$ is contained in both $\mathrm{lk}(a)$ and $\mathrm{lk}(b)$. □

Here are some consequences of these convexity results.

Proposition 3.6. *Let Γ be an incomplete, triangle-free graph with no separating clique that admits a FIDL-Λ. Then every cycle of Γ has even length, every cycle of length greater than 6 has a 1 or 2-chord, and an induced cycle of length 6 occurs only as the rim of a bicycle wheel subgraph of Γ.*

Proof. The graph Γ is bipartite, since its vertices are 2-colored $\mathtt{r}/\mathtt{b}$ according to which component $\Lambda_\mathtt{r}$ or $\Lambda_\mathtt{b}$ of Λ they belong. Thus, Γ has no odd cycles. Suppose $\gamma := c_0, c_1, \dots, c_{n-1}$ is a cycle of length $n > 4$ with no 1 or 2-chords. We always take subscripts modulo n, without further comment.

Construct a Λ-loop at c_0 by taking $[c_0, c_2]_\Lambda + [c_2, c_4]_\Lambda + \cdots + [c_{n-2}, c_n]_\Lambda$. This is a loop in a tree, so it is degenerate. In particular, each edge is crossed an even number of times. By Lemma 3.3, $[c_{2m}, c_{2m+2}]_\Lambda \subset \mathrm{lk}(c_{2m+1})$. Since the cycle has no 1 or 2-chords, for odd $j > i$, we have that $\mathrm{lk}(c_i)$ and $\mathrm{lk}(c_j)$ intersect only if $j = i + 2$ or $i = 1$ and $j = n - 1$, so only $[c_{2m-2}, c_{2m}]_\Lambda$ and $[c_{2m+2}, c_{2m+4}]_\Lambda$ potentially share edges with $[c_{2m}, c_{2m+2}]_\Lambda$. Since a geodesic uses an edge once or not at all, to have all of the edges of $[c_{2m}, c_{2m+2}]_\Lambda$ crossed evenly in total by the loop, we need that each edge of $[c_{2m}, c_{2m+2}]_\Lambda$ is also contained in exactly one of $[c_{2m-2}, c_{2m}]_\Lambda$ and $[c_{2m+2}, c_{2m+4}]_\Lambda$. Thus, there is $x \in [c_{2m}, c_{2m+2}]_\Lambda$ such that

$$[c_{2m}, x]_\Lambda = [c_{2m-2}, c_{2m}]_\Lambda \cap [c_{2m}, c_{2m+2}]_\Lambda,$$
$$[x, c_{2m+2}]_\Lambda = [c_{2m}, c_{2m+2}]_\Lambda \cap [c_{2m+2}, c_{2m+4}]_\Lambda.$$

Such an x is in $\mathrm{lk}(c_{2m-1}) \cap \mathrm{lk}(c_{2m+1}) \cap \mathrm{lk}(c_{2m+3})$. But then

$$d_\Gamma(c_{2m-1}, c_{2m+3}) = 2,$$

so since γ has no 2-chords, $d_\gamma(c_{2m-1}, c_{2m+3}) = 2$, so $n = 6$. The same argument, reversing evens and odds, shows there exists $y \in \mathrm{lk}(c_0) \cap \mathrm{lk}(c_2) \cap \mathrm{lk}(c_4)$.

Consider the cycle $\gamma' := y, c_0, c_1, x, c_3, c_4$. By condition $\mathcal{R}_4$, there is a square $\{c_0, v\} * \{c_1, w\}$ with $v, w \in \mathrm{hull}_\Lambda(\gamma') = \{x, c_0, c_4\} \sqcup \{y, c_1, c_3\}$, so $v \in \{x, c_4\}$ and $w \in \{y, c_3\}$. But since γ has no chords, c_1 is not adjacent to c_4 and c_3 is not adjacent to c_0, so $v = x$ and $w = y$, implying x and y are adjacent. Thus, γ is the rim of a bicycle wheel with hub $\{x, y\}$. □

Proposition 3.7. *Let Γ be an incomplete, triangle-free graph with no separating clique. Suppose Γ has an uncrossed cut $\binom{a}{b}$. Let Γ_i be the components of $\Gamma - \binom{a}{b}$, and let $\bar{\Gamma}_i := \Gamma_i \cup \binom{a}{b}$. Then Γ admits a FIDL-Λ if and only if each $\bar{\Gamma}_i$ admits a FIDL-Λ_i that contains an edge $a \bullet\!\!-\!\!\bullet b$.*

Proof. The "only if" direction follows easily from Lemma 3.2 and Corollary 3.4.

For the converse, suppose $\Lambda_{\Gamma_i} := \Lambda_{\Gamma_i,\mathtt{r}} \cup \Lambda_{\Gamma_i,\mathtt{b}}$ contains an edge $a \bullet\!\!-\!\!\bullet b$ in $\Lambda_{\Gamma_i,\mathtt{r}}$ for all i. This is the only intersection of any of the $\mathtt{r}$ trees, since $\binom{a}{b} = \cap_i \bar{\Gamma}_i$, so $\Lambda_\mathtt{r} := \bigcup_i \Lambda_{\Gamma_i,\mathtt{r}}$ is a tree.

For $\Lambda_\mathtt{b}$, there are two cases. If $\binom{a}{b} = \{a, b\}$ is a cut pair, then the $\Lambda_{\Gamma_i,\mathtt{b}}$ are disjoint. For each i, choose $c_i \in \Gamma_i \cap \mathrm{lk}(a) \cap \mathrm{lk}(b)$, which exists by Lemma 3.1. Define $\Lambda_\mathtt{b}$ as the union of the $\Lambda_{\Gamma_i,\mathtt{b}}$, together with a tree spanning the c_i. If $\binom{a}{b} = \{a, b, c\}$ is a 2-path cut triple, then c is a vertex in all $\Lambda_{\Gamma_i,\mathtt{b}}$, so

$$\Lambda_\mathtt{b} := \bigcup_i \Lambda_{\Gamma_i,\mathtt{b}}$$

is a tree. In this case, set each $c_i := c$, so that we can write a common argument.

Let $\Lambda := \Lambda_\mathtt{r} \sqcup \Lambda_\mathtt{b}$. Condition $\mathcal{R}_1$ we arranged by constructing trees. Conditions $\mathcal{R}_2$ and $\mathcal{F}_1$ are immediate. We check conditions $\mathcal{R}_3$ and $\mathcal{R}_4$. The only interesting cases are squares and cycles that include vertices from different components of $\Gamma - \binom{a}{b}$.

Suppose $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$ are the diagonals of a square. Since $\binom{a}{b}$ is a cut, the square is $\{x_1, x_2\} * \{a, b\}$. Since the only $\Lambda_\mathtt{b}$ connection between Γ_1 and Γ_2 is through c_1 and c_2, we have

$$\begin{aligned}\mathrm{hull}_\Lambda\{a, b\} * \mathrm{hull}_\Lambda\{x_1, x_2\} \\ = \{a, b\} * (\mathrm{hull}_{\Lambda_{\Gamma_1,\mathtt{b}}}\{x_1, c_1\} \cup \mathrm{hull}_{\Lambda_{\Gamma_2,\mathtt{b}}}\{c_2, x_2\} \cup \mathrm{hull}_{\Lambda_\mathtt{b}}\{c_1, c_2\}).\end{aligned}$$

This join is indeed contained in Γ, because $x_i, c_i \in \mathrm{lk}_{\bar{\Gamma}_i}(a) \cap \mathrm{lk}_{\bar{\Gamma}_i}(b)$, and by Corollary 3.5, $\mathrm{lk}_{\bar{\Gamma}_i}(a) \cap \mathrm{lk}_{\bar{\Gamma}_i}(b)$ is $\Lambda_{\bar{\Gamma}_i}$-convex, so all of $\mathrm{hull}_{\Lambda_{\Gamma_i,\mathfrak{b}}}\{x_i, c_i\}$ are common neighbors of a and b, whereas $\mathrm{hull}_{\Lambda_\mathfrak{b}}\{c_1, c_2\} \subset \mathrm{lk}(a) \cap \mathrm{lk}(b)$ by construction. Thus, $\mathcal{R}_3$ is satisfied.

If $\mathcal{R}_4$ is violated, there is a shortest cycle γ containing an edge e that is not included in a square with vertices in $\mathrm{hull}_\Lambda(\gamma)$. We may assume such a cycle is induced, since if it has vertices that are adjacent in Γ and not in γ, we could use that edge to cut γ into two strictly shorter cycles whose union contains all the edges of γ. In particular, e is in one of these shorter cycles, γ'. But $\mathrm{hull}_\Lambda(\gamma') \subset \mathrm{hull}_\Lambda(\gamma)$, so e is not included in a square with vertices in $\mathrm{hull}_\Lambda(\gamma')$, contradicting that γ was a shortest counterexample.

So consider an induced cycle γ that contains, without loss of generality, vertices from Γ_1 and Γ_2. Since the cycle crosses the cut, it contains a and b. Since it is induced, either $\gamma \cap \bar{\Gamma}_1 = a \leftrightarrow c_1 \leftrightarrow b$, or $c_1 \notin \gamma$. In the first case, both edges $a \leftrightarrow c_1$ and $c_1 \leftrightarrow b$ are in the square $\{c_1, c_2\} * \{a, b\}$, and $c_2 \in \mathrm{hull}_\Lambda(\gamma)$ since the only $\Lambda_\mathfrak{b}$-connection between Γ_1 and Γ_2 is through c_2. On the other hand, if $c_1 \notin \gamma$, then there is a cycle γ' made from $\gamma \cap \bar{\Gamma}_1$ and the segment $a \leftrightarrow c_1 \leftrightarrow b$ that is completely contained in $\bar{\Gamma}_1$, so its edges all participate in squares with vertices in $\mathrm{hull}_{\Lambda_{\Gamma_1}}(\gamma')$, which is contained in $\mathrm{hull}_{\Lambda_\Gamma}(\gamma)$, since the vertex $c_1 = \gamma' - \gamma$ is in the $\Lambda_\mathfrak{b}$-hull of γ, being the only $\Lambda_\mathfrak{b}$-connection between $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. This shows that all $\bar{\Gamma}_1$-edges of γ are contained in a square with vertices in $\mathrm{hull}_\Lambda(\gamma)$. □

4 Satellite-dismantlability and the coning algorithm

In this section we take a graph with a FIDL-Λ apart and then put it back together again.

Theorem 4.1. *Let Γ be an incomplete, triangle-free graph with no separating clique that admits a FIDL-Λ. Then Γ is satellite-dismantlable to a square through a sequence of graphs $\Gamma_0 = \Gamma \supset \Gamma_1 \supset \cdots \supset \Gamma_n$ such that, for all i, Γ_i is incomplete, triangle-free with no separating clique and has $\Lambda \cap \Gamma_i$ as a FIDL-Λ, with the satellite vertex $\Gamma_i - \Gamma_{i+1}$ being a leaf in $\Lambda \cap \Gamma_i$.*

Proof. Our goal is to find a leaf v of Λ such that v is a satellite in Γ and such that $\Gamma_1 := \Gamma - \{v\}$ has the desired properties. The proof is easy if Γ is a suspension, so assume not.

Identify Δ with its image in $\boxslash(\Gamma)$, as in Remark 2.6. This graph is connected, since W_Γ and A_Δ are one-ended. We claim that every vertex of $\boxslash(\Gamma)$ is adjacent to a vertex of Δ. Suppose that $\{a, b\} \in \boxslash(\Gamma) - \Delta$. By definition, $\{a, b\}$ is a diagonal of some square, so a and b have at least two common neighbors. By Corollary 3.5,

$\mathrm{lk}(a) \cap \mathrm{lk}(b)$ is Λ-convex, so it contains a subtree of Λ with at least one edge. Any such edge gives a vertex of Δ adjacent to $\{a, b\}$ in $\boxslash(\Gamma)$. Note that this shows Γ is strongly $\mathcal{CFS}$.

The vertex v is a leaf of Λ if and only if it has a unique neighbor v' in Λ. Equivalently, $\{v, v'\}$ is the unique vertex of Δ containing v. We claim that v is a satellite of v'. Let $w \in \mathrm{lk}(v)$. Since $\{v, v'\}$ is the diagonal of some square, $|\mathrm{lk}(v) \cap \mathrm{lk}(v')| \geq 2$, so there exists $c \in (\mathrm{lk}(v) \cap \mathrm{lk}(v')) - \{w\}$. Since the edge $v \mathrel{\bullet\!-\!\bullet} c$ is not a separating clique, there is a path from v' to w that does not go through v or c, so there is a cycle γ that goes from v to c to v' to w and then back across the edge between v and w. By $\mathcal{R}_4$, there is a square $\{v, x\} * \{w, u\}$ with $x, u \in \mathrm{hull}_\Lambda(\gamma)$. But then $\mathcal{R}_3$ implies $\{w\} * \mathrm{hull}_\Lambda\{v, x\} \subset \Gamma$. Since v was a Λ-leaf, $v' \in \mathrm{hull}_\Lambda\{v, x\}$, so w is adjacent to v'. Thus, $\mathrm{lk}(v) \subset \mathrm{lk}(v')$.

The claim that $\Lambda - \{v\}$ is a FIDL-Λ is automatic; all of the defining properties are inherited from Λ. The key for this is that, since only a leaf of Λ was deleted, convex sets in $\Lambda - \{v\}$ are still convex in Λ.

We must show it is possible to choose a leaf of Λ such that $\Gamma - \{v\}$ is not separated by a clique. First, let v be a Λ-leaf and $\{v, v'\}$ the unique vertex of Δ containing v, and suppose that $\{v, v'\}$ is not a cut vertex of Δ. Then $\boxslash(\Gamma) - \{v, v'\}$ consists of leaves of $\boxslash(\Gamma)$ that were connected only to $\{v, v'\}$, plus one additional component Ω containing $\Delta - \{v, v'\}$. The reason for this is that Γ being triangle-free implies $\boxslash(\Gamma)$ is triangle-free, so if $\{a, b\}$ is adjacent to $\{v, v'\}$, but is not a leaf of $\boxslash(\Gamma)$, then it is also adjacent to some other vertex $\{c, d\}$, which is adjacent to a vertex of Δ, but not to $\{v, v'\}$.

We further note that Ω has support $\Gamma - \{v\}$. Since Γ is not a suspension, v' is not a Λ-leaf, so it is contained in at least one other vertex of Δ, so $v' \in \mathrm{supp}(\Omega)$. For other vertices, we only worry about those appearing in a leaf $\{a, b\}$ of $\boxslash(\Gamma)$ connected to $\{v, v'\}$. Since $\{v, v'\}$ is not a cut vertex of Δ, $\{a, b\} \notin \Delta$. But Λ spans Γ, so there is some vertex of Δ with a in its support, and likewise for b.

Now consider $\boxslash(\Gamma - \{v\})$. If $\{a, b\}$ is a leaf of $\boxslash(\Gamma)$ connected only to $\{v, v'\}$, then the only square of Γ with $\{a, b\}$ as a diagonal is $\{a, b\} * \{v, v'\}$, so $\{a, b\}$ is not the diagonal of a square in $\Gamma - \{v\}$ and does not appear as a vertex of $\boxslash(\Gamma - \{v\})$. Since $\{v, v'\}$ was the unique vertex of $\boxslash(\Gamma)$ containing v,

$$\boxslash(\Gamma - \{v\}) = \Omega,$$

which is connected, by hypothesis, and has full support in $\Gamma - \{v\}$. Thus, $\Gamma - \{v\}$ is strongly CFS, and Lemma 2.5 says $\Gamma - \{v\}$ has no separating clique.

Now we argue that there always exists a Λ-leaf not giving a cut vertex of Δ, so we can find it and apply the previous argument. Suppose the first chosen Λ-leaf v does give a cut vertex $\{v, v'\}$ of Δ. This implies that Γ has a cut $\binom{v}{v'}$, since W_Γ and A_Δ split over a two-ended subgroup commensurable to $\langle v v' \rangle$. Let $v_0 := v$ and

$v_0' := v'$. By Corollary 3.4, for each component Γ' of $\Gamma - \binom{v_0}{v_0'}$, the intersection of each component of Λ with $\bar{\Gamma}'$ is Λ-convex, so Γ' contains at least one Λ-leaf v_1. If v_1 uniquely appears as $\{v_1, v_1'\}$ in Δ and $\{v_1, v_1'\}$ is not a cut vertex of Δ, we are done. Otherwise, $\binom{v_1}{v_1'}$ is a cut of Γ. Since Γ has no crossing cuts, $\binom{v_0}{v_0'} - \binom{v_1}{v_1'}$ is a non-empty set contained in a single component of $\Gamma - \binom{v_1}{v_1'}$. Choose a component Γ'' of $\Gamma - \binom{v_1}{v_1'}$ not containing $\binom{v_0}{v_0'} - \binom{v_1}{v_1'}$ and repeat, always choosing a complementary component of the most recent cut that does not contain the previous cuts, so that the size of the components strictly decreases at each step. If v_{i+1} is the lone vertex in its component of $\Gamma - \binom{v_i}{v_i'}$, then it cannot be part of a cut, since the cut would cross $\binom{v_i}{v_i'}$, so eventually, this process produces a Λ-leaf that does not appear in a cut vertex of Δ. □

Corollary 4.2. *If Γ is an incomplete, triangle-free graph with no separating clique and no satellite vertex, then Γ does not admit a FIDL-Λ.*

Also, in the proof of Theorem 4.1, we found that every vertex of $\boxslash(\Gamma)$ is adjacent to a vertex of the connected graph Δ, so $\boxslash(\Gamma)$ is connected, hence we have the following corollary.

Corollary 4.3. *If Γ is an incomplete, triangle-free graph with no separating clique and it admits a FIDL-Λ, then it is strongly $\mathcal{CFS}$.*

Now we go in the other direction.

Coning Algorithm 4.4. We perform the following inductive procedure to build a graph Γ with an associated graph $\Lambda \leq \Gamma^c$ with two connected components.

(1) The initial graph Γ_0 is a square, the associated graph $\Lambda_0 = \Gamma_0^c$ is the complement graph of Γ_0.

(2) Build a sequence of pairs $((\Gamma_0, \Lambda_0), (\Gamma_1, \Lambda_1), \ldots, (\Gamma_n, \Lambda_n))$ by applying the induction step: given (Γ_i, Λ_i), pick a vertex v_i and a set $N_i \subseteq \mathrm{lk}_{\Gamma_i}(v_i)$ of at least two vertices that is Λ_i-convex. Define Γ_{i+1} by coning-off N_i to x_{i+1}, and define Λ_{i+1} by adding an edge from v_i to x_{i+1} to Λ_i.

Note that, at each step, x_{i+1} is a satellite of v_i in Γ_{i+1}.

Theorem 4.5. *If the pair (Γ, Λ) can be constructed by Coning Algorithm 4.4, then Γ is incomplete, triangle-free and has no separating cliques, and Λ is a FIDL-Λ for Γ.*

Proof. The proof is by induction on n. The initial graph $\Theta(\Gamma_0, \Lambda_0)$ satisfies all the conditions. Assume $\Theta(\Gamma_i, \Lambda_i)$ does too. We show that $\Theta(\Gamma_{i+1}, \Lambda_{i+1})$ satisfies $\mathcal{R}_3$ and $\mathcal{R}_4$. The other conditions are easy to verify.

Since $\Theta(\Gamma_i, \Lambda_i)$ satisfies condition $\mathcal{R}_3$, we only need to check squares in Γ_{i+1} containing the new vertex x_{i+1}. By construction of Λ_{i+1}, any such square is of the form $\{x_{i+1}, l\} * \{n, n'\}$ with $n, n' \in N_i$, and since $N_i = \mathrm{lk}_{\Gamma_{i+1}}(x_{i+1})$ is Λ_i-convex, $\{x_{i+1}\} * \mathrm{hull}_{\Lambda_{i+1}}\{n, n'\} \subset \Gamma_{i+1}$.

The vertex x_{i+1} is only connected to v_i in Λ_{i+1}. Hence,

$$\mathrm{hull}_{\Lambda_{i+1}}\{x_{i+1}, l\} - \{x_{i+1}\} = \mathrm{hull}_{\Lambda_{i+1}}\{v_i, l\} = \mathrm{hull}_{\Lambda_i}\{v_i, l\}.$$

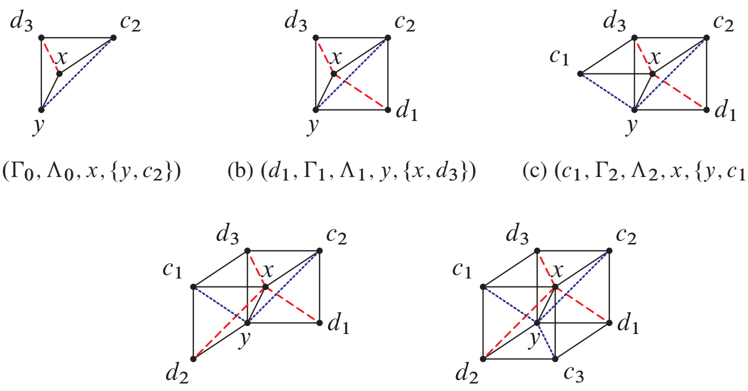
Since Γ_i is triangle-free, v_i and l are not adjacent, so $\{v_i, l\} * \{n, n'\} \subseteq \Gamma_i$ is a square. Condition $\mathcal{R}_3$ for $\Theta(\Gamma_i, \Lambda_i)$ implies $\mathrm{hull}_{\Lambda_i}\{v_i, l\} * \mathrm{hull}_{\Lambda_i}\{n, n'\} \subset \Gamma_i$. Condition $\mathcal{R}_3$ is satisfied, since

$$\begin{aligned}\mathrm{hull}_{\Lambda_{i+1}}&\{x_{i+1}, l\} * \mathrm{hull}_{\Lambda_{i+1}}\{n, n'\}\\ &= \{x_{i+1}\} * \mathrm{hull}_{\Lambda_i}\{n, n'\} \cup \mathrm{hull}_{\Lambda_i}\{v_i, l\} * \mathrm{hull}_{\Lambda_i}\{n, n'\} \subset \Gamma_{i+1}\end{aligned}$$

Since $\Theta(\Gamma_i, \Lambda_i)$ satisfies condition $\mathcal{R}_4$, we only need to check cycles containing x_{i+1}. Let γ be a cycle containing x_{i+1}. Since $\mathrm{lk}_{\Gamma_{i+1}}(x_{i+1}) = N_i$, γ is of the form $\gamma = (x_{i+1}, n, l_1, \ldots, l_k, n')$ for $n, n' \in N_i$. The edges incident to x_{i+1} are contained in the square $\{x_{i+1}, v_i\} * \{n, n'\}$, all of whose vertices are in γ, except possibly v_i. But x_{i+1} is a leaf of Λ_{i+1} connected only to v_i, so v_i is certainly in $\mathrm{hull}_{\Lambda_{i+1}}(\gamma)$. For the remaining edges of γ, replace γ by $\gamma' = (v_i, n, l_1, \ldots, l_k, n')$, which is a loop in Γ_i that can be split into at most two cycles in Γ_i. Condition $\mathcal{R}_4$ for $\Theta(\Gamma_i, \Lambda_i)$ implies each of these edges belongs to a square with vertices in the Λ_i-hull of its cycle, which is a subset of the Λ_{i+1}-hull of γ. □

Example 4.6. The graph Γ of Example 2.7 that is the 1-skeleton of a 3-cube with one space diagonal has a FIDL-Λ. The pair (Γ, Λ) can be constructed by Coning Algorithm 4.4. A coning sequence $(x_i, \Gamma_i, \Lambda_i, v_i, N_i)$ is illustrated in Figure 3.

Next we want to combine Theorem 4.1 and Theorem 4.5 to get necessary and sufficient conditions for the existence of a FIDL-Λ, phrased only in terms of Γ. Notice that Theorem 4.1 has a conclusion that is stronger than just existence of a satellite-dismantling sequence; it also says something like that Λ is compatible with all the terms in the sequence. Similarly, Coning Algorithm 4.4 requires more than just the existence of the inverse of a dismantling sequence; it requires link Λ-convexity at each step. We want to express these additional restrictions without reference to Λ, so we fix a satellite-dismantling sequence of Γ and define a technical condition (†). Theorem 4.7 consists of showing that (†) is sufficient to allow us to make the choices of the coning algorithm, and that existence of a FIDL-Λ compatible with this dismantling sequence implies (†), so (†) is also



(a) $(\Gamma_0, \Lambda_0, x, \{y, c_2\})$ (b) $(d_1, \Gamma_1, \Lambda_1, y, \{x, d_3\})$ (c) $(c_1, \Gamma_2, \Lambda_2, x, \{y, c_1\})$

(d) $(d_2, \Gamma_3, \Lambda_3, y, \{x, d_1, d_2\})$ (e) $(c_3, \Gamma_4, \Lambda_4)$

Figure 3. Example of a coning sequence.

necessary. Section 5 deals with turning Theorem 4.7 into a search algorithm. Section 6 says a few words about the practical performance of such an algorithm, and gives an example that shows that the satisfiability of (†) can depend on the choice of satellite-dismantling sequence.

Theorem 4.7. *Let Γ be an incomplete, triangle-free graph without separating cliques. The graph Γ admits a FIDL-Λ if and only if it admits a satellite-dismantling sequence $\Gamma = \Gamma_n \supset \Gamma_{n-1} \supset \cdots \supset \Gamma_0$ such that Γ_0 is a square, each Γ_i is incomplete and triangle-free with no separating clique, and the following condition is satisfied. For $0 \leq i < n$, let $\{x_{i+1}\} := \Gamma_{i+1} - \Gamma_i$ and $N_i := \mathrm{lk}_{\Gamma_{i+1}}(x_{i+1})$. Then*

$$\text{for all } i < n, \text{ there exists } v_i \in V_i := \{v \in \Gamma_i \mid N_i \subset \mathrm{lk}_{\Gamma_i}(v)\},$$
$$\text{such that, for all } j > i, \text{ if } x_{i+1} \in N_j \text{ and } N_j \cap \Gamma_i \neq \emptyset, \text{ then } v_i \in N_j. \quad (\dagger)$$

Proof. Suppose that we have a satellite-dismantling sequence for Γ satisfying the given conditions. Let $\Lambda_0 := \Gamma_0^c$, and construct Λ_{i+1} from Λ_i by choosing v_i satisfying (†), and adding a Λ-edge from v_i to x_{i+1}. To apply Theorem 4.5, we must verify that N_j is Λ_j-convex for each j. Given j, let $k_j \geq 0$ be the minimum index such that $N_j \cap \Gamma_{k_j} \neq \emptyset$. The vertices of N_j are pairwise at distance 2 from one another, so they all have the same color; assume it is b. If $y \in N_j$, then either $y \in \Gamma_{k_j}$ or $y = x_{i+1}$ for some $k_j \leq i < j$. In the second case, by (†), y is Λ_j-adjacent to $v_i \in N_j$, which is in a lower stratum. In this way, (†) implies that every vertex in N_j can be Λ_j-connected through vertices in N_j to a vertex in $N_j \cap \Gamma_{k_j}$.

If $k_j > 0$, then $\Gamma_{k_j} - \Gamma_{k_j-1}$ is a single vertex, and if $k_j = 0$, then the two vertices $\Lambda_{0,\mathtt{b}}$ are connected by an edge. Thus, N_j is Λ_j-convex.

Conversely, if Γ admits a FIDL-Λ, then by Theorem 4.1, it admits a satellite-dismantling sequence, leading to a square through graphs with the desired properties. For each satellite x_{i+1}, there is a unique Λ_{i+1}-edge at x_{i+1} connecting it to a $v_i \in V_i$. We check that this v_i satisfies (†).

Suppose that, for some j, there is an i such that $x_{i+1} \in N_j$ and there exists $y \in \Gamma_i \cap N_j$. Assume $N_j \subset \Lambda_{\mathtt{r}}$. Lemma 3.3 says $N_j := \mathrm{lk}_{\Gamma_{j+1}}(x_{j+1})$ is Λ_{j+1}-convex, but $\mathrm{hull}_{\Lambda_j}(N_j) = \mathrm{hull}_{\Lambda_{j+1}}(N_j)$, so N_j is Λ_j-convex. Now we have two points x_{i+1} and y that are contained in two different subtrees of $\Lambda_{j,\mathtt{r}}$: one is $N_j \cap \Lambda_{j,\mathtt{r}}$ and the other is $\Lambda_{i+1,\mathtt{r}}$. Thus, the $\Lambda_{j,\mathtt{r}}$-geodesic from x_{i+1} to y, the $N_j \cap \Lambda_{j,\mathtt{r}}$-geodesic from x_{i+1} to y and the $\Lambda_{i+1,\mathtt{r}}$-geodesic from x_{i+1} to y coincide. We know that the $\Lambda_{i+1,\mathtt{r}}$-geodesic from x_{i+1} to y goes through v_i, since x_{i+1} is a leaf of $\Lambda_{i+1,\mathtt{r}}$ connected only to v_i. Thus, $v_i \in N_j$, and (†) is satisfied. □

We give a family of examples to which Theorem 4.7 applies.

Proposition 4.8. *Let T be a finite tree, with at least one edge, whose vertices are labeled by natural numbers $n(v)$ such that 1 occurs only on leaves, and if T is a single edge, then both labels are greater than 1. Let Γ be the graph that has, for each $v \in T$, $n(v)$ vertices $(v,0),\dots,(v,n(v)-1)$. Connect (v,i) to (w,j) by an edge if and only if v is adjacent to w in T. Then Γ admits a FIDL-Λ.*

Proof. We describe a satellite-dismantling and then say why it satisfies Theorem 4.7.

If $v \in T$ with $n(v) > 1$, then for $i+1 \le n(v)-1$, $(v,i+1)$ is a satellite of (v,i), coning off $\{(w,j) \mid w \in \mathrm{lk}_T(v),\ 0 \le j < n(w)\}$. Pick any such v and iteratively remove $(v,n(v)-1),\ \dots,\ (v,2)$. Repeat for all v with $n(v) > 2$. This reduces us to the case that n is bounded by 2.

If Γ is not a square and there is a leaf v of T attached to w with $n(v) = 1$, then there exists $x \in T$ at distance 2 from v. In Γ, $(v,0)$ is a satellite of $(x,0)$ coning-off $\{(w,0),(w,1)\}$. In this way, remove all T-leaves with n-value 1, until either we reach a square or have smaller Γ and T with $n \equiv 2$.

Suppose $n \equiv 2$ and Γ is not a square, so T is not a single edge. If v is a leaf of T attached to w, then $(v,1)$ is a satellite of $(v,0)$ coning-off $\{(w,0),(w,1)\}$. Thus, we reduce n to 1 on each of the leaves of T, and then rerun the case in the previous paragraph.

Iterating gives a satellite-dismantling reducing Γ to a square. Furthermore, in the dismantling, we specified for each satellite which vertex we were considering it as the satellite of, which allows an explicit construction of Λ. It is observed that the coned-off set is Λ-convex at each step, so this is a FIDL-Λ. □

Remark. The proof relies on the fact that T is a tree because, after reducing to $n \leq 2$, the dismantling proceeds by removing T-leaves. Proposition 4.8 cannot be true for arbitrary graphs, because if the underlying graph has a long isometrically embedded loop, then so does Γ, which prevents the existence of a FIDL-Λ, by Proposition 3.6. Interestingly, an arbitrary connected graph with $n \equiv 2$ does produce a Γ that defines a RACG commensurable to a RAAG, by the construction of Davis and Januszkiewicz [8], so this class of examples highlights differences between Davis and Januszkiewicz's and Dani and Levcovitz's approaches.

Figure 4 works out an example of a tree as in Proposition 4.8 in detail.

5 A search algorithm

Definition 5.1. A *FIDL-Λ for Γ relative to* a collection of pairs of vertices

$$\{\{p_0, q_0\}, \dots, \{p_n, q_n\}\}$$

of Γ is a FIDL-Λ that contains each of the pairs $\{p_i, q_i\}$ as an edge of Λ.

Relative Search Algorithm 5.2. Given an incomplete, triangle-free graph Γ without separating cliques, find a FIDL-Λ relative to $\{\{p_0, q_0\}, \dots, \{p_n, q_n\}\}$ or decide that one does not exist as follows.

(1) If Γ is not strongly $\mathcal{CFS}$, stop; no FIDL-Λ exists.

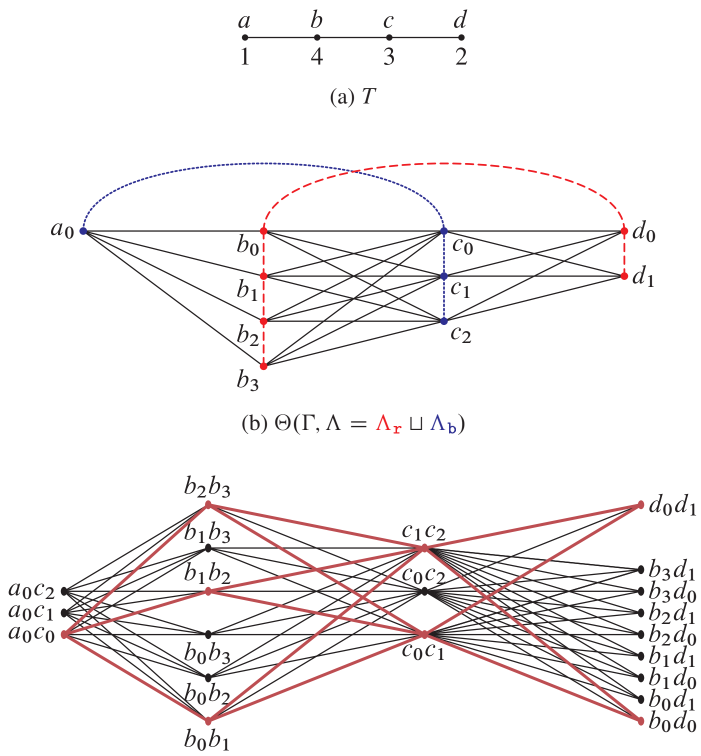
(2) If Γ contains a cycle violating Proposition 3.6, stop; no FIDL-Λ exists.

(3) If there exist $x_0, x_1, \dots, x_{n-1}$ such that, for each $i < n$, there exists j with $\{x_i, x_{i+1}\} = \{p_j, q_j\}$, then stop; no relative FIDL-Λ exists.

(4) Enumerate satellite-dismantling sequences reducing Γ to a square through graphs without separating cliques. If none exist, stop; no FIDL-Λ exists.

(5) For each such satellite-dismantling sequence, check, in the notation of Theorem 4.7, if condition (†) is satisfied. Moreover, if we assume for each k that, for all i, $q_k \in \Gamma_i \Rightarrow p_k \in \Gamma_i$, then we require for each k that either $\{p_k, q_k\}$ is a diagonal of the square Γ_0 or that, for i such that $q_k = x_{i+1}$, we have that $v_i = p_k$ is a choice for $v_i \in V_i$ satisfying condition (†).

(6) If a suitable satellite-dismantling sequence is found, then Theorem 4.7 provides the relative FIDL-Λ. Otherwise, no relative FIDL-Λ exists.



(a) T

(b) $\Theta(\Gamma, \Lambda = \Lambda_{\mathtt{r}} \sqcup \Lambda_{\mathtt{b}})$

(c) Δ sitting in $\boxslash(\Gamma)$

Figure 4. An example of Proposition 4.8, showing a tree T with vertices labeled by natural numbers, the corresponding graph Γ and bicolored Λ making up $\Theta(\Gamma, \Lambda)$, and the resulting presentation graph Δ of a finite index visual RAAG subgroup sitting as a subgraph in $\boxslash(\Gamma)$, with every vertex of $\boxslash(\Gamma)$ adjacent to a vertex of Δ.

Proof. Item (1) is Corollary 4.3. If the given $\{\{p_0, q_0\}, \ldots, \{p_n, q_n\}\}$ contains a cycle as described in (3), then the resulting Λ contains a cycle, violating $\mathcal{R}_1$. Item (4) is Theorem 4.1. Item (5) describes how it is possible to achieve the $\{p_k, q_k\}$ as edges of a FIDL-Λ from the argument of Theorem 4.7. □

Global Search Algorithm 5.3. Given an incomplete, triangle-free graph Γ without separating cliques, find a FIDL-Λ or decide that one does not exist as follows.

(1) If Γ is not strongly $\mathcal{CFS}$, stop; no FIDL-Λ exists.

(2) If Γ contains a cycle violating Proposition 3.6, stop; no FIDL-Λ exists.

(3) Compute the JSJ graph of cylinders for W_Γ in terms of Γ as described in [9, Theorem 3.29] (recall Theorem 2.9). If it has a hanging vertex, stop; W_Γ is not even quasiisometric to a RAAG.

(4) For each subgraph Γ' corresponding to a rigid vertex of the graph of cylinders, use Relative Search Algorithm 5.2 to find a FIDL-Λ for Γ' relative to the pairs $\{p, q\} \subset \Gamma'$ such that Γ has a cut $\binom{p}{q}$. If this search fails for any rigid vertex, stop; no FIDL-Λ exists.

(5) If all rigid vertices have relative FIDL-Λ, then they can be assembled into a FIDL-Λ of Γ.

Proof. Proposition 4.43 in [10] shows that if Γ' is the subgraph corresponding to a rigid vertex, then it has no separating cliques and is $\mathcal{CFS}$. Iterating the cutting direction of Proposition 3.7 implies that if Γ has a FIDL-Λ, then its restriction to Γ' is a FIDL-Λ for Γ' relative to the cuts. Together with Corollary 4.3, this explains the necessity of items (3) and (4).

Torsion-generated groups do not surject to $\mathbb{Z}$, so the underlying graph of the graph of cylinders is a tree. Iterating the assembly direction of Proposition 3.7 over the cuts combines the relative FIDL-Λ's of the subgraphs of the rigid vertices into a FIDL-Λ for Γ, establishing item (5). □

6 Performance

We have not analyzed the complexity of the naive "enumerate and check Λ" vs satellite-dismantling algorithms. We have an older implementation of the naive algorithm and of the one in this paper, and observe that the satellite-dismantling algorithm performs faster on batches of smallish (at most 12 vertices) graphs. However, this is not a fair comparison, as the new algorithm also incorporates other results from this paper, such as that the presence of a long embedded cycle obstructs the existence of a FIDL-Λ, and that if W_Γ has a non-trivial JSJ decomposition, then it suffices to patch together FIDL-Λ for the rigid components. These could have also been used to improve the naive algorithm. Nevertheless, we conjecture that the satellite-dismantling algorithm has better generic-case complexity than the naive algorithm, because it can fail fast: most graphs do not admit a FIDL-Λ and the naive algorithm must enumerate and test all Λ to confirm none work, but the satellite-dismantling algorithm can quit immediately if the graph has no satellite-dismantling sequence.

One might also wonder about the efficiency of the search for a suitable satellite-dismantling sequence. For example, if Γ is a triangle-free strongly $\mathcal{CFS}$ graph that has a satellite-dismantling sequence to a square through strongly $\mathcal{CFS}$ graphs

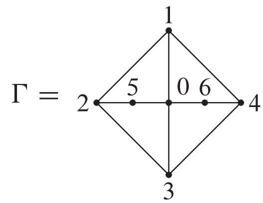


Figure 5. Graph for Example 6.1.

and if v is some satellite vertex of Γ such that $\Gamma - \{v\}$ is strongly $\mathcal{CFS}$, does $\Gamma - \{v\}$ admit a satellite-dismantling sequence to a square through strongly $\mathcal{CFS}$ graphs? That is, does finding a full sequence depend on choosing satellites in the right order? If the order does not matter, this would speed up the search for satellite-dismantling sequences: upon the first failure to extend a dismantling sequence all the way to a square, we could immediately quit rather than backtracking to try different dismantling sequences. However, once we know that dismantling sequences exist, the story is different; satisfying condition (†) of Theorem 4.7 does depend on the chosen sequence, as can be seen in the following example.

Example 6.1. Consider the graph shown in Figure 5. The vertex sequence 6, 5, 0 yields a satellite-dismantling sequence reducing Γ to the square

$$\Gamma_0 = \{1, 3\} * \{2, 4\},$$

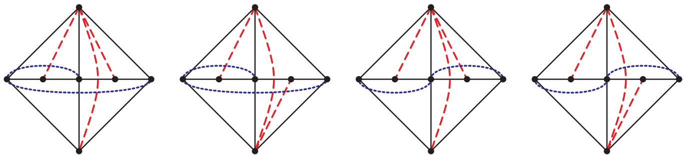
such that all of the Γ_i are strongly $\mathcal{CFS}$, and

$$x_1 = 0, \quad N_0 = \{1, 3\}, \quad V_0 = \{2, 4\},$$
$$x_2 = 5, \quad N_1 = \{0, 2\}, \quad V_1 = \{1, 3\},$$
$$x_3 = 6, \quad N_2 = \{0, 4\}, \quad V_2 = \{1, 3\}.$$

For $i = 0$, $x_{i+1} = 0 \in N_1 \cap N_2$, both of which intersect Γ_i, so to satisfy condition (†) of Theorem 4.7, we would need to choose $v_0 \in V_0 \cap N_1 \cap N_2 = \emptyset$. If we reconstruct Γ by a coning sequence from this data, we see that there are two possibilities up to symmetry, shown in Figure 6 (a) and (b), and both contain a vertex with non-Λ-convex link.

Consider instead the vertex sequence 6, 5, 4, which yields a satellite-dismantling sequence reducing Γ to the square $\Gamma_0 = \{0, 2\} * \{1, 3\}$ with all Γ_i strongly $\mathcal{CFS}$, and

$$x_1 = 4, \quad N_0 = \{1, 3\}, \quad V_0 = \{0, 2\},$$
$$x_2 = 5, \quad N_1 = \{0, 2\}, \quad V_1 = \{1, 3\},$$
$$x_3 = 6, \quad N_2 = \{0, 4\}, \quad V_2 = \{1, 3\}.$$



(a) (b) (c) (d)

Figure 6. Some potential Λ for Γ from reverse coning of a satellite-dismantling. Cases (a) and (b) contain a vertex with a non-Λ-convex link. Cases (c) and (d) give valid FIDL-Λ.

For $i = 0$, $x_{i+1} = 4 \in N_2$ and $N_2 \cap \Gamma_i \neq \emptyset$, but $x_{i+1} \notin N_1$, so we satisfy (†) by choosing $v_0 \in V_0 \cap N_2 = \{0\}$. For $i = 1, 2$, (†) imposes no condition except $v_i \in V_i$. The two possibilities, up to symmetry, shown in Figure 6 (c) and (d), are both FIDL-Λs.

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