

Open mappings of locally compact groups

Michael G. Cowling, Karl H. Hofmann and Sidney A. Morris*

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Abstract. The aim of this note is to insert in the literature some easy but apparently not widely known facts about morphisms of locally compact groups, all of which are concerned with the openness of the morphism.

1 Main result

We are interested in *morphisms of locally compact groups*, by which we mean continuous homomorphisms between locally compact Hausdorff topological groups, and in particular when these are *open*.

Examples 1. To set the scene, we recall a few standard examples and counterexamples of such morphisms $\phi: G \rightarrow H$.

- Let \mathbb{Z}_p be the compact additive group of p -adic integers and \mathbb{R} the additive group of real numbers. Let D be the discrete subgroup $\{(m, m) : m \in \mathbb{Z}\}$ of $G := \mathbb{Z}_p \times \mathbb{R}$. Finally, define $H := G/D$, the compact p -adic solenoid \mathbb{T}_p (see [6, Example 1.28 and Exercise E1.11]), and let $\phi: G \rightarrow H$ be the quotient morphism. Then ϕ is a surjective open morphism of locally compact groups mapping the identity component $G_0 = \{0\} \times \mathbb{R} \cong \mathbb{R}$ onto the dense arc component properly contained in the compact connected group $H = H_0$ (see [6, Exercise E1.11 (iii) and (iv)]).
- Let H be any nondiscrete locally compact group and let G be H_d , the underlying group of H with the discrete topology. Then the identity map $\phi: G \rightarrow H$ is a nonopen bijective morphism.
- Let G be any locally compact noncompact abelian group, H its Bohr compactification, and $\phi: G \rightarrow H$ the natural standard morphism (see [6, Chapter 8, Part 7]). Then ϕ is a nonopen injective morphism with dense image properly contained in H . This is related to the previous example by Fourier duality.

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We write $\mathcal{O}(G)$ for the space of open subgroups (that are automatically closed) of G , and $G \leq H$ (or $H \geq G$) to indicate that G is a subgroup of H . We recall that if U is a subset of a locally compact group G that is symmetric (meaning that $U^{-1} = U$), then U generates a subgroup $\langle U \rangle$ of G equal setwise to $\bigcup_{n \in \mathbb{N}} U^n$. If U is open, then $\langle U \rangle$ is open, hence also closed, and if U is also relatively compact, then $\langle U \rangle$ is σ -compact.

Theorem 2. *Let $\phi: G \rightarrow H$ be a morphism of locally compact groups. If $\phi(G_1)$ is open in H for all G_1 in $\mathcal{O}(G)$, then ϕ is an open mapping.*

Remark 3. If ϕ is open, then trivially $\phi(G_1)$ is open in H for all open subgroups G_1 of G , and so the theorem may also be stated as an equivalence.

Proof. Take a symmetric open subset U of G ; then $G_1 := \langle U \rangle$ is an open subgroup, and hence also closed, and G_1 is σ -compact if U is also relatively compact.

To show that $\phi: G \rightarrow H$ is open, it is necessary and sufficient to show that the restricted surjective morphism $\phi|_{G_1}: G_1 \rightarrow \phi(G_1)$ is open. Indeed, if ϕ is open, then $\phi|_{G_1}$ is trivially open; conversely, every open set U in G is a union $\bigcup_{x \in U} xU_x$, where U_x is an open subset of G_1 ; if $\phi|_{G_1}$ is open, then $\phi(xU_x)$ is open, and so $\phi(U)$ is open.

The Open Mapping Theorem for locally compact groups [6, Exercise EA1.21] uses Baire category to show that, if $\phi: G \rightarrow H$ is a surjective morphism of locally compact groups and G is σ -compact, then ϕ is open. We apply this result to $\phi|_{G_1}$ and the theorem is proved. □

We recall that a subgroup G_1 of G is open if and only if it contains the identity component G_0 of G and G_1/G_0 is open in the totally disconnected group G/G_0 . Further, these open subgroups G_1/G_0 form a basis of the identity neighbourhoods of G/G_0 , and

$$G_0 = \bigcap_{G_1 \in \mathcal{O}(G)} G_1.$$

By a theorem of van Dantzig [9], whose proof is not very difficult, the G_1/G_0 that are both compact and open also form a basis of the identity neighbourhoods of G/G_0 .

If $\phi: G \rightarrow H$ is a morphism, then $\phi(G_0)$ is connected, so $\phi(G_0) \leq H_0$. Hence ϕ induces a canonical morphism $\gamma: G/G_0 \rightarrow H/H_0$, given by

$$\gamma(g) = \phi(g)H_0/H_0 \quad \text{for all } g \in G.$$

Corollary 4. *Assume that $\phi: G \rightarrow H$ is a morphism of locally compact groups and define $\gamma: G/G_0 \rightarrow H/H_0$ as above. Then the following are equivalent:*

- (1) ϕ is open;
- (2) $\gamma: G/G_0 \rightarrow H/H_0$ is open and $H_0 = \bigcap_{G_1 \in \mathcal{O}(G)} \phi(G_1)$.

Proof. Assume that (1) holds. For all $G_1 \in \mathcal{O}(G)$, we have $\phi(G_1) \in \mathcal{O}(H)$ and hence $H_0 \leq \phi(G_1)$. Further, if $G_0 \leq G_1 \leq G$ and $G_1/G_0 \in \mathcal{O}(G/G_0)$, then we have $G_1 \in \mathcal{O}(G)$, and so

$$\gamma(G_1/G_0) = \phi(G_1)H_0/H_0 = \phi(G_1)/H_0,$$

and $\gamma(G_1/G_0)$ is open in H/H_0 , that is, γ is open. Further, $\phi^{-1}(H_1) \in \mathcal{O}(G)$ for all $H_1 \in \mathcal{O}(H)$, and so

$$H_0 \leq \bigcap_{G_1 \in \mathcal{O}(G)} \phi(G_1) \leq \bigcap_{H_1 \in \mathcal{O}(H)} H_1 = H_0.$$

It follows that (2) holds.

Next, assume that Γ is open and $H_0 \leq \phi(G_1)$ for all $G_1 \in \mathcal{O}(G)$. If $G_1 \in \mathcal{O}(G)$, then $G_1/G_0 \in \mathcal{O}(G/G_0)$, so $\gamma(G_1/G_0) \in \mathcal{O}(H/H_0)$. Since $H_0 \leq \phi(G_1)$,

$$\gamma(G_1/G_0) = \phi(G_1)H_0/H_0 = \phi(G_1)/H_0,$$

whence $\phi(G_1) \in \mathcal{O}(G)$. By Theorem 2, ϕ is open, as claimed. □

Examples show that “simpler” versions of this corollary fail. On the one hand, if G_0 and H_0 are both open subgroups, then γ is trivially open. In this case, ϕ is open if and only if the restricted map $\phi|_{G_0}$ is open, and this holds if and only if $H_0 = \phi(G_0)$. On the other hand, Example 1 (a) shows that there are open maps $\phi: G \rightarrow H$ for which $\phi(G_0) \neq H_0$.

Next, we consider where the chief difficulties of the theorem lie. Assume that $\phi: G \rightarrow H$ is a morphism, and define $K := \ker(\phi)$ and $H_1 := \phi(G)^-$. Then ϕ factorises as the composition $\iota \circ \beta \circ \pi$, where π is the canonical projection of G onto G/K , $\beta: \phi(G) \rightarrow H_1$ is the injection of a dense subgroup of H_1 into H_1 , and $\iota: H_1 \rightarrow H$ is the injection of the closed subgroup H_1 into H .

Since π is open by definition, ϕ is open if and only if the canonical induced morphism $\psi: G/K \rightarrow H$ is open. Thus, in considering questions of openness of morphisms, there is no significant loss of generality in restricting our attention to the case where ϕ is injective and π is trivial.

Likewise, if H_1 is not open, it is not possible that $\phi(U)$ can be open for any nonempty open subset U of G , and there is no real loss of generality in restricting our attention to the case in which H_1 is open. We may then assume that $H = H_1$, that is, that ι is trivial.

Finally, if ι and π are trivial and ϕ sends open subgroups to open subgroups, then $\phi(G)$ is open and dense, and hence ϕ is surjective. Thus the heart of the matter is what happens with groups G and H that are isomorphic as abstract groups, and the next corollary is actually an equivalent version of Theorem 2.

Corollary 5. *Let G and H be locally compact groups, and let $\phi: G \rightarrow H$ be a continuous bijection of G onto H . Assume that $\phi(G_1) \in \mathcal{O}(H)$ for all $G_1 \in \mathcal{O}(G)$. Then ϕ is an isomorphism of locally compact groups, that is, $G \cong H$.*

Proof. By Theorem 2, the bijective morphism ϕ is open and thus is an isomorphism. \square

2 Another version

In this section, we present a measure theoretic proof of Theorem 2 and related results. We begin with a simple lemma.

Lemma 6. *Let $\phi: G \rightarrow H$ be a morphism of locally compact groups. If ϕ is open, then $\phi(U)$ has nonzero measure in H for all nonempty open subsets U of G . If ϕ is not open, then $\phi(V)$ has null measure in H for all σ -compact subsets V of G .*

Proof. If U and V are nonempty relatively compact (or σ -compact) open sets in G , then V may be covered by finitely (or countably) many translates of U and vice versa, and $\phi(V)$ may be covered by finitely (or countably) many translates of $\phi(U)$ and vice versa. It follows that $\phi(U)$ has null measure if and only if $\phi(V)$ has null measure.

In light of this observation and the fact that the Haar measure of a nonempty open set is never 0, it suffices to prove that if U is a nonempty relatively compact open set in G and $\phi(U)$ has positive Haar measure in H , then $\phi(U)$ is open in H .

Given any $z \in U$, $z^{-1}U$ is a relatively compact open neighbourhood of the identity in G , and by [3, p. 18], it is possible to find a relatively compact open neighbourhood V of the identity in G that is symmetric ($V^{-1} = V$) such that $V^2 \subseteq z^{-1}U$. It will suffice to show that $\phi(V^2)$ contains an open neighbourhood W of the identity in H , for then $\phi(z)W$ is an open neighbourhood of $\phi(z)$ in H that is contained in $\phi(zV^2)$, and hence in $\phi(U)$.

Since V is relatively compact and ϕ is continuous, $\phi(V)$ is relatively compact in H and has finite Haar measure, which is not 0 by our initial observation, since the measure of $\phi(U)$ is not 0. Then the convolution on H of the characteristic function of $\phi(V)$ with itself, given by the formula

$$\chi_{\phi(V)} * \chi_{\phi(V)}(x) = \int_H \chi_{\phi(V)}(y) \chi_{\phi(V)}(y^{-1}x) \, dy,$$

is a continuous function on H (see [3, (20.16)]) that does not vanish at the identity but vanishes outside $\phi(V^2)$, and $\{y \in H : \chi_{\phi(V)} * \chi_{\phi(V)}(y) \neq 0\}$ is the desired set W . \square

Lemma 6 has the following corollaries, the first of which we already know from Theorem 2.

Corollary 7. *Let $\phi: G \rightarrow H$ be a morphism of locally compact groups. If $\phi(G_1)$ is open in H for all compactly generated G_1 in $\mathcal{O}(G)$, then ϕ is an open mapping.*

Proof. Assume that ϕ is not open, and take a nonempty relatively compact symmetric open subset U of G . Then U generates an open subgroup $\langle U \rangle$ of G , equal to $\bigcup_{n \in \mathbb{N}} U^n$. Since U^n is a nonempty relatively compact open subset of G for all positive integers n and ϕ is not open, each $\phi(U^n)$ has measure 0 by Lemma 6, whence $\phi(\langle U \rangle)$ has measure 0, and so $\phi(\langle U \rangle)$ cannot be open. \square

Corollary 8. *Let $\phi: G \rightarrow H$ be a morphism of locally compact groups, and suppose that there is a compact subset K of G such that $\phi(K)$ has positive Haar measure in H . Then ϕ is open.*

3 Comments

Our arguments and the proof of the Open Mapping Theorem require that we deal with a property of subsets of G that is closed under taking countable unions, which is why Baire category or Haar measure play a role.

The reference [7] provides an Open Mapping Theorem for almost connected pro-Lie groups [7, Theorem 8.60, p. 365] and an Alternative Open Mapping Theorem [7, Theorem 11.79, p. 517]; earlier discussion of open mapping theorems may be found in [4, 5]. This result continues to attract interest: see, for example, [8].

We thank our referee for the following observation. In the study of the structure of locally compact groups G , for the compact Hausdorff ‘‘Chabauty spaces $\mathcal{S}(G)$ ’’ of closed subgroups of G , any continuous morphism $\phi: G \rightarrow H$ between locally compact groups G and H functorially induces continuous maps

$$\mathcal{S}(\phi): \mathcal{S}(G) \rightarrow \mathcal{S}(H),$$

given by $\mathcal{S}(\phi)(L) = \phi(L)$. Accordingly, the following application of our Theorem 2 above is of considerable interest.

Corollary 9. *For profinite groups G, H , the following conditions are equivalent:*

- (1) ϕ is open;
- (2) $\mathcal{S}(\phi)$ is open.

Proof. Proposition 5 in [2] shows that (1) implies (2).

For the converse, let O be an open subgroup of G . In view of [1, Proposition 32], $\mathcal{S}(O)$ is open in $\mathcal{S}(G)$ and therefore $\mathcal{S}(\phi)(\mathcal{S}(O))$ is open in $S(H)$. As $\mathcal{S}(\phi)(\mathcal{S}(O)) = \mathcal{S}(\phi(O))$, also by [1, Proposition 32], $\phi(O)$ is open in H , and so, by Theorem 2, ϕ is open. \square

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Author information

Corresponding author:

Sidney A. Morris, School of Mathematical and Physical Sciences,
La Trobe University, Melbourne, Bundoora, VIC 3086, Australia.

E-mail: morris.sidney@gmail.com

Michael G. Cowling, School of Mathematics and Statistics,
University of New South Wales, Sydney, NSW 2052, Australia.
E-mail: m.cowling@unsw.edu.au

Karl H. Hofmann, Fachbereich Mathematik, Technische Universität Darmstadt,
Schlossgartenstraße 7, 64289 Darmstadt, Germany.
E-mail: hofmann@mathematik.tu-darmstadt.de