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Finite normal subgroups of strongly verbally closed groups

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Abstract. In a recent paper by A. A. Klyachko, V. Y. Miroshnichenko, and A. Y. Olshanskii, it is proven that the center of any finite strongly verbally closed group is a direct factor. In this paper, we extend this result to the case of finite normal subgroups of any strongly verbally closed group. It follows that finitely generated nilpotent groups with nonabelian torsion subgroups are not strongly verbally closed.

1 Introduction

A subgroup H of a group G is called *verbally closed* [14] if any equation of the form

$$w(x_1, x_2, \dots, x_n) = h,$$

where w is an element of the free group $F(x_1, \dots, x_n)$ and $h \in H$, having solutions in G has a solution in H. If each system of equations with coefficients from H,

$$\{w_1(x_1, \dots) = 1, \dots, w_m(x_1, \dots) = 1\},$$

where $w_i \in H * F(x_1, \dots, x_n)$ (and $*$ means the free product), having solutions in G has a solution in H, then the subgroup H is said to be *algebraically closed* in G. Note that if the subgroup H is algebraically closed in the group G, then it is verbally closed in G.

A group G is called *strongly verbally closed* if it is algebraically closed in any group containing G as a verbally closed subgroup. Thus, verbal closedness (as well as algebraic closedness) is a property of a subgroup, while strong verbal closedness is a property of an abstract group. The class of strongly verbally closed groups is fairly wide. For example, it includes

- all abelian groups [12],
- all free groups [7],
- all virtually free groups containing no nontrivial finite normal subgroups [7, 8],

- all groups decomposing nontrivially into a free product [13],
- fundamental groups of all connected surfaces except the Klein bottle [6, 12],
- all finite groups with nonabelian monolith [9],
- the infinite dihedral group [8] and any finite dihedral group whose order is not divisible by 8 (see [9]),
- all acylindrically hyperbolic groups with no nontrivial finite normal subgroups [2].

The class of non-strongly-verbally-closed groups is fairly wide too. Among such groups are the following:

- the already mentioned fundamental group of the Klein bottle [6],
- the discrete Heisenberg group [9],
- any finite group whose center is not its direct factor (in particular, any finite nonabelian nilpotent group) [7, 9, 17].

Proving the strong verbal closedness (or otherwise) of a group is not easy. In [9], for example, a question is raised.

Question 1. *Does there exist a finitely generated nilpotent nonabelian strongly verbally closed group?*

A negative answer to this question would yield a broad generalization of the last two examples of non-strongly-verbally-closed groups mentioned above. In this paper, we give a partial answer to this question. More precisely, we use Theorem 3.6 to establish the following result.

Theorem 1.1. *Let G be a finitely generated nilpotent group with nonabelian torsion subgroup. Then G is not strongly verbally closed.*

Some further results towards a negative answer to Question 1 are proved in the final section of the paper. For example, we prove that some finitely generated nilpotent nonabelian groups with abelian torsion subgroups are not strongly verbally closed.

A property that is stronger than the strong verbal closedness is the property of being a strong retract [9]. A group H is called a *strong retract* if it is a retract of any group $G \geqslant H$ from the variety generated by the group H (recall that the variety generated by a class of groups $\mathcal{K}$ is the class of all groups satisfying all identities that hold in all groups from $\mathcal{K}$; see [15]). This definition gives raise to the following question from [9].

Question 2. *What is an arbitrary finite strong retract?*

In [9], some examples of strong retracts are provided. In this paper, we describe the structure of all nilpotent strong retracts. In particular, we prove the following result.

Theorem 1.2. *Let G be a nilpotent strong retract. Then G is abelian. Moreover, G is either divisible, or it has bounded period and its decomposition as a direct sum of primary cyclic factors has the property that the orders of any distinct summands are either equal or coprime.*

Below, we provide a brief list of *notation* we use. If x, y are elements of some group, then the symbol $[x, y]$ denotes their commutator $x^{-1}y^{-1}xy$. The symbol $\mathrm{ord}(x)$ denotes the order of an element x of a group G. The center of a group G is denoted by $Z(G)$, and its commutator subgroup is denoted by G'. The centralizer of a subset X of a group G is denoted by $C(X)$, and if H is a subgroup of G, then $C_H(X) = C(X) \cap H$. The symbol $\langle\langle X\rangle\rangle$ stands for the normal closure of a subset X of a group G (that is the intersection of all normal subgroups of G containing X). The free group with a basis X is denoted as $F(X)$ or F_n in case X has $n \in \mathbb{N}$ elements. The identity mapping from X to itself is denoted by id. We use the symbol $H \cong G$ to express the fact that groups H and G are isomorphic. The symbol $H \leqslant G$ denotes the fact that a group H is a subgroup of G. The symbol $H \trianglelefteq G$ denotes the fact that H is a normal subgroup of G.

2 Nilpotent strong retracts

Let us recall some terminology:

- a subgroup H of a group G is called a *retract* if there exists an endomorphism $\varphi: G \to H$ such that $\varphi \circ \varphi = \varphi$,
- a *variety of groups* is the class of all groups $\mathcal{K}$ satisfying a given set of identities X,
- the variety generated by a group G is designated by $\mathbf{var}\, G$,
- a group G is called *divisible* if, for any $g \in G$ and $n \in \mathbb{N}$, the equation $x^n = g$ has a solution in G,
- the *period of a group G* is the least number $n \in \mathbb{N}$ such that $x^n = 1$ for any $x \in G$. If such a number exists, then G is a group of *bounded period.*

Note that if G is an abelian group, then $H \leqslant G$ is its retract if and only if H is a direct summand of G. This means that the property of being a strong retract for the abelian group G is equivalent to the property of G being a direct summand of any group $H \in \mathbf{var}\, G$ containing G. For the subsequent discussion, we need the description of all varieties of abelian groups (see [3, Paragraph 18, Exercise 7]).

Lemma 2.1. *Varieties of abelian groups are precisely the following classes of groups:* (1) *the class of all abelian groups;* (2) *the class of all abelian groups with period dividing some fixed positive integer n.*

Proof. Let $\mathcal{K}$ be a variety of abelian groups. Since any set X of identities follows from the set of identities X', consisting of relation $x^n = 1$, where $n \geqslant 0$, and of commutator relations, i.e. relations of the form $w(x_1, \dots, x_k) = 1$, where $w(x_1, \dots, x_k) \in F_k$ and $k \in \mathbb{N}$ (see [15, Theorem 12.12]), $\mathcal{K}$ is the class of all groups satisfying the identities $x^n = 1$ and $[x, y] = 1$, where $n \geqslant 0$. This completes the proof. □

The following is true of divisible abelian groups (see, for example, [10, §23]).

Lemma 2.2. *If G is a divisible abelian group and H is an abelian group such that $G \leqslant H$, then G is a direct summand of H.*

To begin with, consider the case when G is not a group of bounded period.

Proposition 2.3. *An abelian group G of unbounded period is a strong retract if and only if it is divisible.*

Proof. Sufficiency follows from Lemma 2.2. Let G be an abelian group of unbounded period. From Lemma 2.1, it follows that $\mathbf{var}\, G$ is the class of all abelian groups. In particular, $\mathbf{var}\, G$ contains a divisible group H containing G (see [10, §23]). Though, if G is not divisible itself, it is not a direct summand of H (as direct summands of a divisible group are divisible themselves; see [10, §23]), so G is not a strong retract. □

Let us move on to abelian groups of bounded period. Prüfer's first theorem provides a complete description of these groups [10, §24].

Proposition 2.4. *An abelian group G of bounded period d is a direct sum of primary cyclic groups, i.e. $G \cong \bigoplus_{i \in I} \mathbb{Z}_{p_i^{k_i}}$, where p_i are prime numbers and k_i are natural numbers such that $p_i^{k_i} \mid d$, $i \in I$ (I is an index set).*

Now, we are ready to proceed with our description.

Proposition 2.5. *An abelian group G of bounded period is a strong retract if and only if, in its decomposition into the direct sum of primary cyclic groups, the orders of any distinct direct summands are either equal or coprime. That is,*

$$G \cong \bigoplus_{i=1}^{m} C_{p_i^{k_i}}(n_i),$$

where $C_{p_i^{k_i}}(n_i)$ is equal to the direct sum of n_i copies of the group $\mathbb{Z}_{p_i^{k_i}}$, p_i are distinct prime numbers, m and k_i are positive integers, and n_i are some cardinal numbers.

Proof. Suppose that G cannot be decomposed into such a direct sum. We may assume that

$$G = \bigoplus_{i=1}^{m} \bigoplus_{j \in I_i} \mathbb{Z}_{p_i^{k_j}}, \tag{2.1}$$

where $m \in \mathbb{N}$, $|I_i| = n_i$, and among k_j, $j \in I_i$, there are only finitely many different numbers (since G is a group of bounded period), but there exists $i \in \{1, \ldots, m\}$ such that, for some $j_1, j_2 \in I_i$, $k_{j_1} \neq k_{j_2}$.

Consider the group $H = \bigoplus_{i=1}^{m} C_{p_i^{s_i}}(n_i)$, where

$$s_i = \max\{k_j \mid \mathbb{Z}_{p_i^{k_j}} \text{ is a direct summand in decomposition (2.1)}\},$$
$$i = 1, 2, \ldots, m.$$

Since G and H have the same period $\prod_{i=1}^{m} s_i$, it follows from Lemma 2.1 that $H \in \mathbf{var}\, G$.

Consider the injection $f\colon G \to H$, which is defined on each direct summand in (2.1) as follows: let $i \in \{1, \ldots, m\}$, $j \in I_i$, $f\colon \mathbb{Z}_{p_i^{k_j}} \hookrightarrow \mathbb{Z}_{p_i^{s_i}}$, where $\mathbb{Z}_{p_i^{s_i}}$ is the jth summand from the decomposition of $C_{p_i^{s_i}}(n_i)$ into the direct sum. Every direct summand from (2.1) is mapped into the corresponding direct summand of the decomposition of H so that the restriction of f to $\mathbb{Z}_{p_i^{k_j}}$ is a natural injection: if $k_j = s_i$, then it is the identity map; otherwise it is a mapping to the subgroup of $\mathbb{Z}_{p_i^{s_i}}$ of order $p_i^{k_j}$. From the uniqueness of the decomposition of an abelian group of bounded period into a direct sum of primary cyclic groups [3, Theorem 17.4], it follows that $f(G)$ is not a direct summand of H. Thus, G is not a strong retract.

Now, suppose that G has the decomposition from the statement of the proposition. Let $H \in \mathbf{var}\, G$ and let $f\colon G \hookrightarrow H$ be a monomorphism. Since any monomorphism preserves the order of an element, the p_ith component of G is mapped into the p_ith component of H under f, so it suffices to prove the proposition only for the case $G = C_{p^k}(n)$, where p is prime, $k \in \mathbb{N}$, and n is some cardinal number.

Let us show that there exists a subgroup $X \leqslant H$ such that $H = f(G) \oplus X$. Let

$$M = \{Y \leqslant H \mid Y \cap f(G) = \{0\}\}$$

be the set of all subgroups of H having trivial intersection with $f(G)$. Note that $\{0\} \in M$. An order on M is introduced as follows: for $X, Y \in M$, $X \preceq Y$ if X is a subgroup of Y. It can be verified directly that this is an order on M. Any chain $\{Y_\alpha\} \subseteq M$ of subgroups having trivial intersection with $f(G)$ is bounded by an element $Y \in M$, where $Y = \bigcup_\alpha Y_\alpha$. So, by Zorn's lemma, M contains a maximal element X: $X \leqslant H$, $X \cap f(G) = \{0\}$, and X is not a subgroup of any bigger (with respect to the given ordering) subgroup satisfying this property.

From $X \cap f(G) = \{0\}$, it follows that $f(G) + X = f(G) \oplus X$. It remains to prove that $H = f(G) + X$. Let $h \in H$. Then there exists such $k \in \mathbb{N}$ that $kh \in f(G) + X$. Indeed, otherwise $\langle h \rangle \cap (f(G) + X) = \{0\}$, which means that $(\langle h \rangle + X) \cap f(G) = \{0\}$, leading to a contradiction with the maximality of X.

Let s be the smallest such number. Replacing h by a suitable power, if necessary, we may assume that $s = 1$ or s is a prime. Two cases are possible:

(1) $s = p$. Then $ph = f(g) + x$ for some $g \in G$, $x \in X$. If $g = pg_1$, $g_1 \in G$ (g_1 may be equal to zero), then $ph - f(pg_1) = x$. However, from the fact that $h - f(g_1) \notin X$ (as $h \notin f(G) + X$), it can be obtained that

$$(X + \langle h - f(g_1) \rangle) \cap f(G) = \{0\},$$

which leads to a contradiction with the maximality of X. Consequently, $g \neq pg_1$ for any $g_1 \in G$. As $g \neq 0$, $\mathrm{ord}(g) = p^k$. Though, $\mathrm{ord}(ph) = p^r < p^k$, so

$$p^r(ph) = 0 = p^r(f(g)) + p^r x.$$

As the sum $f(G) + X$ is direct, $p^r f(g) = p^r x = 0$, which means that $p^r g = 0$, which is impossible.

(2) $s \neq p$. For abelian groups of period p, the mapping $g \mapsto sg$ is an automorphism, so, as $sh = f(g) + x$ for some $g \in G$, $x \in X$, there exist such $g_1 \in G$, $x_1 \in X$ that $g = sg_1$, $x = sx_1$. Thus, $s(h - f(g_1) - x_1) = 0$. No nontrivial element of H has order s, so $h = f(g_1) + x_1$.

As a result, $H = f(G) \oplus X$, and G is a strong retract. □

Proposition 2.6. *The center of a strong retract is its direct factor.*

Proof. Let G be a strong retract. The center of any group is a normal subgroup, so it suffices to prove that $Z(G)$ is a retract of G. Consider the central product of G with its copy $\tilde{G}$ with joined center

$$K = G \underset{Z(G)=Z(\tilde{G})}{\times} \tilde{G} = (G \times \tilde{G})/\{(g, g^{-1}) \mid g \in Z(G)\} \in \mathbf{var}\, G.$$

The group $\tilde{G}$ is isomorphic to the group G, so it is a strong retract too. Let ρ be a retraction from K to its subgroup $\tilde{G}$. From the fact that, in the group K, the subgroup G commutes with the subgroup $\tilde{G}$, we obtain $\rho(G) \leqslant Z(G)$. By definition of the retraction, $\rho(g) = g$ for any element $g \in Z(G)$. Thus, the restriction of ρ to the subgroup G of the group K is the desired retraction to $Z(G)$. □

The following simple proposition shows that consideration of nilpotent groups does not yield any new strong retracts.

Proposition 2.7. *Let G be a nilpotent strong retract. Then G is abelian.*

Proof. Any nontrivial normal subgroup of a nilpotent group intersects the center of this group nontrivially (see [5, Theorem 16.2.3]). By combining this with Proposition 2.6, we deduce that any nilpotent strong retract is equal to its center. □

This completes the proof of Theorem 1.2. In the next paragraph, we show that many nilpotent groups are not even strongly verbally closed.

3 Finite normal subgroups of strongly verbally closed groups

We say that a group presentation $\langle X \mid R\rangle$ is *finitely presented* over a group presentation $\langle Y \mid S\rangle$ if there exist finite sets A and B such that $\langle X \mid R\rangle \cong \langle X' \mid R'\rangle$, where $X' = Y \cup A$, $R' = S \cup B$.

The following lemma reveals that this definition is, in fact, a group property (which means it does not depend on the choice of a group presentation), so it makes sense to speak about the finite presentability of one group over the other group.

Lemma 3.1. *Suppose that a group presentation $\langle X \mid R\rangle$ is finitely presented over a group presentation $\langle Y \mid S\rangle$ and $\langle Y \mid S\rangle \cong \langle Y' \mid S'\rangle$. Then $\langle X \mid R\rangle$ is finitely presented over $\langle Y' \mid S'\rangle$.*

Proof. We may assume that $X = Y \cup A$ and $R = S \cup B$ for some finite sets A and B. It can be easily shown (for similar fact, refer to [11, Chapter II, Proposition 2.1]) that groups defined by group presentations $\langle Y \mid S\rangle$ and $\langle Y' \mid S'\rangle$ are isomorphic if and only if the presentation $\langle Y' \mid S'\rangle$ can be obtained from $\langle Y \mid S\rangle$ by applying a finite number of *Tietze transformations* (and their inverses):

- adding to the set S an arbitrary set $T \subseteq \langle\langle S\rangle\rangle \trianglelefteq F(Y)$,
- adding to the set Y an arbitrary set $\tilde{Y}$ while adding to the set S a set

$$\{\tilde{y} = w_{\tilde{y}} \mid \tilde{y} \in \tilde{Y},\ w_{\tilde{y}} \in F(Y)\}.$$

It is sufficient to prove the lemma only for the case when $\langle Y' \mid S' \rangle$ is obtained from $\langle Y \mid S \rangle$ by applying one Tietze transformation. One can easily verify that, in case of the first transformation, $X' = X$ and $R' = R \cup T$, while in case of the second transformation, $X' = X \cup \tilde{Y}$ and $R' = R \cup \{\tilde{y} = w_{\tilde{y}} \mid \tilde{y} \in \tilde{Y},\ w_{\tilde{y}} \in F(Y)\}$ provide the desired group presentation. □

By virtue of Lemma 3.1, the following definition may be introduced: a group G is *finitely presented* over a group H if there exists a presentation of G that is finitely presented over any presentation of H.

Lemma 3.2. *Suppose that G contains a subgroup H and a finite normal subgroup N such that G/N is finitely presented over $H/(H \cap N)$. Then G is finitely presented over H.*

Proof. This is similar to the proof of Hall's theorem [4] on the preservation of finite presentability of a group under extensions (see also [16, Theorem 2.2.4]).

Write $H = \langle X \mid R \rangle \leqslant G$, and $N = \langle Y \mid S \rangle \trianglelefteq G$, where Y and S are finite. We are assuming that G/N is finitely presented over $H/(H \cap N) \cong \langle X \mid R \cup C \rangle$, where $\langle\langle C \rangle\rangle = H \cap N$ and the set C is finite. Consequently,

$$G/N \cong \langle X \cup A \mid R \cup C \cup B \rangle,$$

where sets A and B are finite.

Let us construct a presentation of G. For the generators, take $\overline{X} \cup \overline{A} \cup \overline{Y}$, where the sets $\overline{X}$, $\overline{A}$, $\overline{Y}$ are in one-to-one correspondence with the sets X, A, Y respectively. The sets R, S, C, and B are in correspondence with the sets $\overline{R}$, $\overline{S}$, $\overline{C}$, and $\overline{B}$ respectively. For the defining relations, take the union of the following sets: $\overline{R}$, $\overline{S}$, $\overline{C}_1$, $\overline{B}_1$, and $\overline{T}$, where

$$\overline{C}_1 = \{c w_c^{-1} \mid c \in \overline{C},\ w_c \in F(\overline{Y})\}, \quad \overline{B}_1 = \{b w_b^{-1} \mid b \in \overline{B},\ w_b \in F(\overline{Y})\},$$
$$\overline{T} = \{a^{-1} y a w_{a,y}^{-1}, a y a^{-1} v_{a,y}^{-1} \mid a \in \overline{A},\ y \in \overline{Y},\ w_{a,y}, v_{a,y} \in F(\overline{Y})\}$$

($c \in \overline{C}$ and $b \in \overline{B}$ are considered as words from $F(\overline{X})$ and from $F(\overline{X} \cup \overline{A})$ respectively). Denote the group with the above mentioned generators and defining relations as $\tilde{G}$,

$$\tilde{G} = \langle \overline{X} \cup \overline{A} \cup \overline{Y} \mid \overline{R} \cup \overline{S} \cup \overline{C}_1 \cup \overline{B}_1 \cup \overline{T} \rangle.$$

Consider a surjective homomorphism $\theta \colon \tilde{G} \to G$, defined by the following bijections $\overline{X} \to X$, $\overline{A} \to A$, $\overline{Y} \to Y$ on the generators (defining relations are mapped into true identities under such a map on generators, so such a homomorphism exists). The restriction $\theta|_K\colon K \to N$ on the subgroup $K = \langle \overline{Y} \rangle \leqslant \tilde{G}$ is an isomorphism as all the relations in the alphabet $\overline{Y}$ in $\tilde{G}$ are consequences of the defining relations $\overline{S}$. Moreover, $K \trianglelefteq \tilde{G}$.

The homomorphism $\tilde{\theta}\colon \tilde{G}/K \to G/N$ generated by θ, is an isomorphism too. Now, let $g \in \ker\theta$. Then $gK \in \ker\tilde{\theta}$, but $\tilde{\theta}$ is an isomorphism, so $g \in K$. Finally, $\theta|_K$ is an isomorphism, so $g = 1$. □

The following lemma provides a criterion for algebraic closedness of a subgroup H of a group G that is finitely presented over H (for similar propositions, refer to [14]).

Lemma 3.3. *Suppose that $H = \langle X \mid R \rangle$ is a subgroup of G, and G is finitely presented over H. The subgroup H is algebraically closed in G if and only if H is a retract of G.*

Proof. Suppose H is algebraically closed in G and

$$A = \{a_1, \dots, a_m\}, \quad B = \{s_1, \dots, s_n\}$$

are the sets from the definition of finite presentability of G over H. The relations $s_i(a_1, \dots, a_m, X) = 1$, $i = 1, \dots, n$, correspond to a system of equations with coefficients from H,

$$\begin{cases} s_1(t_1, \dots, t_m, X) = 1 \\ \qquad\qquad\vdots \\ s_n(t_1, \dots, t_m, X) = 1, \end{cases}$$

which, by condition, has a solution $t_1 = a_1, \dots, t_m = a_m$. By virtue of the algebraic closedness of H in G, this system has a solution $t_1 = h_1, \dots, t_m = h_m$ in H. Mapping $X \sqcup \{a_1, \dots, a_m\} \to H$, $X \ni x \mapsto x$, $a_i \mapsto h_i$ extends to a surjective homomorphism $\varphi\colon G \to H$, as defining relations of G are mapped into true identities under such a mapping of generators (note that R is the set of words in the alphabet X).

This homomorphism is the desired retraction: let $h \in H$, $h = v(x_1, \dots, x_r)$, $x_i \in X$. Applying to this word the homomorphism φ, we get

$$\varphi(h) = v(\varphi(x_1), \dots, \varphi(x_r)) = h.$$

Algebraic closedness of a subgroup H of a group G follows from retractness of H in G for any group G (see [14, Proposition 2.2 (1)]). □

The following lemma was proven in [9]. We will need it to prove Theorem 3.6.

Lemma 3.4. *Let C be a finite elementary abelian p-group (where p is a prime number). For any $k \in \mathbb{N}$, there exists $t \geqslant k$ such that the direct product*

$$P = \bigtimes_{i=1}^{t} C_i$$

of copies C_i of C contains a subgroup R invariant with respect to the diagonal action on P of the endomorphism algebra End C *with the following properties:*

(1) $R \subseteq \bigcup \ker \rho_j$, *where $\rho_j \colon P \to C_j$, $j = 1, \dots, t$, are the natural projections,*

(2) $R \cdot \bigtimes_{j \notin J} C_j = P$ *for any subset $J \subseteq \{1, \dots, t\}$ of cardinality $|J| = k$,*

(3) *moreover, each such J is contained in a set $J' \supseteq J$ such that*

$$P = R \times \Big(\bigtimes_{j \notin J'} C_j \Big),$$

and there exist integers $n_{ij} \in \mathbb{Z}$ such that the projection $\pi \colon P \to \bigtimes_{j \notin J'} C_j$ with the kernel R acts as $C_i \ni c_i \mapsto \prod_{j \notin J'} c_j^{n_{ij}}$, where $c_j \in C_j$ is the element corresponding to c_i under the isomorphism $C_i \cong C \cong C_j$.

Lemma 3.5. *Let G be a group and $N \trianglelefteq G$. If $xC(N) = yC(N)$ for some $x, y \in G$, then x and y act on N (by conjugations) identically.*

Proof. From $xC(N) = yC(N)$, it follows that, for some $c \in C(N)$, $x = yc$. Then, for $n \in N$, we have

$$x^{-1}nx = c^{-1}(y^{-1}ny)c = y^{-1}ny.$$

The last identity is true, as (due to normality) $y^{-1}ny \in N$ and $c \in C(N)$. □

The following theorem provides a generalization of the result from [9] about the center of a finite strongly verbally closed group (see [9, Centre theorem]). The proof is also analogous to the proof of that theorem, with the exception of some nuances.

Theorem 3.6. *Let H be a strongly verbally closed group. For any finite normal subgroup T of H, for any abelian subgroup A of T, normal in H, it is true that $Z(C_T(A))$ is a direct factor of $C_T(A)$, and some complement is normal in H.*

Proof. Let H be such a group, and let $L = C_T(A)$. It suffices, for each prime p, to find a homomorphism $\psi_p \colon L \to Z(L)$ commuting with the conjugation action of H on L (this action is well-defined as $L \trianglelefteq H$) and injective on the p-component

$Z_p(L)$ of the center of L. Then the homomorphism

$$\psi\colon L \to Z(L), \quad x \mapsto \prod_p \pi_p(\psi_p(x)),$$

where $\pi_p\colon Z(L) \to Z_p(L)$ is the projection on the p-component, is injective on $Z(L)$, so its kernel is the desired complement D (normality of D in H follows from the fact that ψ commutes with the action of H on L).

Suppose that there are no such homomorphisms for some prime number p, i.e. every homomorphism $f\colon L \to Z(L)$ commuting with the action of H on L is not injective on $Z_p(L)$. Then it is not injective on the maximal elementary abelian p-subgroup $C \leqslant Z_p(L)$ (it is finite as L is finite). Indeed, if $x \in Z_p(L)$, $x \neq 1$ is an element such that $f(x) = 1$, then, raising it to the appropriate power d, we get $f(x^d) = 1$ and $x^d \in C$, $x^d \neq 1$.

Use Lemma 3.4 (with respect to C) to choose a positive integer t as in the lemma (for some k to be specified later). Then consider the fibered product of t copies of the group H,

$$Q = \{(h_1, \dots, h_t) \mid h_1 L = \dots = h_t L\} \leqslant H^t.$$

First of all, let us show that the subgroup $R \leqslant C^t \leqslant Q$ from Lemma 3.4 is normal in Q. Note that the subgroup R is invariant under the diagonal action of automorphisms $\operatorname{Aut} C \leqslant \operatorname{End} C$. It remains to show that Q acts diagonally on $P = C^t$ by conjugation. This follows from Lemma 3.5.

Indeed, let

$$q = (q_1, \dots, q_t) \in Q,$$
$$p = (p_1, \dots, p_t) \in P.$$

As $q_1 L = q_2 L = \dots = q_t L$, according to Lemma 3.5, $q^{-1} p q = \tilde{q}^{-1} p \tilde{q}$, where $\tilde{q} = (q_1, \dots, q_1)$. It means that the conjugation action of Q on P is diagonal. On the other hand, the diagonal action by conjugation induces an endomorphism of C^t (due to normality of $C \trianglelefteq H$), and R is invariant with respect to the diagonal action of such endomorphisms, leading to normality of R in Q.

Put $G = Q/R$. First, let us show that H embeds into G. The group H embeds into Q diagonally: $h \mapsto (h, \dots, h)$, $h \in H$. This homomorphism serves as an embedding into G as well, as all projections of any nontrivial diagonal element of Q are nontrivial (and R is contained in the union of the kernels of these projections).

Now, let us prove the verbal closedness of this diagonal subgroup (denote it as H too) in G. Consider an equation

$$w(x_1, \dots, x_n) = (h, \dots, h)$$

having a solution in G and let $\tilde{x}_1, \ldots, \tilde{x}_n$ be a preimage (in Q) of a solution $x_1, \ldots, x_n$. Then (in Q)

$$w(\tilde{x}_1, \ldots, \tilde{x}_n) = (hc_1, \ldots, hc_t),$$

where $(c_1, \ldots, c_t) \in R$. By the first property in Lemma 3.4, $c_i = 1$ for some i. This means that, in H (the group itself), $w(\tilde{x}_1^i, \ldots, \tilde{x}_n^i) = h$, where $\tilde{x}_j^i$ is the ith coordinate of the vector $\tilde{x}_j$, $j = 1, \ldots, n$.

Let us take $y_j = (\tilde{x}_j^i, \ldots, \tilde{x}_j^i)$, $j = 1, \ldots, n$. Then, in $H \leqslant G$, the following is true:

$$w(y_1, \ldots, y_n) = (h, \ldots, h),$$

which proves the verbal closedness of H in G.

Let $U \leqslant L$ and define

$$U_i := \{(1, \ldots, 1, u, 1, \ldots, 1) \mid u \in U\} \leqslant Q, \quad i = 1, \ldots, t$$

(coordinate u stands on the ith place). It remains to prove that H is not algebraically closed in G.

Claim 3.7. *The group Q is finitely presented over its subgroup H.*

Proof. According to Lemma 3.2, it is sufficient to show that $Q/(L_1 \times \cdots \times L_t)$ is finitely presented over $H/\tilde{L}$, where $\tilde{L} = \{(l, \ldots, l) \mid l \in L\}$. However,

$$Q = H \cdot (L_1 \times \cdots \times L_t),$$

so $Q/(L_1 \times \cdots \times L_t)$ is isomorphic to $H/\tilde{L}$, and the result follows. □

From Lemma 3.3 and Claim 3.7, it follows that it suffices to show that H is not a retract of G. Let $\rho: G \to H$ be a hypothetical retraction, and let $\hat{\rho}: Q \to H$ be its composition with the natural epimorphism $Q \to Q/R = G$. Henceforth, all centralizers and other subgroups we refer to relate to Q.

Let us verify that $\hat{\rho}(L_i) \leqslant C_T(C_T(L)) \leqslant L$ for every i. First, we prove the left inclusion. Let $h \in C_T(L)$. Then h commutes with every element from L; consequently, h, as an element of Q, commutes with L_i. Applying the retraction $\hat{\rho}$ to this identity, we get that $\hat{\rho}(h)$ $(= h)$ commutes with the subgroup $\hat{\rho}(L_i)$, which (by definition of the centralizer) proves the inclusion. The second inclusion follows from the fact that $L = C_T(A) = C(A) \cap T$, which means that

$$C_T(C_T(L)) \leqslant C_T(A \cap T) = C_T(A) = L.$$

The first inclusion here is true as $C(L) \geqslant A$, and the claim follows since $A \leqslant T$.

On the other hand, for $i \neq j$, the mutual commutator subgroup $[L_i, L_j]$ is trivial (for $i \neq j$, L_i and L_j are contained in different components of the fibered product). This means that the image of this mutual commutator subgroup is trivial too: $[\hat{\rho}(L_i), \hat{\rho}(L_j)] = \{1\}$. Consequently,

$$\Big[L_i, \prod_{j \neq i} L_j\Big] = \{1\} \quad \text{and} \quad \Big[\hat{\rho}(L_i), \prod_{j \neq i} \hat{\rho}(L_j)\Big] = \{1\}.$$

If $\hat{\rho}(L_i) = \hat{\rho}(L_l)$ for some $i \neq l$, then (by the virtue of well-known commutator identities) $[\hat{\rho}(L_i), \prod_j \hat{\rho}(L_j)] = \{1\}$, which means that $\hat{\rho}(L_i) \leqslant C_T(L)$ (as $L = \hat{\rho}(L) \leqslant \prod_j \hat{\rho}(L_j)$).

Thereby, if, for some different i and j, $\hat{\rho}(L_i) = \hat{\rho}(L_j)$, then $\hat{\rho}(L_i) \leqslant C_T(L)$. From here and from the inclusion we proved earlier, we get

$$\hat{\rho}(L_i) \leqslant L \cap C_T(L) = Z(L).$$

Let us take k in Lemma 3.4 to be the number of all subgroups of T, and let J be the set of all *exclusive* numbers i, namely such that, for any $l \neq i$, $\hat{\rho}(L_i) \neq \hat{\rho}(L_l)$. Since, among $\hat{\rho}(L_i) \leqslant T$, there are no more than k different subgroups, $|J| \leqslant k$. Thus, from property (3) of Lemma 3.4, we have a decomposition

$$\mathop{\times}_{i=1}^{t} C_i = R \times \Big(\mathop{\times}_{i \in I} C_i\Big),$$

where $I \subseteq \{1, \dots, t\} \setminus J$ is some set of non-exclusive elements. Again, according to property (3) of Lemma 3.4, the projection $\pi \colon \times_{i=1}^{t} C_i \to \times_{i \in I} C_i$ onto the second factor of this decomposition is defined by an integer matrix (n_{ij}), namely, for $c_i \in C_i$, $\pi \colon c_i \mapsto \prod_{j \in I} c_j^{n_{ij}}$, where c_j are elements corresponding to c_i under the isomorphism $C_i \cong C \cong C_j$.

This means that the restriction of π to $C = \{(c, \dots, c) \mid c \in C \leqslant H\}$ is defined as follows:

$$\hat{\pi} \colon (c, \dots, c) \mapsto \prod_{j \in I} c_j^{m_j}, \quad m_j = \sum_i n_{ij}.$$

Here c_j are elements corresponding to c under the isomorphism $C \cong C_j$.

Then (as $i \in I$ are non-exclusive, we have $\hat{\rho}(L_i) \leqslant Z(L)$) consider the composition

$$\Psi \colon C \leqslant Q \to Z(L), \quad c \overset{\pi}{\mapsto} \prod_{j \in I} c_j^{m_j} \overset{\hat{\rho}}{\mapsto} \prod_{j \in I} \hat{\rho}(c_j^{m_j}).$$

This extends to a homomorphism $\Phi \colon L \to Z(L)$ defined by the similar formula

$$\Phi \colon g \mapsto \prod_{j \in I} \hat{\rho}(g_j^{m_j}),$$

where $g \in L$, and $g_j \in L_j$ are elements corresponding to g. Obviously, it is an extension of Ψ and a homomorphism since, for $j \in I$, $\hat{\rho}(L_j) \leqslant Z(L)$ and the group $Z(L)$ is abelian. This homomorphism commutes with the conjugation action of H on L. Indeed, let $g \in H$ and let $\mathfrak{g}$ be the action of g on L by conjugation, namely, for $x \in L$, $\mathfrak{g}(x) = g^{-1}xg$. Let us show that $\Phi \circ \mathfrak{g} = \mathfrak{g} \circ \Phi$. Let $h \in L$. Then

$$\Phi(\mathfrak{g}(h)) = \prod_{j \in I} \hat{\rho}(g^{-1} h_j^{m_j} g) = \prod_{j \in I} g^{-1} \hat{\rho}(h_j^{m_j}) g = \mathfrak{g}(\Phi(h)).$$

The penultimate equality holds since $\hat{\rho}$ is a retraction to H, so it acts identically on H itself. By the assumption we made in the beginning, the kernel of this homomorphism has nontrivial intersection with C: $\ker \Phi \cap C \neq \{1\}$, so the restriction $\Psi = \Phi|_C$ has a nontrivial kernel too.

On the other hand, Ψ is the identity map since $\Psi = \hat{\rho}|_C \circ \pi|_C = \hat{\rho}|_C \circ \hat{\pi} = \hat{\rho}|_C$ (the final equality is true since $\hat{\pi}$ is a projection "forgetting" the R-coordinate, and $\hat{\rho}(R) = \{1\}$ is a composition of the natural homomorphism to the quotient group and of the retraction to H) and $\hat{\rho}|_C = \mathrm{id}$, as $\hat{\rho}$ is the retraction from Q to H, so it acts trivially on C. The contradiction thus obtained completes the proof. □

We can now prove Theorem 1.1.

Corollary 3.8. *Let G be a finitely generated nilpotent group with a nonabelian torsion subgroup. Then G is not strongly verbally closed.*

Proof. Let T be the torsion subgroup of G and set $A = Z(T) \neq \{1\}$. Since T is nilpotent and nonabelian, every nontrivial normal subgroup of T has a nontrivial intersection with A (see [5, Theorem 16.2.3]), so A is not a direct factor of T. □

As an immediate corollary, we recover the Centre theorem from [9].

Corollary 3.9. *Let G be a finite group such that $Z(G)$ is not a direct factor of G. Then G is not strongly verbally closed.*

Theorem 3.6 does not cover the case of finitely generated nilpotent nonabelian groups with abelian torsion subgroups, and it is still unknown whether there are strongly verbally closed groups among such groups. As discussed in the next section, at present, we can only provide a partial answer to this question.

4 Nilpotent non-strongly-verbally-closed groups

Let us recall that the *discrete Heisenberg group* is the free nilpotent group of nilpotency class two with two free generators. It can be easily verified that this group admits a faithful representation in the group of upper triangular matrices of size 3 by 3.

Proposition 4.1. *Let H be the discrete Heisenberg group with a and b being its free generators, and let N be the following subgroup:*

$$N = \langle\langle a^{\alpha}, [a,b]^{n} \rangle\rangle,$$

where α, n are non-negative integers. Then the group $G = H/N$ is strongly verbally closed if and only if $\gcd(\alpha, n) = 1$.

Proof. Let $T(G)$ be the torsion subgroup of G. The center of the group H is equal to its commutator subgroup, and it is isomorphic to the infinite cyclic group. Suppose $T(G) = \{1\}$, so $(\alpha, n) = (0, 0)$ or $(\alpha, n) = (0, 1)$, and, respectively, $G = H$ or G is abelian. The non-strong-verbal-closedness of H was proven in [9], while the strong verbal closedness of abelian groups was proven in [12].

If $\gcd(\alpha, n) = 1$, then, once again, G is abelian since $[a,b]^{\alpha} = [a^{\alpha}, b] \in N$; consequently, it is strongly verbally closed.

Consider the case $\gcd(\alpha, n) = d \neq 1$. Without loss of generality, we may assume that α and n are minimal such that $a^{\alpha} \in N$, $[a,b]^{n} \in N$. Consider the central product of G with its copy $\tilde{G}$ with joined commutator subgroup,

$$K = G \underset{G'=\tilde{G}'}{\times} \tilde{G} = (G \times \tilde{G})/\{(c, c^{-1}) \mid c \in G'\}.$$

The subgroup G is not algebraically closed in K since G is not a retract of K. Indeed, let ρ be a hypothetical retraction. The subgroup G commutes with $\tilde{G}$ in K, so $\rho(\tilde{G}) \leqslant Z(G)$ and $\rho(\tilde{G}') = \{1\}$, which leads to a contradiction with the definition of retraction.

However, G is verbally closed in K. Indeed, let $w \in F(t_1, \dots, t_s)$ be some word and

$$w((h_1 N, h_1' N), \dots, (h_s N, h_s' N)) = (hN, N)$$

for some $hN, h_i N \in G$, $h_i' N \in \tilde{G}$. Then, for some $cN \in G'$, the following holds:

$$\begin{cases} w(h_1', \dots, h_s')N = cN, \\ w(h_1, \dots, h_s)N = hc^{-1}N. \end{cases}$$

By an automorphism of the free group, the word w can be reduced to a *normal form* [9]: $w(t_1,\ldots,t_s) = t_1^m w'(t_1,\ldots,t_s)$, where $m \in \mathbb{N}$, $w' \in F_s'$. From the first equation, we get $cN \in G' \cap \varphi(G^s)$, where

$$\varphi\colon G^s \to G, \quad (g_1,\ldots,g_s) \mapsto w(g_1,\ldots,g_s)$$

is a verbal mapping. This means that, for some $w_1, w_2 \in N$, in H, it is true that

$$\begin{cases} w(h_1',\ldots,h_s') = cw_1, \\ w(h_1,\ldots,h_s) = hc^{-1}w_2. \end{cases}$$

Let us show that, in G, the identity $(aN)^x = [aN, bN]^z$ does not hold for $x \notin \alpha\mathbb{Z}$. Suppose otherwise. Then, in H, we have

$$a^x[a,b]^{-z} = b^{-k}a^{-l}a^{\alpha t}[a,b]^{ns}a^l b^k$$

for some $k, l, t, s \in \mathbb{Z}$. After some reductions, we get $a^{x-\alpha t} = [a,b]^{ns+z+\alpha tk}$, which means that $x = \alpha t \in \alpha\mathbb{Z}$, a contradiction. Thus, $h_1' = [a,b]^\gamma$ for some $\gamma \in \mathbb{Z}$, and, consequently, $cw_1 \in H'$. By [9], proof of the Heisenberg-group theorem, for any verbal mapping φ in the discrete Heisenberg group, for any $g \in \varphi(H^s)$, the coset $g(\varphi(H^s) \cap H')$ is contained in $\varphi(H^s)$. Thus, for some $g_1,\ldots,g_s \in H$, $w(g_1,\ldots,g_s) = w(h_1,\ldots,h_s)cw_1$. This means that

$$w(g_1,\ldots,g_s) = hw_3$$

for some $w_3 \in W$, and in G,

$$w(g_1,\ldots,g_s)N = hN,$$

which proves verbal closedness of G in K. □

Finally, let us prove that the higher-dimensional Heisenberg groups over any field are not strongly verbally closed. First, recall that the *Heisenberg group of dimension* $2n+1 \geqslant 3$ *over a field* K is the group of upper triangular matrices of the kind

$$H_n(K) = \left\{ T(\bar{a}, \bar{b}, c) = \begin{pmatrix} 1 & \bar{a} & c \\ 0 & I_n & \bar{b} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, \bar{a}, (\bar{b})^\top \in K^n,\ c \in K \right\},$$

where I_n is the identity matrix of size n, and for $x \in K^n$, the symbol $x^\top$ stands for the transpose vector of a vector x.

Proposition 4.2. *The group $H_n(K)$ is not strongly verbally closed.*

Proof. Consider the central product of $H_n(K)$ with its copy $\tilde{H}_n(K)$ with joined commutator subgroup,

$$G = H_n(K) \underset{H_n(K)' = \tilde{H}_n(K)'}{\times} \tilde{H}_n(K).$$

Denote with the symbols H and $\tilde{H}$ the first and the second factor of this central product respectively. Let us show that H is not algebraically closed in G. The group H is linear, and, consequently, it is *equationally noetherian* (see [1, Theorem B1]), so it is algebraically closed in G if and only if it is a retract of every finitely generated over H subgroup of G (see [8]). In particular, of such a subgroup of G,

$$\bar{H} = \langle H, (1, h_1), \ldots, (1, h_n), (1, g_1), \ldots, (1, g_n) \rangle,$$

where

$$h_i = \begin{pmatrix} 1 & \bar{a}_i & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & \bar{b}_i \\ 0 & 0 & 1 \end{pmatrix},$$

where $\bar{a}_i = (0, \ldots, 1, \ldots, 0) = (\bar{b}_i)^{\mathsf{T}}$ (unit is on the ith place). Thus,

$$N = \langle h_1, \ldots, h_n, g_1, \ldots, g_n \rangle$$

is a subgroup of $\tilde{H}_n(K)$, isomorphic to the discrete Heisenberg group of dimension $2n + 1$. Let ρ be a hypothetical retraction. Since in the group G, the subgroup H commutes with N, we get that $\rho(N') = \{1\}$, which leads to a contradiction with the definition of retraction.

Nevertheless, the subgroup H is verbally closed in G. To see this, let $w \in F_s$ be some word (without loss of generality, this word is in the normal form we established earlier), and let $\varphi\colon H^s \to H$ be the verbal mapping associated with this word. Suppose that, for some $h_i, h \in H$, $h_i' \in \tilde{H}$, $c \in H'$,

$$\begin{cases} w(h_1', \ldots, h_s') = c, \\ w(h_1, \ldots, h_s) = hc^{-1}. \end{cases}$$

In general, on matrices $g_i = T(\bar{a}_i, \bar{b}_i, c_i)$, $i = 1, \ldots, s$, the mapping φ acts as follows:

$$\varphi(g_1, \ldots, g_s) = \begin{pmatrix} 1 & m\bar{a}_1 & mc_1 + f(\bar{a}_1, \ldots, \bar{a}_s; \bar{b}_1, \ldots, \bar{b}_s) \\ 0 & I_n & m\bar{b}_1 \\ 0 & 0 & 1 \end{pmatrix},$$

where $f\colon (K^n)^s \times (K^n)^s \to K$ is some function linear in every argument. The image of f is either trivial or is equal to K, which leads to

$$\varphi(H_n(K)^s) = \begin{cases} \{1\} & \text{if } m = 0 \text{ and the image of } f \text{ is trivial,} \\ (H_n(K))' & \text{if } m = 0 \text{ and the image of } f \text{ is equal to } K, \\ H_n(K) & \text{if } m \neq 0. \end{cases}$$

Then $\varphi(H_n(K)^s) \cap (H_n(K))' \leqslant H_n(K)$ and, for every element $h \in \varphi(H_n(K)^s)$, it is true that

$$h(\varphi(H_n(K)^s) \cap (H_n(K))') \subseteq \varphi(H_n(K)^s),$$

whence verbal closedness follows. □

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