

Redundant relators in cyclic presentations of groups

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Abstract. A cyclic presentation of a group is a presentation with an equal number of generators and relators that admits a particular cyclic symmetry. We characterise the orientable, non-orientable, and redundant cyclic presentations and obtain concise refinements of these presentations. We show that the Tits alternative holds for the class of groups defined by redundant cyclic presentations and show that if the number of generators of the cyclic presentation is greater than two, then the corresponding group is large. Generalising and extending earlier results of the authors, we describe the star graphs of orientable and non-orientable cyclic presentations and classify the cyclic presentations whose star graph components are pairwise isomorphic incidence graphs of generalised polygons, thus classifying the so-called (m, k, ν) -special cyclic presentations.

1 Introduction

A group presentation is said to be *redundant* if it contains a freely redundant relator, that is, if it has a relator that is either freely trivial or that is freely conjugate to another relator or to the inverse of another relator. A presentation with no freely redundant relators is called *concise*, and a concise presentation obtained by removing freely redundant relators from a presentation is called a *concise refinement* of that presentation [9]. A *cyclic presentation* is a presentation with an equal number of generators and relators that admits a particular cyclic symmetry, and the corresponding group is called a *cyclically presented group* [22]. Redundant cyclic presentations are either *orientable* if no relator is a cyclic permutation of the inverse of any other relator, or *non-orientable* otherwise [2].

In this paper, we classify the redundant cyclic presentations, we show that the class of groups defined by such presentations satisfies the Tits alternative, and as a companion piece to [8], we classify the (redundant and concise) (m, k, ν) -special cyclic presentations (that is, the cyclic presentations with length k relators, where the star graph has ν isomorphic components, each of which is the incidence graph of a generalised m -gon).

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The paper is organised as follows. In Section 2, we give relevant definitions and background. In Section 3, we observe that redundant cyclic presentations are ubiquitous, in the sense that, for any $n \geq 2$ and any non-empty word u in the free group of rank n , it is possible to construct both orientable and non-orientable redundant cyclic presentations using these n, u . We classify the orientable (redundant and concise) presentations and classify the non-orientable cyclic presentations; in each case, we obtain concise refinements of these presentations. In Section 4, we show that if a redundant cyclic presentation has more than two generators, then the group it defines is large, and in particular, motivated by [13, Problem 2] (which asks which groups defined by (m, k) -special cyclic presentations are large), we observe that groups defined by redundant (m, k, ν) -special cyclic presentations are large. We show that the Tits alternative holds for the class of groups defined by redundant cyclic presentations, and we investigate which redundant cyclic presentations with two generators define groups that contain a non-abelian free subgroup, providing a classification in the orientable case. In Section 5, we generalise [8, Theorem 3.3] to describe the star graphs of orientable and non-orientable cyclic presentations. In Section 6, we extend the classification in [8, Sections 5, 6] to classify the (redundant and concise) (m, k, ν) -special cyclic presentations. In particular, we show that if a cyclic presentation is (m, k, ν) -special, where $m \geq 3$, then $m = 3$, the presentation is orientable, and the defining word is positive or negative, and we show that if a redundant cyclic presentation is (m, k, ν) -special, then $1/m + 2/k < 1$ and hence defines a non-elementary hyperbolic group.

2 Preliminaries

2.1 Presentations and cyclic presentations of groups

Given a positive integer n , let F_n be the free group with basis $X = \{x_0, \dots, x_{n-1}\}$. A non-empty word $w \in F_n$ is said to be *positive* or *negative* if all of the exponents of generators are positive or negative, respectively. (In particular, generators x_i are positive, and their inverses x_i^{-1} are negative.) We shall say that a word w of length at least 2 is *alternating* if it has no subword of the form $(x_i x_j)^{\pm 1}$ and that it is *cyclically alternating* if it has no cyclic subword of that form. Thus a word is cyclically alternating if and only if it is alternating and has even length. A word w is *reduced* if it does not contain a subword of the form $x_i x_i^{-1}$ or $x_i^{-1} x_i$; it is *cyclically reduced* if all cyclic permutations of it are reduced. If $w \in F_n$ is a cyclically reduced, non-empty, word, then the unique word $v \in F_n$ such that $w = v^p$ with p maximal is called the *root* of w . We shall write $l(w)$ to denote the length of w in F_n . Throughout this article, equality of words will refer to equality of elements of the free group F_n .

When considering group presentations $\langle X \mid R \rangle$, R will be a set of relators (so does not contain any relator more than once), all of whose elements are cyclically reduced. Let $\theta: F_n \rightarrow F_n$ be the *shift automorphism* given by $\theta(x_i) = x_{i+1}$, where (as throughout this article) subscripts are taken mod n . Given a non-empty cyclically reduced word w representing an element in F_n , the words $\theta^i(w)$, where $0 \leq i < n$ and $1 \leq i < n$, are the *shifts* and *proper shifts* of w , respectively, the presentation

$$P_n(w) = \langle x_0, \dots, x_{n-1} \mid w, \theta(w), \dots, \theta^{n-1}(w) \rangle$$

is a *cyclic presentation*, the group $G_n(w)$ it defines is a *cyclically presented group*, and w is the *defining word* [22]. Without loss of generality, we may assume that the generator x_0 is a letter of w , and we make this assumption throughout this article. Then $P_n(w)$ is said to be *irreducible* if the greatest common divisor of n and the subscripts of the generators that appear in w is equal to 1 [11]. The shift automorphism θ satisfies $\theta^n = 1$, and the resulting $\mathbb{Z}_n$ -action on $G_n(w)$ determines the *shift extension* $E = G_n(w) \rtimes_{\theta} \mathbb{Z}_n$, which admits a presentation of the form $\langle x, t \mid t^n, W(x, t) \rangle$, where $W(x, t)$ is obtained by rewriting w in terms of the substitutions $x_i = t^i x t^{-i}$, $0 \leq i < n$ (see, for example, [23, Theorem 4]).

For $1 \leq t \leq n$, we define the *t-truncation* of $P_n(w)$ to be the presentation

$$P_{n,t}(w) = \langle x_0, \dots, x_{n-1} \mid w, \theta(w), \dots, \theta^{t-1}(w) \rangle$$

and denote by $G_{n,t}(w)$ the group that it defines. Let $\phi: F_n \rightarrow F_n$ be the cyclic permutation function that cyclically permutes a word by one generator (or the inverse of a generator), with inverse ϕ^{-1} . That is, $\phi(x_{p_1}^{\epsilon_1} x_{p_2}^{\epsilon_2} \dots x_{p_k}^{\epsilon_k}) = x_{p_2}^{\epsilon_2} \dots x_{p_k}^{\epsilon_k} x_{p_1}^{\epsilon_1}$ ($0 \leq p_i < n$, $\epsilon_i = \pm 1$, $1 \leq i \leq k$). Then ϕ and θ commute, and if w is a cyclically reduced word of length k , then $\phi^k(w) = w$. It follows that if w is a cyclically reduced word of length k that is equal to a cyclic permutation of the shift $\theta^h(w)$, then a cyclic permutation of w is equal to the shift $\theta^{(n,h)}(w)$, and if a shift of w is equal to a cyclic permutation $\phi^t(w)$ of w , then a shift of w is equal to the cyclic permutation $\phi^{(k,t)}(w)$. Consider, for example, $P_6(x_0 x_1 x_2 x_3 x_4 x_5)$ with $h = 4$ and $t = 4$. (To see the first claim, let $\alpha, \beta \in \mathbb{Z}$ satisfy $\alpha h + \beta n = (n, h)$, and observe that w is equal to a cyclic permutation of

$$\theta^{\alpha h}(w) = \theta^{\alpha h + \beta n}(w) = \theta^{(n,h)}(w).$$

Similar arguments hold for the second claim.) Moreover, $\phi^r(v^p) = (\phi^r(v))^p$ and $\theta^h(v^p) = (\theta^h(v))^p$ for any $h, r, p \geq 1$, so if $w = v^p$, where v is the root of w , then $\theta^h(w) = \phi^r(w)$ if and only if $\theta^h(v) = \phi^r(v)$. We will write $\iota(w)$, $\tau(w)$ to denote the initial and terminal letters of a word w , respectively.

Following [2, page 160], given a group presentation $P = \langle X \mid R \rangle$, an element $r \in R$ is said to be *freely redundant* if it is freely trivial or if there exists another element $s \in R$ such that r and s are elements of the free group with basis X and either r is freely conjugate to s or r is freely conjugate to s^{-1} (that is, either r is a cyclic permutation of s or of s^{-1}). A presentation is said to be *redundant* if it contains a freely redundant relator and is *concise* otherwise [9, page 4]. (Concise presentations are referred to as *slender presentations* in [3, page 5] or *irredundant presentations* in [28, page 82].) If a presentation P' is obtained from a presentation P by removing freely redundant relators, then we say that P' is a *refinement* of P , and if, in addition, P' is concise, we say that it is a *concise refinement* of P (compare [9, page 4]). A cyclic presentation $P_n(w)$ is *orientable* if w is not a cyclic permutation of the inverse of any of its shifts [2, page 155] and is *non-orientable* otherwise. Thus an orientable cyclic presentation is redundant if and only if w is equal to a cyclic permutation of one of its proper shifts, and if a cyclic presentation is non-orientable, then it is redundant. As examples, the cyclic presentation $P_2(x_0x_1)$ is orientable and redundant, $P_3(x_0x_1)$ is orientable and concise, and $P_2(x_0x_1^{-1})$ is non-orientable (and therefore redundant).

The *deficiency* of the presentation $P = \langle X \mid R \rangle$ is defined as

$$\text{def}(P) = |X| - |R|,$$

and the *deficiency* of a group G , $\text{def}(G)$, is defined to be the maximum of the deficiencies of all finite presentations defining G . A group G is *large* if it has a finite index subgroup that has a non-abelian free homomorphic image, and it is *SQ-universal* if every countable group embeds in a quotient of G . Every large group is SQ-universal, and every SQ-universal group contains a non-abelian free subgroup [27]. Moreover, as noted in [32], the converse to neither of these statements holds in the class of cyclically presented groups: the Higman group $G_4(x_0x_1x_0^{-2}x_1^{-1})$ is SQ-universal [29] but has no proper finite index subgroup [18] so is not large, and the group $G_7(x_0x_1x_3)$, identified in [12, Example 3.3] and [20, Example 6.3], contains a non-abelian free subgroup [12] but is just-infinite [13, Theorem 2] so is not SQ-universal. A class of groups is said to satisfy the *Tits alternative* if every group in that class either contains a non-abelian free subgroup or is virtually solvable.

2.2 Star graphs

Let $P = \langle X \mid R \rangle$ be a group presentation and let $\tilde{R}$ denote the symmetrised closure of R , that is, the set of all cyclic permutations of elements in $R \cup R^{-1}$. The *star graph* of P is the undirected vertex-labelled graph Γ where the vertex set is in one-one correspondence with $X \cup X^{-1}$, vertices are labelled by the correspond-

ing element of $X \cup X^{-1}$, and where there is an edge joining vertices labelled x and y for each distinct word $xy^{-1}u$ in $\tilde{R}$ (see [25, page 61]). Such words occur in pairs, that is, $xy^{-1}u \in \tilde{R}$ implies that $yx^{-1}u^{-1} \in \tilde{R}$. These pairs are called *inverse pairs*, and the two edges corresponding to them are identified in Γ . It follows that replacing any relator of a presentation by its root, or removing a redundant relator from a presentation, leaves the star graph unchanged; in particular, the star graphs of a presentation and any concise refinement of it are equal. We refer to vertices in X as *positive* vertices and vertices in X^{-1} as *negative* vertices.

We now set out our graph theoretic terminology. We allow graphs to have loops and to have more than one edge joining a pair of vertices. Given a graph Γ , we write $V(\Gamma)$ to denote its vertex set. If Γ is bipartite with vertex partition $V(\Gamma) = V_1 \cup V_2$, where each edge connects a vertex in V_1 to a vertex in V_2 , then V_1, V_2 are called the *parts* of $V(\Gamma)$. Two adjacent vertices are said to be *neighbours*, and the set of neighbours of a vertex v in a graph Γ is denoted $N_\Gamma(v)$. A graph Γ is *r-regular* if $|N_\Gamma(v)| = r$ for all $v \in V(\Gamma)$, and it is *regular* if it is *r-regular* for some r .

A *path of length l* in Γ is a sequence of vertices $(u = u_0, u_1, \dots, u_l = v)$ with edges $u_i - u_{i+1}$ for each $0 \leq i < l$; it is a *closed path* if $u = v$. The path is *reduced* if the edge $u_{i+1} - u_{i+2}$ is not equal to the edge $u_{i+1} - u_i$ ($0 \leq i < l - 1$). The *distance* $d_\Gamma(u, v)$ between vertices u, v of Γ is $l \geq 0$ if there is a path of length l from u to v , but no shorter path, and $d_\Gamma(u, v) = \infty$ if there is no path from u to v . The *girth*, $\text{girth}(\Gamma)$ of a graph Γ is the length of a reduced closed path of minimal length if Γ contains a reduced closed path, and $\text{girth}(\Gamma) = \infty$ otherwise. The *diameter*, $\text{diam}(\Gamma)$ of a graph Γ is the greatest distance between any pair of vertices of the graph (which may be infinite). If Γ is a graph with finite girth, then $\text{girth}(\Gamma) \leq 2 \text{diam}(\Gamma) + 1$. For an integer $n \geq 2$ and a (multi-) set $A \subseteq \{0, 1, \dots, n - 1\}$, the *circulant graph* $\text{circ}_n(A)$ is the graph with vertices $v_0, \dots, v_{n-1}$ and edges $v_i - v_{i+a}$ for all $0 \leq i < n, a \in A$ (subscripts mod n). We define the graph $\text{circ}'_n(A)$ to be $\text{circ}_n(A)$ if $n/2 \notin A$ and to be $\text{circ}_n(A)$ with exactly one edge $v_i - v_{i+n/2}$ removed for each $n/2 \leq i < n$ otherwise. For later use, we note that $\text{circ}_n(A)$ is the complete bipartite graph $K_{n/2, n/2}$ if and only if $n/2$ is even and $A = \{\pm 1, \pm 3, \dots, \pm(n/2 - 1)\}$, and $\text{circ}'_n(A)$ is the complete bipartite graph $K_{n/2, n/2}$ if and only if $n/2$ is odd and $A = \{\pm 1, \pm 3, \dots, \pm(n/2 - 2), n/2\}$.

2.3 Special presentations

An (m, k, v) -special presentation is a group presentation in which the relators have length k and whose star graph has v isomorphic components, each of which is the incidence graph of a generalised m -gon. Formally, we have the following definition.

Definition 2.1 ([8, Definition 2.1]). Let $m \geq 2$, $k \geq 3$, $v \geq 1$. A finite group presentation $P = \langle X \mid R \rangle$ is said to be (m, k, v) -special if the following conditions hold:

- (a) the star graph Γ of P has v isomorphic components, each of which is a connected, bipartite graph of diameter m and girth $2m$ in which each vertex has degree at least 3;
- (b) each relator $r \in R$ has length k ;
- (c) if $m = 2$, then $k \geq 4$.

This generalises the concept of (m, k) -special cyclic presentations introduced in [13], which corresponds to the case $v = 1$, which in turn generalises the concept of special presentations introduced in [20] (which corresponds to the case $m = k = 3$). The concise cyclic presentations that are (m, k, v) -special were classified in [8]. We refer the reader to [8, 13] for background and further references on properties of (m, k, v) -special presentations and the groups they define.

Note that if a presentation is (m, k, v) -special, then it has at least 3 generators. As in the proof of [13, Theorem 2], a group G defined by an (m, k, v) -special presentation with $2/k + 1/m < 1$ is non-elementary hyperbolic and hence SQ-universal. If $w = v^p$, where v is the root of w , then the star graph Γ of $P_n(w)$ is equal to the star graph of $P_n(v)$. Moreover, if v has length 2, then the vertices of Γ have degree at most 2, so neither $P_n(v)$ nor $P_n(w)$ are (m, k, v) -special. Therefore, $P_n(w)$ is (m, pk, v) -special if and only if $P_n(v)$ is (m, k, v) -special. Thus, in classifying (m, k, v) -special cyclic presentations $P_n(w)$, we can assume that w is not a proper power. For our characterisation of $(3, k, v)$ -special cyclic presentations, we recall that a set of k integers $d_1, \dots, d_k$ is called a *perfect difference set* (of order k) if, among the $k(k-1)$ differences $d_i - d_j$ ($i \neq j$), each of the residues $1, 2, \dots, (k^2 - k) \bmod (k^2 - k + 1)$ occurs exactly once.

3 Classification of redundant cyclic presentations

3.1 Classification of orientable redundant cyclic presentations

Suppose that $P_n(w)$ is orientable and that v is the root of w , and recall from Section 2.1 that a cyclic permutation of w is equal to one of its proper shifts if and only if a cyclic permutation of v is equal to one of its proper shifts. It follows that $P_{n,t}(w)$ is a concise refinement of $P_n(w)$ if and only if $P_{n,t}(v)$ is a concise refinement of $P_n(v)$. Therefore, in this context, we may assume that w is not a proper power.

Every non-empty word $w \in F_n$ has an expression of the form

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad (3.1)$$

for some $u \in F_n$ of length $l(u) \geq 1$ and some $0 \leq h < n$, even if only trivially, with $u = w$ and $h = 0$. However, as we shall see, if $P_n(w)$ is orientable and redundant, then there is an expression (3.1) with $u \neq w$ and $h > 0$. (For example, $P_3(x_0x_1x_2)$ has such an expression with $u = x_0$ and $h = 1$.) Therefore, w satisfies $\phi^{l(w)}(w) = \theta^h(w)$, and hence a cyclic permutation of w is equal to the shift $\theta^{(n,h)}(w)$, which is proper if $0 < h < n$.

Note that, in (3.1), w is positive if and only if u is positive, w is cyclically alternating if and only if u is cyclically alternating, and if $w \neq u$, then w is alternating if and only if it is cyclically alternating. For later use, we record that, given a word w of the form (3.1), every cyclic permutation of w has an expression of the same form, where the length of u and the value of h are preserved.

Lemma 3.1. *Let*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u)$$

for some reduced word u and for some $0 \leq h < n$. Then every cyclic permutation of w is of the form

$$\prod_{i=0}^{n/(n,h)-1} \theta^{ih}(v)$$

for some reduced word v , where $l(v) = l(u)$.

Proof. Let $l = l(u)$ and $0 \leq s < l(w)$, and write

$$s = tl + r, \quad \text{where } 0 \leq t < n/(n, h), \quad 0 \leq r < l.$$

Then

$$\phi^s(w) = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(v), \quad \text{where } v = \theta^{th}(u_2)\theta^{(t+1)h}(u_1),$$

where u_1 is the initial subword of u of length r and u_2 is the terminal subword of u of length $l - r$. □

Theorem 3.2 identifies the word u in an expression (3.1) in terms of the cyclic permutations of w that are equal to a shift of w . Corollary 3.3 shows that if u is chosen to be the shortest possible, then (for the corresponding value of h) the (n, h) -truncation $P_{n,(n,h)}(w)$ is a concise refinement of $P_n(w)$.

Theorem 3.2. *Let $w \in F_n$ be a non-empty cyclically reduced word of length k that is not a proper power and suppose that some shift of w is equal to $\phi^t(w)$ for some $0 \leq t < k$. Then*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u)$$

for some $0 \leq h < n$, where u is the initial subword of w of length (k, t) .

Proof. Since some shift of w is equal to $\phi^t(w)$, we have $\theta^h(w) = \phi^{(k,t)}(w)$ for some $0 \leq h < n$. Let $\lambda = k/(k, t)$ and write $w = \prod_{i=0}^{\lambda-1} u_i$ for some words $u_0, \dots, u_{\lambda-1}$, each of length (k, t) . Then (subscripts mod λ)

$$\prod_{i=0}^{\lambda-1} u_{i+1} = \phi^{(k,t)}(w) = \theta^h(w) = \prod_{i=0}^{\lambda-1} \theta^h(u_i).$$

Thus, for each $0 \leq i < \lambda$, we have $u_{i+1} = \theta^h(u_i)$, so $u_{i+1} = \theta^{(i+1)h}(u_0)$. In particular, we have $u_0 = u_\lambda = \theta^{\lambda h}(u_0)$, so $\lambda h \equiv 0 \pmod{n}$, so $\lambda \equiv 0 \pmod{n/(n, h)}$, so $\lambda = p n/(n, h)$ for some $p \geq 1$. If $p > 1$, then $n/(n, h) < \lambda$ and

$$u_{n/(n,h)} = \theta^{(n/(n,h)) \cdot h}(u_0) = u_0,$$

so w is a proper power, a contradiction. Therefore, $\lambda = n/(n, h)$, and hence

$$w = \prod_{i=0}^{\lambda-1} u_i = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u),$$

where $u = u_0$, as required. □

Corollary 3.3. *Let $w \in F_n$ be a non-empty cyclically reduced word of length k that is not a proper power and let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n.$$

Then $P_{n,(n,h)}(w)$ is a concise refinement of $P_n(w)$.

Proof. Observe first that $\theta^h(w) = \phi^{l(u)}(w)$, so $\theta^{(n,h)}(w)$ is a cyclic permutation of w , so $P_{n,(n,h)}(w)$ is a refinement of $P_n(w)$. Suppose for contradiction that $P_{n,(n,h)}(w)$ is not concise. Then $\theta^s(w) = \phi^r(w)$ for some $0 < s < (n, h)$, $0 \leq r < k$. Let $t = l(u)$ (noting that $t \mid k$), and let $a, b \in \mathbb{Z}$ satisfy $ar + bt = (r, t)$. Then

$$\begin{aligned} \theta^{as+bh}(w) &= \theta^{as}(\theta^{bh}(w)) = \theta^{as}(\phi^{bt}(w)) \\ &= \phi^{bt}(\theta^{as}(w)) = \phi^{bt}(\phi^{ar}(w)) = \phi^{ar+bt}(w) = \phi^{(r,t)}(w). \end{aligned}$$

Theorem 3.2 then implies $(r, t) \geq t$, so $(r, t) = t$, and hence $r = \lambda t$ for some $\lambda \geq 1$. Therefore,

$$\theta^s(w) = \phi^r(w) = \phi^{\lambda t}(w) = \theta^{\lambda h}(w),$$

and hence $s \equiv \lambda h \pmod{n}$, but $0 < s < (n, h)$, a contradiction. $\square$

3.2 Classification of non-orientable cyclic presentations

Given any non-empty word $u \in F_n$, where $n \geq 2$ is even, and the word

$$w = u\theta^{n/2}(u)^{-1} \in F_n,$$

the cyclic presentation $P_n(w)$ is non-orientable. Lemma 3.6 of [2] characterises the non-orientable cyclic presentations $P_n(w)$ as those for which n is even and w is equal to $u\theta^{n/2}(u)^{-1}$ for some reduced word u . Unfortunately, that statement is not quite correct: a presentation that demonstrates this is $P_4(w)$, where $w = x_0x_2^{-1}x_3x_1^{-1}$. (However, the remaining results from [2] appear to be unaffected.) In Theorem 3.4, we instead characterise such presentations as those for which n is even and *some cyclic permutation of w* is equal to $u\theta^{n/2}(u)^{-1}$ for some reduced word u , and we show that the $n/2$ -truncation $P_{n,n/2}(w)$ is a concise refinement of $P_n(w)$.

Theorem 3.4. *Let $w \in F_n$ be a non-empty cyclically reduced word. The cyclic presentation $P_n(w)$ is non-orientable if and only if n is even and w has a cyclic permutation that is equal to $u\theta^{n/2}(u)^{-1}$ for some reduced word $u \in F_n$, in which case $P_{n,n/2}(w)$ is a concise refinement of $P_n(w)$.*

Proof. Suppose that n is even and that w has a cyclic permutation that is equal to $u\theta^{n/2}(u)^{-1}$ for some reduced word u . Then w is equal to a cyclic permutation of $\theta^{n/2}(w^{-1})$, so $P_n(w)$ is non-orientable.

Conversely, suppose $\theta^h(w) = \phi^t(w^{-1})$ for some $0 \leq h < n$ and $0 \leq t < l(w)$. Note that $h \neq 0$ since the only word that is equal to a cyclic permutation of

its inverse is the empty word. Let $w = x_{d_0}^{\epsilon_0} x_{d_1}^{\epsilon_1} \dots x_{d_{k-1}}^{\epsilon_{k-1}}$, where $\epsilon_j \in \{\pm 1\}$ and $0 \leq d_j < n$ for each $0 \leq j < k$, $k \geq 1$. Then

$$\begin{aligned} & x_{h+d_0}^{\epsilon_0} x_{h+d_1}^{\epsilon_1} \dots x_{h+d_{t-1}}^{\epsilon_{t-1}} \cdot x_{h+d_t}^{\epsilon_t} \dots x_{h+d_{k-1}}^{\epsilon_{k-1}} \\ &= x_{d_{t-1}}^{-\epsilon_{t-1}} x_{d_{t-2}}^{-\epsilon_{t-2}} \dots x_{d_0}^{-\epsilon_0} \cdot x_{d_{k-1}}^{-\epsilon_{k-1}} x_{d_{k-2}}^{-\epsilon_{k-2}} \dots x_{d_t}^{-\epsilon_t}. \end{aligned}$$

Hence, by comparing exponents and subscripts, we have

$$d_j \equiv h + d_{t-1-j} \pmod{n}, \quad \epsilon_j = -\epsilon_{t-1-j} \quad \text{for all } 0 \leq j < t, \quad (3.2)$$

$$d_j \equiv h + d_{k+t-1-j} \pmod{n}, \quad \epsilon_j = -\epsilon_{k+t-1-j} \quad \text{for all } t \leq j < k. \quad (3.3)$$

In particular, since $d_0 \equiv (h + d_{t-1}) \pmod{n}$ and $d_{t-1} \equiv (h + d_0) \pmod{n}$, we have $2h \equiv 0 \pmod{n}$, and so $n = 2h$. If t is odd, then (3.2) implies that $\epsilon_{(t-1)/2} = 0$, a contradiction; thus t is even, $t = 2\mu \geq 0$, say. Similarly, it follows that $k - t$ is even, $k - t = 2\nu \geq 0$, say. Therefore, eliminating $t = 2\mu$ and $k = 2\mu + 2\nu$, equations (3.2), (3.3) become

$$d_j \equiv h + d_{2\mu-1-j} \pmod{n}, \quad \epsilon_j = -\epsilon_{2\mu-1-j} \quad \text{for all } 0 \leq j < 2\mu,$$

$$d_j \equiv h + d_{4\mu+2\nu-1-j} \pmod{n}, \quad \epsilon_j = -\epsilon_{4\mu+2\nu-1-j} \quad \text{for all } 2\mu \leq j < 2\mu + 2\nu,$$

respectively. Therefore,

$$\begin{aligned} w &= \prod_{j=0}^{\mu-1} x_{d_j}^{\epsilon_j} \cdot \prod_{j=\mu}^{2\mu-1} x_{d_j}^{\epsilon_j} \cdot \prod_{j=2\mu}^{2\mu+\nu-1} x_{d_j}^{\epsilon_j} \cdot \prod_{j=2\mu+\nu}^{2\mu+2\nu-1} x_{d_j}^{\epsilon_j} \\ &= \prod_{j=0}^{\mu-1} x_{d_j}^{\epsilon_j} \cdot \prod_{j=0}^{\mu-1} x_{h+d_{\mu-j-1}}^{-\epsilon_{\mu-j-1}} \cdot \prod_{j=2\mu}^{2\mu+\nu-1} x_{d_j}^{\epsilon_j} \cdot \prod_{j=2\mu}^{2\mu+\nu-1} x_{h+d_{4\mu+\nu-1-j}}^{-\epsilon_{4\mu+\nu-1-j}} \\ &= \prod_{j=0}^{\mu-1} x_{d_j}^{\epsilon_j} \cdot \left(\prod_{j=0}^{\mu-1} x_{h+d_j}^{\epsilon_j} \right)^{-1} \cdot \prod_{j=2\mu}^{2\mu+\nu-1} x_{d_j}^{\epsilon_j} \cdot \left(\prod_{j=2\mu}^{2\mu+\nu-1} x_{h+d_j}^{\epsilon_j} \right)^{-1} \\ &= u_1 \theta^h(u_1)^{-1} \cdot u_2 \theta^h(u_2)^{-1}, \end{aligned}$$

where

$$u_1 = \prod_{j=0}^{\mu-1} x_{d_j}^{\epsilon_j}, \quad u_2 = \prod_{j=2\mu}^{2\mu+\nu-1} x_{d_j}^{\epsilon_j}.$$

Setting $u = \theta^h(u_1^{-1})u_2$, we have that $u\theta^h(u)^{-1}$ is a cyclic permutation of w , as required.

Thus we have that $P_{n,n/2}(w)$ is a refinement of $P_n(w)$. Suppose for contradiction that $P_{n,n/2}(w)$ is not concise; then $u\theta^{n/2}(u^{-1})$ is a cyclic permutation

of $\theta^i(u\theta^{n/2}(u^{-1}))$ or of $\theta^i(\theta^{n/2}(u)u^{-1})$ for some $1 \leq i < n/2$, but in the latter case, $2i \equiv 0 \pmod n$ (as above), a contradiction. Therefore, Theorem 3.2 implies

$$u\theta^{n/2}(u^{-1}) = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(v)$$

for some reduced word v and some $1 \leq h < n$. Therefore, v is an initial subword of u and $\theta^{n-h}(v)$ is a terminal subword of $\theta^{n/2}(u^{-1})$. Hence $\theta^{n-h}(v) = \theta^{n/2}(v^{-1})$, so $v^{-1} = \theta^{n/2-h}(v)$. Let $x_t^{\epsilon_t}, x_\tau^{\epsilon_\tau}$ ($0 \leq \iota, \tau < n$, $\epsilon_\iota, \epsilon_\tau \in \{\pm 1\}$) be the initial and terminal letters of v , respectively. Then, by comparing initial and terminal letters of v^{-1} and $\theta^{n/2-h}(v)$, we have

$$x_\tau^{-\epsilon_\tau} = x_{\iota+n/2-h}^{\epsilon_\iota} \quad \text{and} \quad x_\iota^{-\epsilon_\iota} = x_{\tau+n/2-h}^{\epsilon_\tau}.$$

In particular, $\tau \equiv \iota + n/2 - h$ and $\iota \equiv \tau + n/2 - h \pmod n$, so $h = n/2$. Therefore, $u\theta^{n/2}(u^{-1}) = v\theta^{n/2}(v)$, so $u = v$ and $\theta^{n/2}(u^{-1}) = \theta^{n/2}(v)$, a contradiction. $\square$

In considering non-orientable cyclic presentations $P_n(w)$ in contexts where relators can be replaced by their cyclic permutations, it thus suffices to consider cyclic presentations $P_n(u\theta^{n/2}(u)^{-1})$ for some reduced word u (even though w itself may not be of the form $u\theta^{n/2}(u)^{-1}$). Note that, for such a w and u , the word w is non-positive, non-negative, and w is cyclically alternating if and only if w is alternating if and only if u is alternating.

4 The Tits alternative

The Tits alternative has been considered for various classes of cyclically presented groups [4, 7, 14, 21, 26, 32]. In this section, we investigate the Tits alternative for the class of groups defined by redundant cyclic presentations. Observe that groups that are both solvable and virtually abelian arise within this class: the group $\mathbb{Z}$ has orientable and non-orientable redundant cyclic presentations $P_2(x_0x_1), P_2(x_0x_1^{-1})$; the group $\mathbb{Z}^2$ has the non-orientable cyclic presentation $P_2(x_0x_1x_0^{-1}x_1^{-1})$; the Baumslag–Solitar group $BS(1, -1)$ (the fundamental group of the Klein bottle) has the orientable redundant cyclic presentation $P_2(x_0^2x_1^2)$ and the non-orientable, redundant, cyclic presentation $P_2(x_0^2x_1^{-2})$. Groups defined by redundant cyclic presentations have positive deficiency; by [1], a virtually solvable group of positive deficiency has deficiency one, and a group of deficiency one is solvable if and only if it is isomorphic to a group of the form $G_k = \langle a, b \mid bab^{-1} = a^k \rangle$ for some $k \in \mathbb{Z}$ (see [33, Theorem 1]). Note that $G_0 \cong \mathbb{Z}$, $G_1 \cong \mathbb{Z}^2$, and $G_{-1} \cong BS(1, -1)$; we expect that if $k \notin \{-1, 0, 1\}$, then

G_k does not have a cyclic presentation. Note further that $\mathbb{Z}$ and $\mathbb{Z}^2$ are the only abelian one-relator groups [25, Proposition 5.24, page 108] and hence are the only abelian groups defined by redundant cyclic presentations.

Theorem 4.1. *Let $n \geq 2$, let $w \in F_n$ be a cyclically reduced word, and assume that $P_n(w)$ is redundant. If either $n \geq 3$ or $n = 2$ and w is a proper power, then $G_n(w)$ is large. In particular, if $P_n(w)$ is a redundant (m, k, v) -special cyclic presentation, then $G_n(w)$ is large.*

Proof. If $n \geq 3$, then Corollary 3.3 and Theorem 3.4 imply $\text{def}(G) \geq 2$ (as its defining presentation has a concise refinement with n generators and at most $\lfloor n/2 \rfloor$ relators), so $G_n(w)$ is large by [1]. If $n = 2$, then $\text{def}(G) \geq 1$, so if in addition w is a proper power, then $G_n(w)$ is large by [30], [17, page 83]. The final observation follows by noting that every (m, k, v) -special presentation has at least three generators. $\square$

Therefore, if $P_n(w)$ is a redundant cyclic presentation that does not define a large group, then (since there are no redundant cyclic presentations $P_1(w)$) $n = 2$ and w is not a proper power. In the concise case, we note that, by [10], if $n \geq 3$ and w is a proper power, then $G_n(w)$ is large, and if w is a third or higher power, then $G_2(w)$ is large (the infinite dihedral group $G_2(x_0^2)$, being solvable, demonstrates that this does not extend to all proper power defining words w). If $P_2(w)$ is orientable and redundant, then $\theta(w)$ is equal to $\phi^t(w) \neq w$ for some t , so by Theorem 3.2, $w = u\theta(u)$, where $u = u(x_0, x_1)$ is the initial subword of w of length $n/2$. If $P_2(w)$ is non-orientable, then by Theorem 3.4, w is a cyclic permutation of $u\theta(u)^{-1}$ for some reduced word u . That is, $P_2(w)$ is redundant if and only if w is a cyclic permutation of $u(x_0, x_1)u(x_1, x_0)^\epsilon$, where $\epsilon = 1$ in the orientable case and $\epsilon = -1$ in the non-orientable case, and so $G_2(w)$ is the one-relator group $\langle x_0, x_1 \mid u(x_0, x_1)u(x_1, x_0)^\epsilon \rangle$. One-relator groups either contain a non-abelian free subgroup or are solvable (see [24, Theorem 3], [6]), so we have the following.

Corollary 4.2 (The Tits alternative). *If $P_n(w)$ is a redundant cyclic presentation, then $G_n(w)$ either contains a non-abelian free subgroup or is solvable.*

In the orientable case, we can say precisely which presentations define a solvable group.

Corollary 4.3. *Let $P_n(w)$ be an orientable redundant cyclic presentation. Then $G_n(w)$ contains a non-abelian free subgroup unless $n = 2$ and either w is a cyclic permutation of $x_0^\epsilon x_1^\epsilon$ ($\epsilon = \pm 1$), in which case $G \cong \mathbb{Z}$, or w is a cyclic permutation of $x_0^{2\epsilon} x_1^{2\epsilon}$ ($\epsilon = \pm 1$), in which case $G \cong \text{BS}(1, -1)$.*

Proof. As explained above, if G does not contain a non-abelian free subgroup, then $G \cong G_2(w)$, where w is cyclically reduced and is a cyclic permutation of $u(x_0, x_1)u(x_1, x_0)$ for some reduced word $u(x_0, x_1)$.

If $u(x_0, x_1)$ involves exactly one of x_0, x_1 , then w is a cyclic permutation of $x_0^p x_1^p$ for some $p \in \mathbb{Z} \setminus \{0\}$, so $G \cong \langle x_0, x_1 \mid x_0^p x_1^p \rangle$ which is large if $|p| \geq 3$ (as it maps onto $\mathbb{Z}_p * \mathbb{Z}_p$), isomorphic to $\text{BS}(1, -1)$ if $p = \pm 2$, and isomorphic to $\mathbb{Z}$ if $p = \pm 1$. Thus we may assume that $u(x_0, x_1)$ involves both x_0 and x_1 .

Without loss of generality, we may write either

(i) $u(x_0, x_1) = x_0^{k_1} x_1^{k_2} x_0^{k_3} \dots x_1^{k_s}$, where $s \geq 2$ is even, or

(ii) $u(x_0, x_1) = x_0^{k_1} x_1^{k_2} x_0^{k_3} \dots x_0^{k_s}$, where $s \geq 3$ is odd,

and (in each case) each $k_i \in \mathbb{Z} \setminus \{0\}$.

If $s = 2$, then w is a cyclic permutation of $x_0^{k_1+k_2} x_1^{k_1+k_2}$, so

$$G \cong \langle x_0, x_1 \mid x_0^{k_1+k_2} x_1^{k_1+k_2} \rangle,$$

which maps onto

$$G_2(x_0^{k_1+k_2}) \cong \mathbb{Z}_{|k_1+k_2|} * \mathbb{Z}_{|k_1+k_2|},$$

which is large if $|k_1 + k_2| \notin \{1, 2\}$, and G is isomorphic to $\mathbb{Z}$ if $k_1 + k_2 = \pm 1$ and to $\text{BS}(1, -1)$ if $k_1 + k_2 = \pm 2$. Thus we may assume $s \geq 3$.

The shift extension of G is of the form

$$E = \langle x, t \mid t^2, U(x, t)U(xt, t) \rangle = \langle x, t \mid t^2, (U(x, t)t)^2 \rangle,$$

where

$$U(x, t) = \begin{cases} \left(\prod_{i=0}^{(s-3)/2} x^{k_{2i+1}} t x^{k_{2i+2}} t \right) x^{k_s} & \text{if } s \text{ is odd,} \\ \prod_{i=0}^{s/2-1} x^{k_{2i+1}} t x^{k_{2i+2}} t & \text{if } s \text{ is even,} \end{cases}$$

and hence $U(x, t)t$ is a cyclic permutation of

$$\begin{cases} \prod_{i=1}^s x^{k_i} t & \text{if } s \text{ is odd,} \\ x^{k_s+k_1} t \cdot \prod_{i=2}^{s-1} x^{k_i} t & \text{if } s \text{ is even.} \end{cases}$$

Note that if s is even, then $k_s + k_1 \neq 0$ since w is cyclically reduced. By [15, Theorem 8] or [16, Lemma 7.3.3.1], E contains a non-abelian free subgroup, and hence so does G , as required. $\square$

To consider the remaining non-orientable cases, we must consider the groups $G_2(u(x_0, x_1)u(x_1, x_0)^{-1})$.

Lemma 4.4. *Let $u(x_0, x_1)$ be a reduced word and let*

$$G = \langle x_0, x_1 \mid u(x_0, x_1)u(x_1, x_0)^{-1} \rangle,$$

where $u(x_0, x_1) = v(x_0, x_1)^t$ for some $t \geq 2$. Then G contains a non-abelian free subgroup except if $t = 2$ and either $v(x_0, x_1) = x_0^{\pm 1}$ or $x_1^{\pm 1}$, in which case G is isomorphic to $\text{BS}(1, -1)$.

Proof. We may assume that v is the root of u . The group G maps onto

$$\langle x_0, x_1 \mid v(x_0, x_1)^t, v(x_1, x_0)^t \rangle,$$

which is large if $t > 2$ by [10]; thus we may assume $t = 2$. If $v(x_0, x_1)$ involves exactly one of x_0, x_1 , then $G \cong \langle x_0, x_1 \mid x_0^2 x_1^{-2} \rangle \cong \text{BS}(1, -1)$. Suppose then that $v(x_0, x_1)$ involves both x_0 and x_1 . Without loss of generality, we may write either

- (i) $v(x_0, x_1) = x_0^{k_1} x_1^{k_2} x_0^{k_3} \dots x_1^{k_s}$, where $s \geq 2$ is even, or
- (ii) $v(x_0, x_1) = x_0^{k_1} x_1^{k_2} x_0^{k_3} \dots x_0^{k_s}$, where $s \geq 3$ is odd,

and (in each case) each $k_i \in \mathbb{Z} \setminus \{0\}$.

The group G maps onto

$$H = \langle x_0, x_1 \mid v(x_0, x_1)^2, v(x_1, x_0)^2 \rangle = G_2(v(x_0, x_1)^2).$$

Then the shift extension of H is of the form $E = \langle x, t \mid t^2, V(x, t)^2 \rangle$, where

- $V(x, t) = (x^{k_1} t) \dots (x^{k_s} t)$ in case (i) (so $s \geq 2$) and
- $V(x, t) = (x^{k_s + k_1} t) \dots (x^{k_{s-1}} t)$ in case (ii) (so $s - 1 \geq 2$).

By [15, Theorem 8] or [16, Lemma 7.3.3.1], E contains a non-abelian free subgroup, and hence so does H , and hence G . □

Thus it remains to consider groups of the form

$$G = \langle x_0, x_1 \mid u(x_0, x_1)u(x_1, x_0)^{-1} \rangle,$$

where $u(x_0, x_1)$ is not a proper power. If $u(x_0, x_1) = x_0^{\pm 1}$ or $x_1^{\pm 1}$, then $G \cong \mathbb{Z}$; if $u(x_0, x_1) = (x_0 x_1^{\pm 1})^{\pm 1}$ or $(x_1 x_0^{\pm 1})^{\pm 1}$, then $G \cong \mathbb{Z}^2$. We expect that, in all other cases, G contains a non-abelian free subgroup. As examples in this direction,

- (1) it follows from [5, Corollary 3.4] that if $G \not\cong \mathbb{Z}^2$ and $G^{\text{ab}} \cong \mathbb{Z}^2$, then G contains a non-abelian free subgroup; and

- (2) if $u(x_0, x_1) = x_0 x_1 x_0 \dots x_1 x_0$ is of odd length $k \geq 3$, then the shift extension of G is the group

$$E = \langle x, t \mid t^2, (xt)^k (x^{-1}t^{-1})^k \rangle \cong \langle y, t \mid t^2, y^k t y^{-k} t \rangle,$$

which maps onto $\langle y, t \mid t^2, y^k \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_k$, so is large.

In summary, we pose the following conjecture.

Conjecture 4.5. *If $P_2(w)$ is a non-orientable cyclic presentation, then $G_2(w)$ either contains a non-abelian free subgroup or is isomorphic to $\mathbb{Z}$, $\mathbb{Z}^2$ or $\text{BS}(1, -1)$.*

5 Star graphs of cyclic presentations

In [8, Theorem 3.3], the star graphs of concise cyclic presentations were described. In this section, we generalise that result to describe the star graphs of (possibly redundant) cyclic presentations.

5.1 Star graphs of orientable cyclic presentations

As in [8], it is convenient to express results concerning star graphs in terms of multisets of differences of subscripts in length-two subwords of a particular subword of w (in the redundant case) or in length-two cyclic subwords of w (in the concise case).

Definition 5.1. Suppose $P_n(w)$ is orientable and let u be the shortest subword of w such that

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n.$$

Let $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{Q}^+, \mathcal{Q}^-$ be the multisets defined as follows:

$$\mathcal{A} = \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } u\theta^h(u), \text{ with multiplicities}\},$$

$$\mathcal{B} = \{b \mid \text{there is a subword } x_i^{-1} x_{i+b} \text{ of } u\theta^h(u), \text{ with multiplicities}\},$$

$$\mathcal{Q}^+ = \{q \mid \text{there is a subword } x_i x_{i+q} \text{ of } u\theta^h(u), \text{ with multiplicities}\},$$

$$\mathcal{Q}^- = \{q \mid \text{there is a subword } x_{i+q}^{-1} x_i^{-1} \text{ of } u\theta^h(u), \text{ with multiplicities}\},$$

$$\mathcal{Q} = \mathcal{Q}^+ \cup \mathcal{Q}^-,$$

where $\cup$ denotes multiset sum and the entries are taken mod n . Note that $\mathcal{Q} = \emptyset$ if and only if u is cyclically alternating, and observe that the following congruence

holds (compare [8, (2)]):

$$\sum_{a \in \mathcal{A}} a + \sum_{b \in \mathcal{B}} b + \sum_{q \in \mathcal{Q}^+} q - \sum_{q \in \mathcal{Q}^-} q \equiv h \pmod{n} \quad (5.1)$$

and that if $P_n(w)$ is concise, then the sets $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{Q}^+, \mathcal{Q}^-$ are the sets defined in [8].

Theorem 5.2 describes the star graph of a (redundant or concise) orientable cyclic presentation $P_n(w)$. We use this in obtaining our classification of the orientable redundant (m, k, ν) -special cyclic presentations, and in Example 6.1, we use it to construct orientable redundant cyclic presentations with particular star graphs. Recall that the star graph of $P_n(w)$ is equal to the star graph of $P_n(v)$, where v is the root of w , so we may assume that w is not a proper power. In first studying the proof, the reader may find it helpful to consider the star graphs of the presentations $P_6(x_0x_2^{-1}x_1^{-1}x_3x_5^{-1}x_4^{-1})$ and $P_6(x_0x_3^{-1}x_1x_2^{-1}x_3x_0^{-1}x_4x_5^{-1})$.

Theorem 5.2. *Suppose $P_n(w)$ is orientable, where w is not a proper power, and let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n,$$

and let Γ be the star graph of $P_n(w)$. Let

$$d_{\mathcal{A}} = \gcd(n, a \ (a \in \mathcal{A})), \quad d_{\mathcal{B}} = \gcd(n, b \ (b \in \mathcal{B})),$$

and if u is not cyclically alternating, let $q_0 \in \mathcal{Q}$ and set

$$d = \gcd(n, a \ (a \in \mathcal{A}), b \ (b \in \mathcal{B}), q - q_0 \ (q \in \mathcal{Q})).$$

Then Γ is $l(u)$ -regular and has vertices x_i, x_i^{-1} ($0 \leq i < n$) and edges $x_i - x_{i+a}, x_i^{-1} - x_{i+b}^{-1}, x_i - x_{i+q}^{-1}$ for all $a \in \mathcal{A}, b \in \mathcal{B}, q \in \mathcal{Q}, 0 \leq i < n$.

(a) *If u is not cyclically alternating, then the graph Γ has d connected components $\Gamma_0, \dots, \Gamma_{d-1}$, where, for $0 \leq j < d$, the graph Γ_j is the induced subgraph of Γ with vertex sets $V(\Gamma_j) = V(\Gamma_j^+) \cup V(\Gamma_j^-)$, where Γ_j^+ and Γ_j^- are the induced subgraphs of Γ with vertex sets*

$$\begin{aligned} V(\Gamma_j^+) &= \{x_{j+td} \mid 0 \leq t < n/d\}, \\ V(\Gamma_j^-) &= \{x_{j+td+q_0}^{-1} \mid 0 \leq t < n/d\} \end{aligned}$$

(subscripts mod n). In particular,

$$|V(\Gamma_j^+)| = |V(\Gamma_j^-)| = n/d \quad \text{for all } 0 \leq j < d$$

and the subscripts of the positive (respectively negative) vertices in any component are congruent mod d .

- (b) If u is cyclically alternating, then the graph Γ has $d_{\mathcal{A}} + d_{\mathcal{B}}$ connected components $\Gamma_0^+, \dots, \Gamma_{d_{\mathcal{A}}-1}^+, \Gamma_0^-, \dots, \Gamma_{d_{\mathcal{B}}-1}^-$, which are, respectively, the induced labelled subgraphs of Γ with vertex sets

$$V(\Gamma_j^+) = \{x_{j+td_{\mathcal{A}}} \mid 0 \leq t < n/d_{\mathcal{A}}\},$$

$$V(\Gamma_j^-) = \{x_{j+td_{\mathcal{B}}}^{-1} \mid 0 \leq t < n/d_{\mathcal{B}}\}$$

(subscripts mod n). Moreover, each component Γ_j^+ is isomorphic to the circulant graph $\text{circ}_{n/d_{\mathcal{A}}}(\{a/d_{\mathcal{A}} \mid (a \in \mathcal{A})\})$, and each component Γ_j^- is isomorphic to $\text{circ}_{n/d_{\mathcal{B}}}(\{b/d_{\mathcal{B}} \mid (b \in \mathcal{B})\})$.

Proof. By Corollary 3.3, the star graph of $P_n(w)$ is equal to the star graph of the concise (n, h) -truncation $P_{n, (n, h)}(w)$ of $P_n(w)$. Let $P = P_{n, (n, h)}(w)$ and $Q = P_n(u)$ with star graphs Γ and Λ , respectively. Observe that each relator of P has length $n/(n, h) \cdot l(u)$, so the sum of the lengths of the relators of P is equal to $(n/(n, h) \cdot l(u)) \cdot (n, h) = nl(u)$, which is the sum of the lengths of the relators of Q . By Corollary 3.3, since u is chosen to be the shortest possible, the presentation Q is a concise cyclic presentation, and so its star graph is described in [8, Theorem 3.3]. Our strategy is to identify the differences between Γ and Λ . We use the multisets $\mathcal{A}, \mathcal{B}, \mathcal{Q}$ of Definition 5.1 for each presentation P, Q , denoting them $\mathcal{A}_P, \mathcal{B}_P, \mathcal{Q}_P$ (for P) and $\mathcal{A}_Q, \mathcal{B}_Q, \mathcal{Q}_Q$ (for Q). These multisets are related as follows (where $\cup$ and $\setminus$ denote multiset sum and multiset difference, respectively):

$$\mathcal{A}_Q = \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } u, \text{ with multiplicities}\}$$

$$\cup \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } \tau(u)\iota(u)\},$$

$$\mathcal{A}_P = \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } u, \text{ with multiplicities}\}$$

$$\cup \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } \tau(u)\theta^h(\iota(u))\}$$

$$= (\mathcal{A}_Q \setminus \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } \tau(u)\iota(u)\})$$

$$\cup \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } \tau(u)\theta^h(\iota(u))\}.$$

Similarly,

$$\begin{aligned}\mathcal{B}_P &= (\mathcal{B}_Q \setminus \{b \mid \text{there is a subword } x_i^{-1}x_{i+b} \text{ of } \tau(u)\iota(u)\}) \\ &\quad \cup \{b \mid \text{there is a subword } x_i^{-1}x_{i+b} \text{ of } \tau(u)\theta^h(\iota(u))\}, \\ \mathcal{Q}_P &= (\mathcal{Q}_Q \setminus \{q \mid \text{there is a subword } x_i x_{i+q} \text{ or } x_{i+q}^{-1}x_i^{-1} \text{ of } \tau(u)\iota(u)\}) \\ &\quad \cup \{q \mid \text{there is a subword } x_i x_{i+q} \text{ or } x_{i+q}^{-1}x_i^{-1} \text{ of } \tau(u)\theta^h(\iota(u))\}.\end{aligned}$$

The graph Γ is obtained from Λ by, for each $0 \leq i < n$, removing exactly one edge of the form $\theta^i(\tau(u)) - \theta^i(\iota(u))^{-1}$ and then adjoining one edge of the form $\theta^i(\tau(u)) - \theta^{i+h}(\iota(u))^{-1}$. By [8, Theorem 3.3], Λ is $l(u)$ -regular and has vertices x_i, x_i^{-1} and edges $x_i - x_{i+a}, x_i^{-1} - x_{i+b}^{-1}, x_i - x_{i+q}^{-1}$ for all $a \in \mathcal{A}_Q, b \in \mathcal{B}_Q, q \in \mathcal{Q}_Q, 0 \leq i < n$. Therefore, Γ is also $l(u)$ -regular and has vertices x_i, x_i^{-1} and the same (multi-)set of edges as Λ , but with the substitutions described above applied. That is, Γ has edges $x_i - x_{i+a}, x_i^{-1} - x_{i+b}^{-1}, x_i - x_{i+q}^{-1}$ for all $a \in \mathcal{A}_P, b \in \mathcal{B}_P, q \in \mathcal{Q}_P, 0 \leq i < n$, completing the proof of first part of the statement. With this description of Γ in place, statements (a) and (b) follow as in the proof of [8, Theorem 3.3]. $\square$

Analogously to [8, Corollary 3.4], we have the following immediate corollary.

Corollary 5.3. *Let $P_n(w)$ be an orientable cyclic presentation in which w is of length k and is not a proper power, and let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n.$$

Then

- (a) $P_n(w)$ is $(3, k, v)$ -special if and only if $l(u)^2 - l(u) + 1 = n/v$ and each component of the star graph is the incidence graph of a projective plane of order $l(u) - 1$;
- (b) $P_n(w)$ is $(2, k, v)$ -special if and only if $l(u) = n/v$ and each component of the star graph is the complete bipartite graph $K_{l(u), l(u)}$.

Analogously to [8, Corollary 4.2], we have the following.

Corollary 5.4. *Let $P_n(w)$ be an orientable cyclic presentation in which w is of length k and is not a proper power, and let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n.$$

Suppose that u has length at least 3 and let Γ be the star graph of $P_n(w)$. If u is a non-positive, non-negative, word of length 3, then $\text{girth}(\Gamma) \leq 8$; otherwise, $\text{girth}(\Gamma) \leq 6$.

Proof. If $l(u) \geq 4$, then by a modification of [8, Lemma 4.1, Corollary 4.2], we have $\text{girth}(\Gamma) \leq 6$. (More precisely, parts (a), (c), (d), (e), (f) of [8, Lemma 4.1] hold for the star graph Γ of the, possibly redundant, cyclic presentation $P_n(w)$ if “ w ” is replaced by “ u ” and “cyclic subword” is replaced by “subword”; then the argument of [8, Corollary 4.2] carries through as before if “cyclic subword” is replaced by “subword”.) Thus we may assume $l(u) = 3$. By inversion and cyclic permutation, and noting that, by Lemma 3.1, every cyclic permutation of w has the form in the statement, we may assume that either (a) $u = x_0 x_p x_{p+q}$ or (b) $u = x_0 x_p x_{p+q}^{-1}$ for some $(0 \leq p, q < n)$. In case (a), Γ has edges

$$x_i - x_{i+p}^{-1}, x_i - x_{i+q}^{-1}, x_i - x_{i+h-p-q}^{-1} \quad (0 \leq i < n).$$

Then Γ contains the reduced closed path

$$x_0 - x_p^{-1} - x_{p-q} - x_{h-2q}^{-1} - x_{h-2q-p} - x_{h-q-p}^{-1} - x_0$$

of length 6, so $\text{girth}(\Gamma) \leq 6$. In case (b), Γ has edges

$$x_i - x_{i+p}^{-1}, x_i - x_{i+q}, x_i^{-1} - x_{i+h-p-q}^{-1} \quad (0 \leq i < n).$$

Then Γ contains the reduced closed path

$$x_0 - x_p^{-1} - x_{h-q}^{-1} - x_{h-p-q} - x_{h-p} - x_h^{-1} - x_{p+q}^{-1} - x_q - x_0$$

of length 8, so $\text{girth}(\Gamma) \leq 8$. □

As in [8, Theorem A], it follows from [19] that if $l(u) \geq 3$ and if $P_n(w)$ satisfies the small cancellation condition $T(q)$, where $q \geq 7$, then $l(u) = 3$ and u is non-positive and non-negative.

The following corollary is analogous to the final statement of [8, Theorem A].

Corollary 5.5. *Suppose $P_n(w)$ is orientable. If $P_n(w)$ is (m, k, v) -special with $m \geq 3$, then $m = 3$ and w is positive or negative.*

Proof. Let u be the shortest subword of w such that

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n.$$

If $l(u) < 3$, then the star graph of $P_n(w)$ is at most 2-regular, so $P_n(w)$ is not special, so we may assume $l(u) \geq 3$. If u is positive or negative, then by Corollary 5.4, $\text{girth}(\Gamma) \leq 6$, so $m = 3$, as required. Suppose then u is non-positive and non-negative and that $P_n(w)$ is (m, k, v) -special, where $m \in \{3, 4\}$ by Corollary 5.4. Then there exists $a \in \mathcal{A}$. Then, letting $t = n/(n, a)$,

$$x_0 - x_a - x_{2a} - \cdots - x_{(t-1)a} - x_0$$

is a reduced closed path of length t , which is therefore even (since Γ is bipartite). Then (noting that v divides (n, a) by Theorem 5.2) $n/v = t \cdot ((n, a)/v)$ is even. Since each component of Γ is the incidence graph of a generalised m -gon, it has 14 vertices if $m = 3$, and 30 vertices if $m = 4$ (see [31, Corollary 1.5.5]). Therefore, $n/v = 7$ or 15, a contradiction. $\square$

5.2 Star graphs of non-orientable cyclic presentations

We now define multisets of differences of subscripts in length 2 subwords of a particular cyclic subword of w for the non-orientable case.

Definition 5.6. Suppose $w = u\theta^{n/2}(u)^{-1}$. Let $\bar{\mathcal{A}}', \bar{\mathcal{B}}', \bar{\mathcal{Q}}$ be the multisets defined as follows:

$$\begin{aligned}\bar{\mathcal{A}}' &= \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } u, \text{ with multiplicities}\}, \\ \bar{\mathcal{B}}' &= \{b \mid \text{there is a subword } x_i^{-1} x_{i+b} \text{ of } u, \text{ with multiplicities}\}, \\ \bar{\mathcal{Q}} &= \{q \mid \text{there is a subword } x_i x_{i+q} \text{ or } x_{i+q}^{-1} x_i^{-1} \text{ of } u, \text{ with multiplicities}\},\end{aligned}$$

and define $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ as follows:

$$(\bar{\mathcal{A}}, \bar{\mathcal{B}}) = \begin{cases} (\bar{\mathcal{A}}', \bar{\mathcal{B}}' \cup \{n/2, n/2\}), & \iota(u) \text{ positive and } \tau(u) \text{ negative,} \\ (\bar{\mathcal{A}}' \cup \{n/2, n/2\}, \bar{\mathcal{B}}'), & \iota(u) \text{ negative and } \tau(u) \text{ positive,} \\ (\bar{\mathcal{A}}' \cup \{n/2\}, \bar{\mathcal{B}}' \cup \{n/2\}) & \text{otherwise.} \end{cases}$$

Note that $\bar{\mathcal{Q}} = \emptyset$ if and only if u is alternating and that, letting v denote the cyclic subword $\theta^{n/2}(\iota(u)^{-1})u\theta^{n/2}(\tau(u))^{-1}$ of w ,

$$\begin{aligned}\bar{\mathcal{A}} &= \{a \mid \text{there is a subword } x_i x_{i+a}^{-1} \text{ of } v, \text{ with multiplicities}\}, \\ \bar{\mathcal{B}} &= \{b \mid \text{there is a subword } x_i^{-1} x_{i+b} \text{ of } v, \text{ with multiplicities}\}.\end{aligned}$$

The following theorem, Theorem 5.7, describes the star graph of a non-orientable cyclic presentation $P_n(u\theta^{n/2}(u^{-1}))$. We use this in obtaining our classification of

the non-orientable redundant (m, k, v) -special cyclic presentations, and in Example 6.1, we use it to construct non-orientable cyclic presentations with particular star graphs. We may assume $u\theta^{n/2}(u^{-1})$ is not a proper power (noting that proper power words of this form exist, such as $(x_0x_1^{-1})^2$ when $n = 2$). In first studying the proof, the reader may find it helpful to consider the star graphs of the presentations $P_4(x_0x_1x_3^{-1}x_2^{-1})$, $P_4(x_1x_3^{-1}x_2^{-1}x_0x_2^{-1}x_0x_1x_3^{-1})$, $P_4(x_0x_1^{-1}x_3x_2^{-1})$, and $P_4(x_0x_1^{-1}x_0x_2^{-1}x_3x_2^{-1})$.

Theorem 5.7. *Let Γ be the star graph of $P_n(w)$, where $w = u\theta^{n/2}(u)^{-1}$ and is not a proper power. Let $d_{\bar{\mathcal{A}}} = \gcd(n, a \ (a \in \bar{\mathcal{A}}))$, $d_{\bar{\mathcal{B}}} = \gcd(n, b \ (b \in \bar{\mathcal{B}}))$, and if u is not alternating, let $q_0 \in \bar{\mathcal{Q}}$ and set*

$$d = \gcd(n, a \ (a \in \bar{\mathcal{A}}), b \ (b \in \bar{\mathcal{B}}), q - q_0 \ (q \in \bar{\mathcal{Q}})).$$

Then Γ is $l(u)$ -regular and has vertices x_i, x_i^{-1} ($0 \leq i < n$) and edges

- $x_i - x_{i+a}, x_i^{-1} - x_{i+b}^{-1}, x_i - x_{i+q}^{-1}$ for all $a \in \bar{\mathcal{A}}', b \in \bar{\mathcal{B}}', q \in \bar{\mathcal{Q}}, 0 \leq i < n$,
- $x_i - x_{i+a}, x_i^{-1} - x_{i+b}^{-1}$ for all $a \in \bar{\mathcal{A}} \setminus \bar{\mathcal{A}}', b \in \bar{\mathcal{B}} \setminus \bar{\mathcal{B}}', 0 \leq i < n/2$.

- (a) *If u is not alternating, then Γ has d connected components $\Gamma_0, \dots, \Gamma_{d-1}$, where, for $0 \leq j < d$, the graph Γ_j is the induced subgraph of Γ with vertex sets $V(\Gamma_j) = V(\Gamma_j^+) \cup V(\Gamma_j^-)$, where Γ_j^+ and Γ_j^- are the induced subgraphs of Γ with vertex sets*

$$V(\Gamma_j^+) = \{x_{j+td} \mid 0 \leq t < n/d\},$$

$$V(\Gamma_j^-) = \{x_{j+td+q_0}^{-1} \mid 0 \leq t < n/d\}$$

(subscripts mod n). In particular,

$$|V(\Gamma_j^+)| = |V(\Gamma_j^-)| = n/d \quad \text{for all } 0 \leq j < d,$$

and the subscripts of the positive (respectively negative) vertices in any component are congruent mod d .

- (b) *If u is alternating, then Γ has $d_{\bar{\mathcal{A}}} + d_{\bar{\mathcal{B}}}$ connected components*

$$\Gamma_0^+, \dots, \Gamma_{d_{\bar{\mathcal{A}}}-1}^+, \quad \Gamma_0^-, \dots, \Gamma_{d_{\bar{\mathcal{B}}}-1}^-.$$

which are, respectively, the induced labelled subgraphs of Γ with vertex sets

$$V(\Gamma_j^+) = \{x_{j+td_{\bar{\mathcal{A}}}} \mid 0 \leq t < n/d_{\bar{\mathcal{A}}}\},$$

$$V(\Gamma_j^-) = \{x_{j+td_{\bar{\mathcal{B}}}}^{-1} \mid 0 \leq t < n/d_{\bar{\mathcal{B}}}\}$$

(subscripts mod n). Moreover, each component Γ_j^+ is isomorphic to the graph

$$\text{circ}'_{n/d_{\bar{\mathcal{A}}}}(\{a/d_{\bar{\mathcal{A}}} \ (a \in \bar{\mathcal{A}})\})$$

and each component Γ_j^- is isomorphic to

$$\text{circ}'_{n/d_{\bar{\mathcal{B}}}}(\{b/d_{\bar{\mathcal{B}}} \ (b \in \bar{\mathcal{B}})\}).$$

Proof. By Theorem 3.4, the star graph of $P_n(w)$ is equal to the star graph of $P_{n,n/2}(w)$, which is concise. Therefore, every length-two cyclic subword of each relator $\theta^i(w)$ ($0 \leq i < n/2$) contributes exactly one edge to the star graph Γ .

Noting that w is of the form $\iota(u) \dots \tau(u) \cdot \theta^{n/2}(\tau(u)^{-1}) \dots \theta^{n/2}(\iota(u)^{-1})$, the length-two cyclic subwords of the $n/2$ relators $\theta^i(w)$ ($0 \leq i < n/2$) consist of the $l(u) - 1$ length-two subwords of $\theta^i(u)$, the $l(u) - 1$ length-two subwords of $\theta^{i+n/2}(u^{-1})$, $\theta^i(\tau(u)) \cdot \theta^{i+n/2}(\tau(u)^{-1})$, and of $\theta^{i+n/2}(\iota(u)^{-1}) \cdot \theta^i(\iota(u))$. These contribute edges $x_i - x_{i+a}$ for each $a \in \bar{\mathcal{A}}'$, $x_i^{-1} - x_{i+b}^{-1}$ for each $b \in \bar{\mathcal{B}}'$, $x_i - x_{i+q}^{-1}$ for each $q \in \bar{\mathcal{Q}}$ ($0 \leq i < n$),

$$\theta^i(\tau(u)) - \theta^{i+n/2}(\tau(u)), \theta^{i+n/2}(\iota(u)^{-1}) - \theta^i(\iota(u)^{-1}) \quad (0 \leq i < n/2).$$

Equivalently, they contribute the edges $x_i - x_{i+a}$ for each $a \in \bar{\mathcal{A}}'$, $x_i^{-1} - x_{i+b}^{-1}$ for each $b \in \bar{\mathcal{B}}'$, $x_i - x_{i+q}^{-1}$ for each $q \in \bar{\mathcal{Q}}$ ($0 \leq i < n$), and the edges $x_i - x_{i+a}$ ($a \in \bar{\mathcal{A}} \setminus \bar{\mathcal{A}}'$), $x_i^{-1} - x_{i+b}^{-1}$ ($b \in \bar{\mathcal{B}} \setminus \bar{\mathcal{B}}'$) ($0 \leq i < n/2$).

Thus each positive vertex has the same degree, and since, in the star graph of any finite presentation, vertices corresponding to a generator and its inverse have the same degree (see [19, Section 2.3.3]), the graph Γ is regular. Moreover, the number of edges of the star graph of a concise presentation that has no proper power relators, and where the relators are cyclically reduced, is equal to the sum of the lengths of the relators, so the number of edges is equal to $nl(u)$, and hence Γ is $l(u)$ -regular.

This completes the proof of first part of the statement. With this description of Γ in place, statements (a) and (b) follow as in the proof of [8, Theorem 3.3]. $\square$

Corollary 5.8. *Let Γ be the star graph of $P_n(u\theta^{n/2}(u)^{-1})$, where $l(u) \geq 2$ and u is reduced, and let $\epsilon_\iota, \epsilon_\tau \in \{1, -1\}$ be the exponents of $\iota(u), \tau(u)$, respectively. Then $\text{girth}(\Gamma) \leq 4$, and if $\epsilon_\iota\epsilon_\tau = -1$, then $\text{girth}(\Gamma) = 2$.*

Proof. If $\epsilon_\iota = 1, \epsilon_\tau = -1$ or $\epsilon_\iota = -1, \epsilon_\tau = 1$, then

$$x_0 - x_{n/2} - x_0 \quad \text{or} \quad x_0^{-1} - x_{n/2}^{-1} - x_0^{-1}$$

is a closed path of length 2. If $\bar{\mathcal{Q}} \neq \emptyset$, then there exists $q \in \bar{\mathcal{Q}}$, so it follows that $x_0 - x_{n/2} - x_{n/2+q}^{-1} - x_q^{-1} - x_0$ is a closed path of length 4. Thus we may assume

$\bar{\mathcal{Q}} = \emptyset$ and $\epsilon_i = \epsilon_\tau$, that is, u is alternating but not cyclically alternating. Then $|\bar{\mathcal{A}}| \geq 2$, $|\bar{\mathcal{B}}| \geq 2$, so $\text{girth}(\Gamma^+) \leq 4$ and $\text{girth}(\Gamma^-) \leq 4$, so $\text{girth}(\Gamma) \leq 4$. $\square$

As in [8, Theorem A], it follows from [19] that if $l(u) \geq 2$, then $P_n(u\theta^{n/2}(u)^{-1})$ does not satisfy the small cancellation condition $T(5)$.

6 (m, k, ν) -special cyclic presentations

The concise (m, k, ν) -special cyclic presentations were classified in [8, Sections 5 and 6]. In this section, we generalise that classification to (possibly redundant) cyclic presentations. By Corollary 5.5 and Corollary 5.8, if a cyclic presentation $P_n(w)$ is (m, k, ν) -special, where $m \geq 3$, then $P_n(w)$ is orientable, w is positive or negative, and $m = 3$. In Section 6.1 (Theorem 6.2), we classify the (redundant and concise) orientable $(3, k, \nu)$ -special cyclic presentations $P_n(w)$, where w is positive. In Section 6.2 (Theorems 6.3–6.5), we classify the (redundant and concise) orientable $(2, k, \nu)$ -special cyclic presentations. In Section 6.3 (Theorems 6.6, 6.7), we classify the non-orientable $(2, k, \nu)$ -special cyclic presentations. Theorems 6.3–6.7 are analogous to [8, Theorems C, D, E], which deal with concise cyclic presentations. The proofs of those theorems proceed by analysing graphs defined in terms of sets $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{Q}^+, \mathcal{Q}^-$. Theorems 6.2–6.5 (which deal with the orientable case) are re-expressions of the corresponding theorems from [8] in terms of the sets $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{Q}^+, \mathcal{Q}^-$ of Definition 5.1. (Note that these statements require $l(u) \geq 3$ to ensure that the degrees of vertices are at least 3, as required by Definition 2.1.) The proofs are similarly analogous to those from [8], so are omitted. A minor exception to this is Theorem 6.3, where, in [8, Theorem C], the conciseness hypothesis provides the further conclusion that $\nu \leq 2$, a condition that does not hold in the redundant case. Theorems 6.6 and 6.7 (which deal with the non-orientable case) have no direct counterparts in [8].

It will follow from the results of Sections 6.1–6.3, together with Theorem 3.2, that if $P_n(w)$ is a redundant (m, k, ν) -special cyclic presentation, then $k \geq 6$, so $2/k + 1/m < 1$, and hence (as described in Section 2.3) the corresponding group $G_n(w)$ is non-elementary hyperbolic.

Example 6.1. This example exhibits the various situations that arise for redundant (m, k, ν) -special cyclic presentations.

- (a) (i) $P_7(\prod_{i=0}^6 \theta^{4i}(x_0^2 x_1))$ is orientable, redundant, $(3, 21, 1)$ -special, and defines the one-relator group $G_{7,1}(\prod_{i=0}^6 \theta^{4i}(x_0^2 x_1))$;
- (ii) $P_{14}(x_0 x_1 x_{10} x_7 x_8 x_3)$ is orientable, redundant, and $(3, 6, 2)$ -special;

- (iii) $P_{21}(\prod_{i=0}^6 \theta^{3i}(x_0x_2x_7))$ is orientable, redundant, and $(3, 21, 3)$ -special. In each case, each component of the star graph is the Heawood graph.
- (b) $P_9(x_0x_1x_5x_3x_4x_8x_6x_7x_2)$ and $P_8(x_0x_1x_3x_6x_2x_3x_5x_0x_4x_5x_7x_2x_6x_7x_1x_4)$ are orientable, redundant, and $(2, 9, 3)$ -special and $(2, 16, 1)$ -special, respectively. In these cases, the defining word is positive.
- (c) $P_8(x_0x_1^{-1}x_6x_3^{-1}x_4x_5^{-1}x_2x_7^{-1})$ is orientable, redundant, and $(2, 8, 2)$ -special, and the defining word is cyclically alternating.
- (d) $P_8(x_0x_2^{-1}x_4x_7x_6x_0^{-1}x_2x_5x_4x_6^{-1}x_0x_3x_2x_4^{-1}x_6x_1)$ is orientable, redundant, and $(2, 16, 2)$ -special, and the defining word is non-positive, non-negative, and non-alternating.
- (e) $P_6(x_0x_1^{-1}x_0x_3^{-1}x_4x_3^{-1})$ is non-orientable and $(2, 6, 2)$ -special, and the defining word is cyclically alternating.
- (f) $P_{12}(x_0x_2^{-1}x_4x_7x_2x_1x_7^{-1}x_8^{-1}x_1^{-1}x_{10}^{-1}x_8x_6^{-1})$ is non-orientable and $(2, 12, 2)$ -special, and the defining word is non-alternating.

The presentations in Example 6.1 were constructed using Theorems 5.2 and 5.7. For example, for the presentation of Example 6.1 (a) (i), we sought a 7-generator redundant presentation with $\mathcal{A} = \mathcal{B} = \emptyset$ and $\mathcal{Q} = \mathcal{Q}^+$ equal to a perfect difference set mod 7, such as $\{0, 1, 3\}$. With

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u),$$

as in Definition 5.1, if we set $u = x_0^2x_1$, then $\mathcal{Q}^+ = \{0, 1, h-1\}$, and so setting $h = 4$ yields the given presentation.

6.1 $(3, k, v)$ -special cyclic presentations

Theorem 6.2 classifies when a (possibly redundant) orientable cyclic presentation $P_n(w)$, in which w is a positive word, is $(3, k, v)$ -special and so generalises [8, Theorem B]; its proof is analogous to the proof of that theorem. See Example 6.1 (a) for examples.

Theorem 6.2. *Suppose $P_n(w)$ is an irreducible, orientable cyclic presentation, where w is a positive word of length $k \geq 3$ and is not a proper power, and let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n.$$

Then $P_n(w)$ is $(3, k, v)$ -special if and only if the following hold:

- (a) $n = vN$, where $N = l(u)^2 - l(u) + 1$;
- (b) $\mathcal{Q}$ is a perfect difference set;
- (c) $q \equiv q' \pmod{v}$ for each pair $q, q' \in \mathcal{Q}$.

We remark that a consequence of conditions (a) and (c) of Theorem 6.2 is that v divides k .

6.2 Orientable $(2, k, v)$ -special cyclic presentations

In this section, we classify the orientable $(2, k, v)$ -special cyclic presentations

$$P_n \left(\prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \right)$$

(where we may assume that u is the shortest possible). We consider the cases u positive, cyclically alternating, and non-positive, non-negative, and not cyclically alternating separately.

u positive

Theorem 6.3 classifies when a (redundant or concise) orientable cyclic presentation

$$P_n \left(\prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \right),$$

in which u is a positive word, is $(2, k, v)$ -special and so generalises [8, Theorem C]. Note that, unlike in [8, Theorem C], v is no longer limited to the values 1, 2. The proof of Theorem 6.3 is analogous to that of [8, Theorem C] except that the argument that leads to the conclusion $v \in \{1, 2\}$ needs to be removed. See Example 6.1 (b) for an example.

Theorem 6.3. *Let $P_n(w)$ be an irreducible, orientable cyclic presentation, where w has length $k \geq 4$ and is not a proper power, let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n,$$

and suppose that u is positive. Then $P_n(w)$ is $(2, k, v)$ -special if and only if $n = vl(u)$, $l(u) \geq 3$, $\mathcal{Q} = \{q_0, v + q_0, 2v + q_0, \dots, (l(u) - 1)v + q_0\}$ for some $0 \leq q_0 < n$ such that $\gcd(q_0, v) = 1$, in which case each component of Γ is isomorphic to $K_{l(u), l(u)}$.

In constructing examples, it is useful to note that, by (5.1), the conditions of Theorem 6.3 imply $l(u)q_0 + [l(u)(l(u) - 1)/2]v \equiv h \pmod{n}$.

u cyclically alternating

Theorem 6.4 classifies when a (redundant or concise) orientable cyclic presentation

$$P_n \left(\prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \right),$$

in which u is a cyclically alternating word, is $(2, k, v)$ -special and so generalises [8, Theorem D]; its proof is analogous to the proof of that theorem. See Example 6.1 (c) for an example.

Theorem 6.4. *Let $P_n(w)$ be an irreducible, orientable cyclic presentation, where w has length $k \geq 4$ and is not a proper power, let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n,$$

and suppose that u is cyclically alternating. Then $P_n(w)$ is $(2, k, v)$ -special if and only if $n = 2l(u)$, $l(u) \geq 3$, $v = 2$, and $\mathcal{A}, \mathcal{B}$ are each sets of the form

$$\{\pm 1, \pm 3, \dots, \pm(n/2 - 1)\}.$$

u non-positive, non-negative, not cyclically alternating

Theorem 6.5 classifies when a (redundant or concise) orientable cyclic presentation

$$P_n \left(\prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \right),$$

in which u is a non-positive, non-negative, and not cyclically alternating word, is $(2, k, v)$ -special and so generalises [8, Theorem E]; its proof is analogous to the proof of that theorem. See Example 6.1 (d) for an example.

Theorem 6.5. *Let $P_n(w)$ be an irreducible, orientable cyclic presentation, where w has length $k \geq 4$ and is not a proper power, and let u be the shortest subword of w such that*

$$w = \prod_{i=0}^{n/(n,h)-1} \theta^{ih}(u) \quad \text{for any } 0 \leq h < n,$$

and suppose that u is non-positive, non-negative, and not cyclically alternating.

Then $P_n(w)$ is $(2, k, v)$ -special if and only if the following hold:

- (a) $n = vl(u)$ and $l(u)$ is divisible by 4;
- (b) $\mathcal{A}, \mathcal{B}$ are each sets of the form $\{\pm v, \pm 3v, \dots, \pm((n/2v) - 1)v\}$;
- (c) $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$ and there exists some $0 \leq q_0 < n$ with $(q_0, v) = 1$ such that $\mathcal{Q} = \{q_0, q_0 + 2v, \dots, q_0 + (n/v - 2)v\}$.

6.3 Non-orientable $(2, k, v)$ -special cyclic presentations

In this section, we classify the $(2, k, v)$ -special cyclic presentations $P_n(w)$, where $w = u\theta^{n/2}(u)^{-1}$.

w alternating

Theorem 6.6 classifies when a cyclic presentation $P_n(w)$, where $w = u\theta^{n/2}(u)^{-1}$ is an alternating word, is $(2, k, v)$ -special and so extends [8, Theorem D]. See Example 6.1 (e) for an example.

Theorem 6.6. *Let $w = u\theta^{n/2}(u)^{-1}$ be a word of length k that is alternating and not a proper power, and suppose that $P_n(w)$ is irreducible. Then $P_n(w)$ is $(2, k, v)$ -special if and only if $l(u) = n/2 \geq 3$ is odd, $v = 2$, and $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ are each sets of the form $\{\pm 1, \pm 3, \dots, \pm(n-4)/2, n/2\}$.*

Proof. Let Γ be the star graph of $P_n(w)$ and let Γ^+, Γ^- be the induced subgraphs of Γ whose vertices are the positive and negative vertices of Γ , respectively, and let $d_{\bar{\mathcal{A}}} = \gcd(n, a \ (a \in \bar{\mathcal{A}}))$, $d_{\bar{\mathcal{B}}} = \gcd(n, b \ (b \in \bar{\mathcal{B}}))$. By Theorem 5.7, Γ^+ has $d_{\bar{\mathcal{A}}}$ components and Γ^- has $d_{\bar{\mathcal{B}}}$ components. In this proof, we use the properties of Γ provided by Theorem 5.7 freely without further reference.

Suppose first that $P_n(w)$ is $(2, k, v)$ -special. By Corollary 5.8, u is not cyclically alternating, so $l(u)$ is odd. Then each component of Γ is isomorphic, so $d_{\bar{\mathcal{A}}} = d_{\bar{\mathcal{B}}}$, and since $P_n(w)$ is irreducible, $1 = (d_{\bar{\mathcal{A}}}, d_{\bar{\mathcal{B}}}) = d_{\bar{\mathcal{A}}} = d_{\bar{\mathcal{B}}}$, so Γ has 2 components, and so $v = 2$. Hence Γ^+, Γ^- are each isomorphic to $K_{n/2, n/2}$, which is $n/2$ -regular, so $l(u) = n/2$ and (by the definition of special presentations) $l(u) \geq 3$. Moreover,

$$K_{n/2, n/2} = \Gamma^+ = \text{circ}'_n(\bar{\mathcal{A}}) \quad \text{and} \quad K_{n/2, n/2} = \Gamma^- = \text{circ}'_n(\bar{\mathcal{B}}),$$

so $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ are each sets of the form $\{\pm 1, \pm 3, \dots, \pm(n-4)/2, n/2\}$, as required.

Conversely, suppose that the given conditions hold. Then $d_{\bar{\mathcal{A}}} = 1$ and the edges of Γ^+ join each even vertex x_i to each odd vertex x_j , and so Γ^+ is the complete bipartite graph $K_{n/2, n/2}$. Similarly, Γ^- is the complete bipartite $K_{n/2, n/2}$. Since the degree of each vertex is $l(u) \geq 3$, the presentation $P_n(w)$ is $(2, k, v)$ -special. $\square$

w non-alternating

Theorem 6.7 classifies when a cyclic presentation $P_n(u\theta^{n/2}(u)^{-1})$, in which u is a non-alternating word, is $(2, k, v)$ -special and so extends [8, Theorem E]. See Example 6.1 (f) for an example.

Theorem 6.7. *Let $w = u\theta^{n/2}(u)^{-1}$ be a word of length k that is not a proper power, where u is a non-alternating word of length at least 3, and suppose that $P_n(w)$ is irreducible. Then $P_n(w)$ is $(2, k, v)$ -special if and only if the following hold:*

- (a) $n = vl(u)$, $l(u) \equiv 2 \pmod{4}$, $l(u) \geq 6$;
- (b) $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ are each sets of the form $\{\pm v, \pm 3v, \dots, \pm(n/(2v) - 2)v, n/2\}$;
- (c) there exists some $0 < q_0 < n$ with $(q_0, v) = 1$ such that

$$\bar{\mathcal{Q}} = \{q_0, q_0 + 2v, \dots, q_0 + (n/v - 2)v\}.$$

Proof. Suppose first that $P_n(w)$ is $(2, k, v)$ -special. Then $\bar{\mathcal{A}}, \bar{\mathcal{B}}, \bar{\mathcal{Q}}$ are sets (i.e. they have no repeated elements), and by Corollary 5.8, we have $\bar{\mathcal{A}} \setminus \bar{\mathcal{A}}' = \{n/2\}$ and $\bar{\mathcal{B}} \setminus \bar{\mathcal{B}}' = \{n/2\}$.

By Theorem 5.7, each component Γ_i of the star graph Γ of $P_n(w)$ is the complete bipartite graph $K_{l(u), l(u)}$, so $n = vl(u)$. Moreover, $v < n/2$, and hence $l(u) > 2$, for otherwise, each component has at most four vertices, so the vertices have degree at most 2, a contradiction. We use properties of Γ provided by Theorem 5.7 freely without further reference. The component Γ_0 has vertex set $V(\Gamma_0) = V(\Gamma_0^+) \cup V(\Gamma_0^-)$, where Γ_0^+, Γ_0^- are the induced labelled subgraphs of Γ_0 with vertex sets

$$\begin{aligned} V(\Gamma_0^+) &= \{x_0, x_v, \dots, x_{(n/v-1)v}\}, \\ V(\Gamma_0^-) &= \{x_{q_0}^{-1}, x_{q_0+v}^{-1}, \dots, x_{q_0+(n/v-1)v}^{-1}\}, \end{aligned}$$

respectively, for some $q_0 \in \bar{\mathcal{Q}}$ (which is non-empty since w is non-alternating). In particular, $v \mid a$ for all $a \in \bar{\mathcal{A}}'$ and $v \mid b$ for all $b \in \bar{\mathcal{B}}'$.

Suppose for contradiction that $v, n - v \notin \bar{\mathcal{A}}$. Then, for each $0 \leq i < n$, vertices x_i, x_{i+v} are not joined by an edge. Therefore, the positive vertices of Γ_0 are all in the same part of Γ_0 , so $\bar{\mathcal{A}} = \emptyset$. By Corollary 5.8, $\iota(u), \tau(u)$ are both positive or both negative, so $n/2 \in \bar{\mathcal{A}}$, a contradiction, so v or $n - v \in \bar{\mathcal{A}}$. Similarly, v or $n - v \in \bar{\mathcal{B}}$. Therefore, Γ_0 contains closed paths

$$\begin{aligned} x_0 - x_v - \dots - x_{(n/v-1)v} - x_0, \\ x_{q_0}^{-1} - x_{q_0+v}^{-1} - \dots - x_{q_0+(n/v-1)v}^{-1} - x_{q_0}^{-1}, \end{aligned}$$

each of length n/v , which is therefore even, since Γ is bipartite. Therefore, the vertices $x_v, x_{3v}, \dots, x_{(n/v-1)v}$ are precisely those positive vertices of Γ_0 that belong to a different part of Γ_0 , to x_0 (and so are neighbours of x_0), and the vertices $x_{q_0+v}^{-1}, x_{q_0+3v}^{-1}, \dots, x_{q_0+(n/v-1)v}^{-1}$ are precisely those negative vertices of Γ_0 that belong to a different part of Γ_0 , to $x_{q_0}^{-1}$ (and so are neighbours of $x_{q_0}^{-1}$). Hence $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ are sets of the form $\{\pm v, \pm 3v, \dots, \pm(n/(2v) - 2)v, n/2\}$. Moreover, the positive vertices and the negative vertices of Γ_0 each induce a complete bipartite graph $K_{n/(2v), n/(2v)}$ which therefore has $(n/(2v))^2$ edges. Since each element of $\bar{\mathcal{A}}'$ contributes n/v edges to Γ_0 and the element $n/2 \in \bar{\mathcal{A}} \setminus \bar{\mathcal{A}}'$ contributes $(n/2)/v$ edges to Γ_0 , we have $n/v(|\bar{\mathcal{A}}| - 1) + (n/v)/2 = (n/(2v))^2$, so

$$|\bar{\mathcal{A}}'| + 1 = |\bar{\mathcal{A}}| = ((n/v) + 2)/4.$$

Similarly,

$$|\bar{\mathcal{B}}'| + 1 = |\bar{\mathcal{B}}| = ((n/v) + 2)/4.$$

In particular, $n/v \equiv 2 \pmod{4}$, so parts (a), (b) hold.

Since $x_{q_0}^{-1}, x_{q_0+v}^{-1}$ belong to different parts of Γ_0 and there is an edge $x_0 - x_{q_0}^{-1}$, the vertices $x_0, x_{q_0+v}^{-1}$ belong to the same part of Γ_0 . Hence

$$\bar{\mathcal{Q}} = \{q_0, q_0 + 2v, \dots, q_0 + (n/v - 2)v\}.$$

Finally, $\gcd(q_0, v)$ divides $\gcd(n, a \ (a \in \bar{\mathcal{A}}), b \ (b \in \bar{\mathcal{B}})) = 1$ since $P_n(w)$ is irreducible, so $\gcd(q_0, v) = 1$, and so (c) holds.

Now suppose that the conditions of the statement hold. Then Γ has v isomorphic components. Consider the component Γ_0 . The set of neighbours of x_{jv} ($0 \leq j < n/v$) is

$$\begin{cases} \{x_v, x_{3v}, \dots, x_{(n/v-1)v}\} \cup \{x_{q_0}^{-1}, x_{q_0+2v}^{-1}, \dots, x_{q_0+(n/v-2)v}^{-1}\} & \text{if } j \text{ is even,} \\ \{x_0, x_{2v}, \dots, x_{(n/v-2)v}\} \cup \{x_{q_0+v}^{-1}, x_{q_0+3v}^{-1}, \dots, x_{q_0+(n/v-1)v}^{-1}\} & \text{if } j \text{ is odd,} \end{cases}$$

and so Γ_0 is bipartite with vertex partition

$$\begin{aligned} & \{x_v, x_{3v}, \dots, x_{(n/v-1)v}, x_{q_0}^{-1}, x_{q_0+2v}^{-1}, \dots, x_{q_0+(n/v-2)v}^{-1}\} \\ & \cup \{x_0, x_{2v}, \dots, x_{(n/v-2)v}, x_{q_0+v}^{-1}, x_{q_0+3v}^{-1}, \dots, x_{q_0+(n/v-1)v}^{-1}\}. \end{aligned}$$

Further, for each $0 \leq j < n/v$, the set of neighbours $N_\Gamma(x_{q_0+jv}^{-1}) = N_\Gamma(x_{jv})$, so Γ_0 is isomorphic to $K_{n/v, n/v}$, as required. $\square$

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