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Lower bound on growth of non-elementary subgroups in relatively hyperbolic groups

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Abstract. This paper proves that, in a non-elementary relatively hyperbolic group, the logarithm growth rate of any non-elementary subgroup has a linear lower bound by the logarithm of the size of the corresponding generating set. As a consequence, any non-elementary subgroup has uniform exponential growth.

1 Introduction

1.1 Results and background

Let S be a finite symmetric generating set of a group H and d_S the corresponding word metric. Write

$$S^{\leq n} := \{h \in H : d_S(1,h) \leq n\} \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

The (logarithm) *growth rate* of H with respect to S is defined to be the following limit:

$$\omega(H,S) := \lim_{n\to\infty} \frac{\log \sharp(S^{\leq n})}{n},$$

which exists since $\sharp(S^{\leq n+m}) \leq \sharp(S^{\leq n}) \cdot \sharp(S^{\leq m})$. In what follows, we always consider finitely generated groups.

The spectrum of growth rates of a group H has attracted lots of research interests,

$$\Omega(H) := \{\omega(H,S) : \sharp S < \infty,\ \langle S\rangle = H\}.$$

For a group with exponential growth, the question of Gromov [14] whether $\Omega(H)$ admits the infimum 0 was open for twenty years and answered negatively by Wilson [25] (see [2] also). He constructed the first examples of groups with non-uniform exponential growth such that a sequence of two-element generating sets has growth rates tending to 0.

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A group H has *uniform exponential growth* if $\inf \Omega(H) > 0$. If a group has uniform exponential growth, it is quite interesting to ask whether $\Omega(H)$ obtains the minimum. Sambusetti [24] showed that the answer was again negative for the free products of any two non-Hopfian groups which are a special class of relatively hyperbolic groups. However, the recent work by Fujiwara–Sela [11] obtains a positive answer for hyperbolic groups by showing the set $\Omega(H)$ is well-ordered, so $\Omega(H)$ admits a minimum. This settles a question of de la Harpe. The starting point of their argument relies on the fact due to Arzhantseva–Lysenok [1] that the growth rate $\omega(H, S)$ is lower bounded by a linear function of the size of S.

Noting the simple fact $\omega(H, S) \le \log \sharp S$, the work of [1, 11] seem to suggest it is worth understanding the following set:

$$\Theta(H) := \left\{ \frac{\omega(H, S)}{\log \sharp S} : \sharp S < \infty,\ \langle S \rangle = H \right\}.$$

Of course, $\Theta(H) \subset [0, 1]$. A number of inquiries could be made about the nature of $\Theta(H)$. For instance, could the set $\Theta(H)$ always be infinite? If it is infinite, what are the accumulation points of the set $\Theta(H)$? The purpose of this paper is not to give complete answers to these questions. Instead, we collect here a few simple observations to motivate further investigations.

A group H has *purely exponential growth* if $\frac{1}{C} \exp(n\omega) \le \sharp S^n \le C \exp(n\omega)$ for some $C = C(S) > 0$ independent of $n \ge 1$. This class of groups includes (relatively) hyperbolic groups and many other groups (see [5, 28] for relevant discussions). By taking $T_n := S^n$, one sees[1] that

$$\frac{\omega(H, T_n)}{\log \sharp T_n} \to 1 \quad \text{as } n \to \infty.$$

Thus, the upper bound 1 is an accumulation point for any group with purely exponential growth. On the other hand, the growth tightness [13] of free groups implies that $1 \in \Theta(H)$ if and only if H is a free group. Thus, it is interesting to ask whether there exist examples with $\Theta(H) \subset [0, 1 - \epsilon]$ for some $\epsilon > 0$.

The examples of Wilson also imply $0 \in \overline{\Omega(H)}$ for certain non-uniform exponential growth groups H. Analogous to the question of uniform exponential growth, we can ask for which groups $\Theta(H)$ admits a positive infimum. In fact, $\inf \Theta(H) > 0$ has been obtained for hyperbolic groups in [1].

The main result of this paper is a generalization of the previous results of Arzhantseva–Lysenok [1] to the class of relatively hyperbolic groups. Since the official introduction in the Gromov 1987 monograph [15], this class of groups has

[1] The authors learned this fact from Alex Furman.

been well-studied in the last thirty years; see [3, 8, 9, 12, 21]. The important examples include Gromov-hyperbolic groups, geometrically finite Kleinian groups (with variable negative curvature), infinitely ended groups, small cancellation quotients of free products, limit groups, to name just a few.

Our main theorem establishes the positive lower bound on $\Theta(H)$ for any non-elementary subgroup in a relatively hyperbolic group. By definition, a subgroup H is said to be *non-elementary* if its limit set contains at least three points. So an elementary subgroup is either virtually cyclic or can be conjugated into a maximal parabolic subgroup. See § 2.2 for details.

Theorem 1.1. *Assume that G is a non-elementary relatively hyperbolic group. Then there exists a constant $\kappa = \kappa(G) \in (0, 1]$ such that, for any non-elementary subgroup H with a finite symmetric generating set S, we have*

$$\omega(H, S) \geq \kappa \cdot \log \sharp S.$$

Remark. Wilson's example exhibits a sequence of 2-generator sets with growth rate tending 0. This shows that the non-elementary assumption of H is necessary: indeed, any group H can be realized as the maximal parabolic subgroup in a free product of H with any nontrivial group.

A group G has *locally uniform exponential growth* if every finitely generated subgroup of exponential growth has uniform exponential growth. Xie [26] has proved that relatively hyperbolic groups have uniform exponential growth. As a direct corollary of Theorem 1.1, we obtain a strengthening of Xie's theorem.

Theorem 1.2. *Any non-elementary finitely generated subgroup H of a non-elementary relatively hyperbolic group G has uniform exponential growth.*

Moreover, $(G, \mathcal{P})$ has locally uniform exponential growth if and only if every maximal parabolic subgroup $P \in \mathcal{P}$ has locally uniform exponential growth.

Recall that non-elementary relatively hyperbolic groups G are growth tight, so non-Hopfian ones cannot realize its infimum of $\Omega(G)$ (see [24, 27]). It is thus interesting to know whether Fujiwara–Sela's result [11] can generalize to torsion-free toral relatively hyperbolic groups [16, 17].[2]

[2] This question has been answered for *equationally Noetherian* relatively hyperbolic groups by Fujiwara's preprint [10] (posted on 2 March 2021, one day earlier than ours on arXiv). One ingredient is Theorem 1.2 which was also obtained independently and simultaneously by him.

1.2 Connection with other works

There has been recent interest in studying the product set growth in various classes of groups, starting in free groups [23], hyperbolic groups and acylindrical hyperbolic groups [7], free product of groups [4], and so on. We refer the reader to [4] for further references and connection with approximate groups.

To be precise, let S be any set in a group G subject to the condition that S does not generate a "small" subgroup. The Helfgott type growth (in the terminology of [4]) wishes to have the following:

$$\sharp(S^3) \geq c \cdot (\sharp S)^{1+\kappa}$$

for some universal $c, \kappa > 0$ depending only on G. By induction, it is easy to see that if a group G has the Helfgott type growth, then Theorem 1.1 holds for this group G. In this sense, Theorem 1.1 could be understood as an asymptotic version of product set growth. Indeed, our proof boils down to a similar product growth with high powers

$$\sharp(S^{i\kappa}) \geq (\sharp S)^i \quad \text{for all } i \in \mathbb{N},$$

for a universal $\kappa > 0$; see formula (5.7). Even though, our Theorem 1.1 cannot be deduced directly from the result [7, Theorem 1.9]. Their result does provide certain product set growth only assuming the acylindrical action on hyperbolic spaces. However, the large displacement assumption imposed on S there is hard to verify in practice.

Very recently, Kropholler–Lyman–Ng obtained Theorem 1.2 independently as [20, Proposition 4.12] during the writing of this paper. Similar to us, they made a variant of Xie's result as Lemma 3.1, and then ran the remaining argument in [26] to get Theorem 1.2.

To conclude the introduction, let us mention briefly the proof of the main theorems. We follow closely the strategy of [1] which appears to us quite robust. On the other hand, we have to deal with several difficulties from the relative case. They are resolved largely by adapting the work of Xie [26] (see Lemma 3.1) and by a strengthening of Koubi's result [19] (see Lemma 3.2). We believe that Lemma 3.2 has independent interest and admits further applications.

Structure of the paper. This paper is organized as follows. Section 2 recalls standard materials in Gromov's hyperbolic geometry, Bowditch–Gromov's definition of relatively hyperbolic groups. As mentioned above, the work of Xie and Koubi are properly adapted and strengthened in Section 3. A notion of loxodromic elements with large injectivity is introduced in Section 4 to streamline the strategy of Arzhantseva–Lysenok. The proof of Theorem 1.1 is then completed in Section 5.

2 Preliminary

Consider an isometric action of G on a metric space (X, d). Let $S \subset G$ be a set of isometries. Let $\ell_x(S) := \max_{s\in S}\{d(x, sx)\}$ for a given point $x \in X$. For a subset $A \subset X$, define

$$\ell_A(S) := \inf_{x\in A} \ell_x(S).$$

Note that $x \in X \mapsto \ell_x(S) \in \mathbb{R}$ is a continuous non-negative function.

2.1 Hyperbolic spaces and Loxodromic elements

Define the Gromov product

$$\langle x, y\rangle_o = \frac{d(x, o) + d(y, o) - d(x, y)}{2} \quad \text{for all } x, y, o \in X.$$

A geodesic metric space X is said to be *hyperbolic* if any geodesic triangle is δ-thin: if $d(o, p) = d(o, q) \le \langle x, y\rangle_o$ for two points $p \in [o, x]$, $q \in [o, y]$, then $d(p, q) \le \delta$. Then, for any $x, y, z, o \in X$, we have

$$\langle x, y\rangle_o \ge \min\{\langle x, z\rangle_o, \langle z, y\rangle_o\} - \delta.$$

Assume that a finitely generated group G acts properly by isometry on a proper hyperbolic space X. Then the induced action of G on the Gromov boundary ∂X of X is a convergence group action. Thus, any infinite order element fixes at least one but at most two points in ∂X. So the elements in G are classified into three mutually exclusive classes: *elliptic isometries* being finite order elements, *parabolic isometries* with only one fixed point and *loxodromic isometries* with exactly two fixed points. See [3] for a detailed discussion about convergence group actions and relevant notions.

Equivalently, an isometry g on a proper hyperbolic space X is loxodromic if it admits a (λ, c)-quasi-geodesic γ for some $\lambda, c > 0$ so that $\gamma, g\gamma$ have finite Hausdorff distance. Such quasi-geodesics shall be referred to as (λ, c)-*quasi-axis*.

Lemma 2.1 ([6, Lemma 9.2.2]). *If an isometry g on X satisfies*

$$d(o, go) \ge 2\langle o, g^2o\rangle_{go} + 6\delta$$

for some point $o \in X$, then g is loxodromic.

In hyperbolic spaces, it is a well-known fact that a sufficiently long local quasi-geodesic is globally a quasi-geodesic. The following statement is a variant of this.

Lemma 2.2 ([1, Lemma 1]). *Let $x_1, x_2, \dots, x_k$ for $k \geq 3$ be points in a δ-hyperbolic space such that, for any $2 \leq i \leq k-2$, we have*

$$\langle x_{i-1}, x_{i+1}\rangle_{x_i} + \langle x_i, x_{i+2}\rangle_{x_{i+1}} \leq d(x_i, x_{i+1}) - 3\delta.$$

Then

$$d(x_1, x_k) \geq \sum_{i=1}^{k-1} d(x_i, x_{i+1}) - 2\sum_{i=2}^{k-1}(\langle x_{i-1}, x_{i+1}\rangle_{x_i} + \delta).$$

The following immediate corollary will be used later.

Corollary 2.3. *Under the assumption of Lemma 2.2, if*

$$\langle x_{i-1}, x_{i+1}\rangle_{x_i} + \langle x_i, x_{i+2}\rangle_{x_{i+1}} \leq \frac{1}{4} d(x_i, x_{i+1}) - \delta,$$

then

$$d(x_1, x_k) \geq \frac{1}{2}\sum_{i=1}^{k-1} d(x_i, x_{i+1}).$$

Lemma 2.4. *If g, h are two isometries satisfying*

$$\frac{1}{4}\min\{d(go, o), d(ho, o)\} \geq L \geq \max\{\langle go, h^{-1}o\rangle_o, \langle g^{-1}o, ho\rangle_o\} + \delta$$

for some point $o \in X$ and some $L > 0$, then

(1) *gh is loxodromic.*

(2) *There exist constants $\lambda, c > 0$ depending only on L and δ such that the concatenated path $\alpha := \bigcup_{i\in\mathbb{Z}} (gh)^i ([o, go] \cdot g[o, ho])$ is a (λ, c)-quasi-geodesic.*

Proof. The proof uses the well-known fact that long local geodesics are global quasi-geodesics. To be precise, applying Lemma 2.2 to the points

$$x_1 = o, \quad x_2 = go, \quad x_3 = gho, \quad \dots, \quad x_{2n+1} = (gh)^n o,$$

we have

$$d(x_1, x_{2n+1}) \geq \sum_{i=1}^{2n} d(x_i, x_{i+1}) - 2\sum_{i=2}^{2n}(\langle x_{i-1}, x_{i+1}\rangle_{x_i} + \delta) \geq \frac{1}{2}\sum_{i=1}^{2n} d(x_i, x_{i+1})$$

for $n \geq 1$. Note that $[o, go] \cdot g[o, ho]$ is a quasi-geodesic with parameters depending only on L, δ. These two facts imply that α is a (λ, c)-quasi-geodesic with constants λ, c depending on L, δ. The proof is complete. □

Define the asymptotic *translation length* of an isometry g as follows:

$$\tau(g) := \lim_{n\to\infty} \frac{d(o, g^n o)}{n}$$

for some (thus any) point $o \in X$.

Lemma 2.5 ([6, Proposition 10.6.4]). *If g is a loxodromic element, then*

$$|\ell_X(g) - \tau(g)| \leq 16\delta.$$

We say that an element $g \in G$ *preserves the orientation* of the bi-infinite quasi-geodesic γ if $\alpha, g\alpha$ have finite Hausdorff distance for any half-ray α of γ. It is clear that a loxodromic element preserves the orientation of any quasi-axis.

Lemma 2.6. *Suppose that a loxodromic isometry g has a (λ, c)-quasi-axis γ for some $\lambda, c > 0$. Then there exists a constant $C = C(\lambda, c, \delta) > 0$ with the following property. For any $x \in \gamma$, there exists $y \in \gamma$ such that $\langle x, gx\rangle_y < C$, $\langle y, gy\rangle_{gx} < C$.*

The following lemma is well known with a proof included for completeness.

Lemma 2.7. *There exists a constant $C = C(\lambda, c, \delta) > 0$ for any $\lambda, c > 0$ with the following property. If a loxodromic element g admits a (λ, c)-quasi-axis γ, then for any $x \in \gamma$, we have $|\ell_X(g) - d(x, gx)| \leq C$.*

Proof. By the Morse Lemma, any two (λ, c)-quasi-axes γ and $g\gamma$ have bounded Hausdorff distance depending only on λ, c and δ. Thus, the inclusion of γ into $\bigcup_{i\in\mathbb{Z}} g^i\gamma$ is a quasi-isometry with constants depending only on λ, c, δ. We can thus assume that the quasi-axis γ is $\langle g\rangle$-invariant.

Let $\pi_\gamma\colon X \to \gamma$ be the shortest projection to a (λ, c)-quasi-geodesic γ. Note that π_γ enjoys the C-contracting for a constant $C = C(\lambda, c, \delta)$: if

$$\operatorname{diam}(\{\pi_\gamma(z), \pi_\gamma(w)\}) > C \quad \text{for some } z, w \in X,$$

then

$$\max\{d(\pi_\gamma(z), [z, w]), d(\pi_\gamma(w), [z, w])\} \leq C.$$

We then derive the following for any $z, w \in X$:

$$\operatorname{diam}(\{\pi_\gamma(z), \pi_\gamma(w)\}) + d(z, \pi_\gamma(z)) + d(w, \pi_\gamma(w)) \leq d(z, w) + 4C.$$

Let $o \in X$ so that $d(o, go) = \ell_X(g)$. Apply the above inequality for $z = o$ and $w = go$. Since γ is $\langle g\rangle$-invariant, $\pi_\gamma(\cdot)$ is $\langle g\rangle$-equivariant. If $d(o, \gamma) > 2C$, we then obtain from the above inequality that $d(\pi_\gamma(o), g\pi_\gamma(o)) < d(o, go)$. This contradicts the choice of o with $d(o, go) = \ell_X(g)$. Thus, $d(o, \gamma) \leq 2C$.

Without loss of generality, replacing o by its projection $\pi_\gamma(o)$, we can assume that o lies on γ. Then $|d(o, go) - \ell_X(g)| \leq 2C$.

Let $x \in \gamma$. Assume that the above constant C also satisfies Lemma 2.6. Up to $\langle g\rangle$-translations to x, we can assume that $\langle o, go\rangle_x \leq C$ and then $\langle x, gx\rangle_{go} \leq C$. By hyperbolicity, $\max\{d(go, [x, gx]), d(x, [o, go])\} \leq C + \delta$. Consequently,

$$\begin{aligned}
&|d(o, go) - d(x, gx)| \\
&\qquad \leq |d(o, x) + d(x, go) - d(x, go) - d(go, gx)| + 4C + 4\delta \\
&\qquad \leq 4C + 4\delta.
\end{aligned}$$

The proof is complete. □

2.2 Elementary subgroups

Recall that G acts properly on a proper hyperbolic space X. The *limit set* ΛH of a subgroup H is the set of accumulation points in ∂X of any H-orbit in X. A subgroup H in G is said to be *elementary* if its limit set contains at most two points. See [3] for relevant discussion.

If ΛH consists of only one point p, then H is said to be a *parabolic subgroup* and p is said to be a *parabolic point*. It is a well-known fact that, in a convergence group action, a loxodromic element cannot fix a parabolic point. The (maximal) parabolic group plays the key role in Definition 2.10 of relatively hyperbolic groups given in the next subsection. In the remainder of this subsection, we first consider the elementary subgroup with exactly two limit points.

Let γ be a quasi-axis for a loxodromic element h. The coarse stabilizer of the axis defined as

$$E(h) = \{g \in G : \text{there exists } r > 0 \text{ such that } \gamma \subset N_r(g\gamma),\ g\gamma \subset N_r(\gamma)\}$$

gives the maximal elementary subgroup containing h. Note that the subgroup

$$E^+(h) := \{g \in G : \text{there exists } n > 0 \text{ such that } gh^n g^{-1} = h^n\}$$

with index at most 2 is precisely the set of orientation-preserving elements in $E(h)$.

Let $E^-(h) = E(h) \setminus E^+(h)$. Let $E^\star(h)$ be the torsion group of $E^+(h)$. The following result actually holds for a contracting element h.

Lemma 2.8. *For a loxodromic element h, the following statements hold.*

(1) $[E(h) : \langle h\rangle] < \infty$.

(2) *We have*

$$E(h) = \{g \in G : \textit{there exists } n > 0 \textit{ such that } (gh^n g^{-1} = h^n) \textit{ or } (gh^n g^{-1} = h^{-n})\}.$$

(3) *$E^\star(h)$ is a finite normal subgroup of $E(g)$.*

(4) *$g^2 \in E^\star(h)$ for any $g \in E^-(h)$.*

Proof. The first two statements are from [28, Lemma 2.11] which holds for any contracting element h. The last two statements follow from [1, Lemma 4] where only assertion (1) is used in the proof. □

Lemma 2.9. *Let $h \in G$ be a loxodromic element admitting a (λ, c)-quasi-axis α. Then there exists some $D = D(\lambda, c, \delta) > 0$ such that*

$$d(go, o) \leq D \quad \text{for any } g \in E^\star(h) \text{ and } o \in \alpha.$$

Proof. By the Morse Lemma, there exists a constant $D > 3\delta$ depending only on λ, c, δ such that any two of $g\alpha, \alpha, g^{-1}\alpha$ have Hausdorff distance at most D. Since $g \in E^\star(h) \subset E^+(h)$ preserves the orientation of α, we can assume further that $d(go, [o, \alpha_+]_\alpha) \leq D$ and $d(g^{-1}o, [o, \alpha_-]_\alpha) \leq D$.

Let $x, y \in \alpha$ such that $d(go, x), d(g^{-1}o, y) \leq D$. Since x, y are on the opposite sides of o on α, we have $d(o, [x, y]) \leq D$ by the Morse Lemma. Thus, $\langle x, y \rangle_o \leq D$, and then $\langle go, g^{-1}o \rangle_o \leq 3D$. If we assume $d(o, go) > 4D$, then we have $d(o, go) \geq 2d(o, g^2 o) + 6\delta$. By Lemma 2.1, g is loxodromic. This is a contradiction, so $d(o, go) \leq 4D$. □

2.3 Relatively hyperbolic groups

The notion of a relative hyperbolicity has a number of equivalent formulations (see [3, 8, 9, 12, 21], etc.). See [18] for a survey of their equivalence. In this paper, we define a relatively hyperbolic group which admits a cusp-uniform action on a hyperbolic space.

Definition 2.10. Suppose G admits a proper and isometric action on a proper hyperbolic space (X, d) such that G does not fix a point in the Gromov boundary ∂X. Denote by $\mathcal{P}$ the set of maximal parabolic subgroups in G. Assume that there is a G-invariant system of disjoint (open) horoballs $\mathbb{U}$ centered at parabolic points of G such that the action of G on the complement called the *neutered space* $X(\mathbb{U}) := X \setminus \mathcal{U}$ is co-compact, where $\mathcal{U} := \bigcup_{U \in \mathbb{U}} U$. Then the pair $(G, \mathcal{P})$ is said to be *relatively hyperbolic*, and the action of G on X is said to be *cusp-uniform*.

We fix a G-invariant system $\mathbb{U}$ of horoballs and the neutered space $X(\mathbb{U})$ on which G acts co-compactly. The following result is proved by [1, Lemma 6] in hyperbolic groups. In the relative case, we follow their arguments closely.

Lemma 2.11. *Let h be a loxodromic element in G such that, for some point $o \in X$ and $\lambda, c > 0$, the path $\alpha = \bigcup_{n\in\mathbb{Z}}[h^n o, h^{n+1} o]$ is a (λ, c)-quasi-geodesic in X. Then, for any given $\theta > 0$, there exists $N = N(\lambda, c, \delta, \theta)$ independent of the point o such that, for any $f \notin E(h)$, we have*

$$\operatorname{diam}(\alpha \cap N_R(f\alpha)) \le N \cdot d(o, ho),$$

where $R := \theta \cdot d(o, ho)$.

Remark. Along with Lemma 3.1, this is the other result which crucially uses cusp-uniform actions.

Proof. First of all, since h is a loxodromic element and cannot fix any parabolic point, we obtain that α cannot be contained inside any horoball $U \in \mathbb{U}$. Thus, the $\langle h\rangle$-invariant set $\alpha \cap X(\mathbb{U})$ is a non-empty unbounded set. Namely, for any $x \in \alpha \cap X(\mathbb{U})$ and any $i \in \mathbb{Z}$, we have $h^i x \in \alpha$.

We argue by contradiction. Assume that $\operatorname{diam}(\alpha \cap N_R(f\alpha)) > N \cdot d(o, ho)$ for a constant N determined below. Let $z, w, z', w' \in \alpha$ such that

$$d(z, w) = \operatorname{diam}(\alpha \cap N_R(f\alpha)) \quad \text{and} \quad d(z, fz'), d(w, fw') \le R.$$

By hyperbolicity, $[z, w]_\alpha$ and $[z', w']_\alpha$ contain subpaths β_1, β_2 respectively (after truncating an R-long segment at both ends) such that $\beta_1, f\beta_2$ have Hausdorff distance at most $C = C(\lambda, c, \delta) > 0$ and, for $i = 1, 2$, we have

$$\operatorname{diam}(\beta_i) \ge \operatorname{diam}(\alpha \cap N_R(f\alpha)) - 2R \ge (N - \theta) d(o, ho).$$

Since α is a (λ, c)-quasi-geodesic, there exists a monotone increasing function $N' = N'(\lambda, c, N, \theta) > 0$ such that β_1 contains at least $(N' + 1)$ translates of $[o, ho]$. Moreover, we have $N' = N'(\lambda, c, N, \theta) \to \infty$ as $N \to \infty$. Thus, β_1 contains $(N' + 1)$ points $x, hx, \ldots, h^{N'} x \in X(\mathbb{U})$. Let $y \in \beta_2$ be a point so that $d(x, fy) \le C$.

Assume that C also satisfies the conclusion of Lemma 2.7 by taking the larger one. Using Lemma 2.5, for $1 \le i \le N'$, we have

$$|d(x, h^i x) - \tau(h^i)| \le C + 16\delta, \quad |d(fy, fh^i y) - \tau(h^i)| \le C + 16\delta,$$

so

$$|d(x, h^i x) - d(fy, fh^i y)| \le 2C + 32\delta.$$

Thus, $d(h^i x, fh^i y) \le 3C + 32\delta$ for each $1 \le i \le N'$.

Set $N(x, y) = \sharp\{g \in G : d(x, gy) \le 3C + 32\delta\} + 1$. Since G acts co-compactly on the C-neighborhood of $X(\mathbb{U})$, we see that $N(x, y)$ over $x, y \in N_C(X(\mathbb{U}))$ is uniformly bounded above by a constant $N_0 > 0$.

Choose $N > 0$ such that $N' = N'(\lambda, c, N, \theta) \geq N_0$, and consequently, we obtain $h^{-i} f h^i = h^{-j} f h^j$ for $1 \leq i \neq j \leq N'$. So $f \in E(h)$ contradicts the assumption. The result is thus proved. □

Finally, let us mention the following result of Osin which holds for loxodromic elements in any acylindrical action on hyperbolic spaces.

Lemma 2.12 ([22, Lemma 6.8]). *There exists a finite number $N_0 > 0$ such that $\sharp E^\star(g) \leq N_0$ for any loxodromic element $g \in G$.*

3 Short loxodromic elements

The goal of this section is to provide short loxodromic elements.

Let $S = S^{-1}$ be a symmetric generating set of a non-elementary group H. Recall that $S^{\leq n_0} := \{h \in H : d_S(1, h) \leq n_0\}$.

The following is a variant of [26, Lemma 5.3].

Lemma 3.1. *For any $M > 0$, there exists a positive integer $n_0 = n_0(M) > 0$ such that, for any finite symmetric generating set S of H, we have $\ell_X(S^{\leq n_0}) > M$.*

Proof. Let $\mathbb{U}$ be an M-separated G-invariant system of horoballs centered at the parabolic points. Recall that the action of G on $X(\mathbb{U})$ is proper and co-compact. Let $K \subset X(\mathbb{U})$ be a compact set such that $\bigcup_{g \in G} g(K) = X(\mathbb{U})$. Fix a point $p \in K$, and let $a = \operatorname{diam}(K)$ depending on M. The proper action implies the set

$$A = \{g \in G : d(g(p), p) \leq 2a + M\}$$

is a finite set. Since G is finitely generated, up to increasing the value of a, we can assume that A generates G.

Consider the finite set $\mathbb{H}$ of conjugates of H which is generated by some finite set $S' \subset A$. Since H is infinite, the proper action of H on X implies that, for every $H' \in \mathbb{H}$, there is some $g_{H'} \in H'$ with $d(g_{H'}(p), p) > M + 2a$. Since A generates G and $\mathbb{H}$ is finite, the integer

$$n_0 := \max\{d_S(1, g_{H'}) : S' \subset A,\ H' := \langle S' \rangle \in \mathbb{H}\}$$

is finite.

Now let S be a finite generating set of H. If $\ell_X(S) > M$, then we are done: $\ell_X(S^{n_0}) \geq \ell_X(S) > M$. If there is some $x \in X$ with $\ell_x(S) \leq M$, then $x \in X(\mathbb{U})$. Indeed, assume that $x \in U$ for some $U \in \mathbb{U}$. By definition of $\ell_x(S) \leq M$, we have $d(s(x), x) \leq M$ for all $s \in S$. The M-separation of $\mathbb{U}$ implies $s(U) = U$

for all $s \in S$, and so the center of U would be fixed by the non-elementary subgroup H: a contradiction. Hence, it follows that $x \in X(\mathbb{U})$.

Recalling that $\bigcup_{g \in G} g(K) = X(\mathbb{U})$, we choose $g \in G$ with $g(x) \in K$. We now show $S' := \{gsg^{-1} : s \in S\} \subset A$. Indeed, for each $s \in S$, we have

$$\begin{aligned} d(p, gsg^{-1}(p)) &\le d(p, g(x)) + d(g(x), gs(x)) + d(gs(x), gsg^{-1}(p)) \\ &\le d(p, g(x)) + d(x, s(x)) + d(g(x), p) \le 2a + M. \end{aligned}$$

Since $S' \subset A$ generates $H' := gHg^{-1} \in \mathbb{H}$, by the definition of n_0, there is some integer $1 \le k \le n_0$ such that $d_{S'}(1, g_{H'}) = k$. Thus,

$$g_{H'} = (gs_1g^{-1}) \cdots (gs_kg^{-1}) = g(s_1 \cdots s_k)g^{-1}$$

for $s_i \in S \cup S^{-1}$. Now, by the triangle inequality, we have

$$\begin{aligned} d(g^{-1}g_{H'}g(x), x) &= d(g_{H'}g(x), g(x)) \\ &\ge d(g_{H'}(p), p) - d(g_{H'}(p), g_{H'}g(x)) - d(g(x), p) \\ &= d(g_{H'}(p), p) - d(p, g(x)) - d(g(x), p) \\ &> M + 2a - a - a = M. \end{aligned}$$

Since $g^{-1}g_{H'}g = s_1 \cdots s_k \in S^{\le n_0}$, it follows that $\ell_x(S^{\le n_0}) > M$. □

The following result improves [19, Proposition 3.2].

Lemma 3.2. *Let X be a δ-hyperbolic geodesic metric space and H a group of isometries of X with a finite symmetric generating set S. Assume that $\ell_X(S) > 28\delta$. Then H contains a loxodromic element $b \in S^{\le 2}$ with the following property.*

Let $o \in X$ such that $|\ell_o(S) - \ell_X(S)| \le \delta$. There exist constants $\lambda_0, c_0, C_0 > 0$ depending only on δ such that

$$d(o, bo) \ge \ell_X(S) - C_0$$

and the path

$$\alpha := \bigcup_{n \in \mathbb{Z}} b^n[o, bo]$$

is a (λ_0, c_0)-quasi-axis for b.

Proof. Set $L_0 = 4\delta$ and then $\ell_o(S) > 7L_0$. Denote by S_0 the (non-empty) set of elements $s \in S$ so that

$$d(o, so) \ge \ell_o(S) - 2L_0 - \delta.$$

Let $t \in S$ such that $\ell_o(S) = d(o, to)$, and $m \in [o, to]$ such that $d(o, m) = L_0$.

The main observation is as follows.

Claim. *There exists an isometry $s \in S_0$ such that s is either loxodromic with*

$$\langle o, s^2 o\rangle_{so} \le L_0$$

or satisfies

$$\max\{\langle to, so\rangle_o, \langle t^{-1}o, s^{-1}o\rangle_o\} \le L_0.$$

Proof of the claim. Assume to the contrary that, for all $s \in S_0$, we have

$$\max\{\langle to, so\rangle_o, \langle t^{-1}o, s^{-1}o\rangle_o\} > L_0. \tag{3.1}$$

Moreover, we have that each $s \in S_0$ is either non-loxodromic or loxodromic with $\langle o, s^2 o\rangle_{so} > L_0$. If $s \in S_0$ is non-loxodromic, by Lemma 2.1, we have

$$\langle o, s^2 o\rangle_{so} \ge \frac{1}{2} d(o, so) - 3\delta \ge \frac{1}{2}(\ell_o(S) - 2L_0 - \delta) - 3\delta \ge L_0.$$

Hence, for each $s \in S_0$, we have $\langle o, s^2 o\rangle_{so} \ge L_0$. In particular, $\langle t^{-1}o, to\rangle_o \ge L_0$.

By (3.1), we have that $\langle t^\star o, s^\star o\rangle_o > L_0$ for $\star \in \{1, -1\}$. Let $m_1, m_2 \in [o, s^\star o]$ for $s \in S_0$ so that $d(o, m_1) = d(s^\star o, m_2) = L_0$. By hyperbolicity,

$$\langle s^\star o, to\rangle_o \ge \min\{\langle s^\star o, t^\star o\rangle_o, \langle t^{-1}o, to\rangle_o\} - \delta \ge L_0 - \delta$$

which by the δ-thin triangle property implies $d(m, m_1) \le 3\delta$. Using again the δ-thin triangle property with $\langle o, (s^\star)^2 o\rangle_{s^\star o} \ge L_0$, we obtain $d(m_2, s^\star m_1) \le \delta$.

We shall derive $\ell_m(S) < \ell_X(S)$, which is a contradiction. Indeed, for each $s \in S_0$,

$$\begin{aligned} d(m, s^\star m) &\le 2d(m, m_1) + d(m_1, s^\star m_1) \\ &\le 7\delta + d(m_1, m_2) \le 7\delta + d(o, s^\star o) - 2L_0 \\ &\le \ell_o(S) - 2L_0 + 7\delta \le \ell_o(S) - \delta. \end{aligned}$$

If $s \in S \setminus S_0$, then $d(o, so) \le \ell_o(S) - 2L_0 - \delta$, and thus

$$d(m, sm) \le 2d(o, m) + d(o, so) \le 2L_0 + d(o, so) < \ell_o(S) - \delta.$$

We obtain the contradiction $\ell_m(S) \le \ell_o(S) - \delta < \ell_X(S)$. The proof of the claim is now complete. □

By the above claim, there exists some $s \in S_0$ such that either

$$\langle so, s^{-1}o\rangle_o + \delta \le L_0 + \delta \le \frac{1}{4} d(o, so)$$

or

$$\max\{\langle to, so\rangle_o, \langle t^{-1}o, s^{-1}o\rangle_o\} + \delta \le L_0 + \delta \le \frac{1}{4}\min\{d(o, so), d(o, to)\}.$$

Accordingly, we apply Lemma 2.4 to $g = h = s$ or $g = t$, $h = s$. Then $b = s$ or $b = t^{-1}s$ is the desired loxodromic element with the (λ_0, c_0)-quasi-axis α. The proof is then completed by Lemma 2.4. □

4 Short loxodromic elements with large injectivity

Let $o \in X$ be a basepoint, and let $\theta \geq 10$, $\lambda, c > 0$ be fixed constants. Denote by $C = C(\lambda, c, \delta)$ the constant such that any quadrilateral with (λ, c)-quasi-geodesic sides is C-thin: any side is contained in the C-neighborhood of the other three sides.

In this section, we make the following assumptions on f and b:

(i) the element b is loxodromic so that $d(o, bo) > C$ and $\alpha := \bigcup_{n \in \mathbb{Z}} b^n[o, bo]$ is a (λ, c)-quasi-axis;

(ii) the element f lies outside $E(b)$ and satisfies $d(o, fo) \leq \theta d(o, bo)$;

(iii) the element b satisfies the conclusion of Lemma 2.11.

For large $n \geq 0$, we shall study when the elements of form $h := fb^n$ are loxodromic and their properties (Lemmas 4.2 and 4.4).

The following immediate estimate will be useful later on:

$$\begin{aligned} \text{for all } n \geq 1, \quad n \cdot d(o, bo) \geq d(o, b^n o) &\geq \lambda^{-1} n \cdot d(o, bo) - c \\ &\geq (\lambda\theta)^{-1} n \cdot d(o, fo) - c. \end{aligned} \tag{4.1}$$

We emphasize that all the statements in this section are proved without involving any group action.

4.1 Loxodromic elements raising to power

Consider the points $x = b^{m-n}o$, $y = b^{-m}o$ for $0 \leq 2m < n$ on α, and then we have $hx = fb^m o \in f\alpha$. Consider two quasi-geodesics α and $f\alpha$ connected by a geodesic $[o, fo]$. To get a quasi-axis of $h = fb^n$, we shall use the next lemma to truncate the subpath containing $[o, fo]$ from the point y to hx of the union $\alpha \cup f\alpha$.

Lemma 4.1. *There exist $m = m(\lambda, c, \theta, \delta)$ and $n_0 = n_0(\lambda, c, \theta, \delta) > 2m$ so that $h = fb^n$ for any $n > n_0(\lambda, c, \theta, \delta)$ has the following property:*

$$\max\{\langle x, hx\rangle_y, \langle y, hy\rangle_{hx}\} \leq C$$

and $d(y, hx) \geq \theta d(o, bo)$.

Proof. Consider the quadrilateral formed by the subpaths

$$[x, o]_\alpha, \quad [o, fo], \quad f[o, b^m o]_\alpha,$$

and the geodesic $[x, hx]$ as depicted in Figure 1.

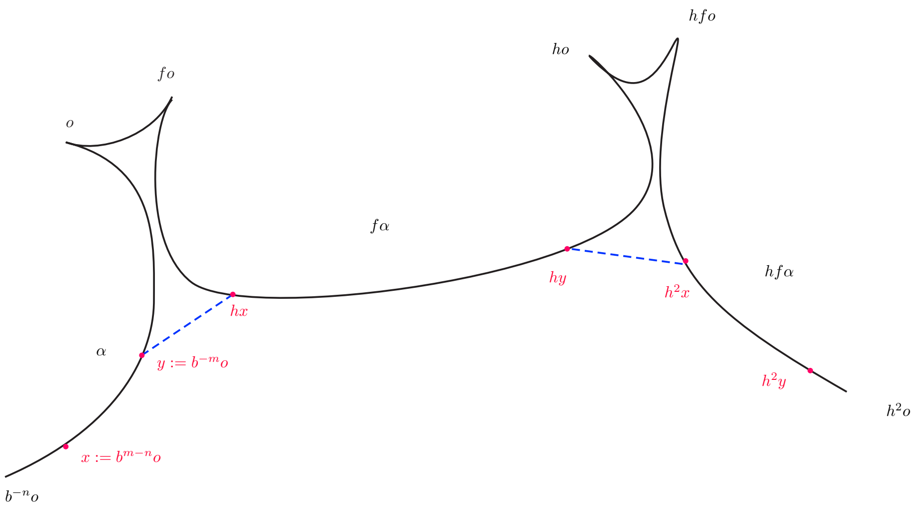

Figure 1. Truncate the quadrilaterals y, o, fo, hx and hy, ho, hfo, h^2x

Set $N = N(\lambda, c, \delta, \theta)$ given by Lemma 2.11. By (4.1), the distance $d(o, b^n o)$ grows linearly. Hence, we can choose the least integer $m = m(\lambda, c, \delta) > N$ and then the least $n_0 = n_0(\lambda, c, \theta, \delta)$ so that the following holds:

$$\min\{d(o, b^m o), d(o, b^{n-2m} o)\} \geq \max\{Nd(o, bo), 10C\} \tag{4.2}$$

for any $n \geq n_0$.

Since $R := \theta d(o, bo) \geq d(o, fo)$ is as assumed, we can obtain

$$d(y, f\alpha) > R.$$

Indeed, assume to the contrary that $d(y, f\alpha) \leq R$. As $d(o, f\alpha) \leq d(o, fo) \leq R$, Lemma 2.11 implies that the diameter of $\alpha \cap N_R(f\alpha)$ is at most $Nd(o, bo)$. However, $\alpha \cap N_R(f\alpha)$ contains two points $y = b^{-m}o, o$ with distance at least $Nd(o, bo)$ by (4.2). This is a contradiction. As $\theta \geq 1$ and $d(o, bo) > C$ is assumed, we have $d(y, f\alpha) > R \geq d(o, bo) > C$. The C-thin quadrilateral property then implies $d(y, [x, hx]) \leq C$, so we obtain $\langle x, hx\rangle_y \leq d(y, [x, hx]) \leq C$.

By symmetry, we can run the above argument for the quadrilateral with vertices y, o, fo, hy and obtain $\langle y, hy\rangle_{hx} \leq C$.

Note that $d(y, hx) \geq d(y, f\alpha) \geq R = \theta d(o, bo)$. The proof is complete. □

In what follows, let $m = m(\lambda, c, \theta, \delta)$ and $n_0 = n_0(\lambda, c, \theta, \delta)$ be given by Lemma 4.1.

Lemma 4.2. *There exist constants $\lambda_1, c_1 > 0$ depending only on $\delta, \lambda, c > 0$ with the following property. For any $n \geq n_0$, the element $h := fb^n$ is loxodromic with a (λ_1, c_1)-quasi-axis β defined as follows:*

$$\beta := \bigcup_{i \in \mathbb{Z}} h^i([x, y]_\alpha [y, hx]). \tag{4.3}$$

Proof. From the hypothesis at the beginning of this section, we have $d(o, bo) > C$ and $\theta \geq 10$. By Lemma 4.1, for $n \geq n_0$, we have

$$\max\{\langle x, hx\rangle_y, \langle y, hy\rangle_x\} \leq C$$

and $d(y, hx) \geq \theta d(o, bo) \geq 10C$. Note that $d(x, y) = d(o, b^{n-2m}o) \geq 10C$ by inequality (4.2). Therefore,

$$\langle x, hx\rangle_y + \langle y, hy\rangle_{hx} \leq 2C \leq \frac{1}{4}\min\{d(x, y), d(y, hx)\} - \delta,$$

so the assumption of Corollary 2.3 is verified for the sequence of points

$$\ldots, h^{-i}x, h^{-i}y, \ldots, x, y, hx, hy, \ldots, h^j x, h^j y, \ldots$$

Hence, there exist the desired constants $\lambda_1, c_1 > 0$ such that β is a (λ_1, c_1)-quasi-geodesic. This proves that h is loxodromic. □

4.2 Large injectivity

The crucial property in constructing free subgroups is the following property of a loxodromic isometry $h = fb^n$.

Definition 4.3. The element h has *injective radius* $L > 0$ if $E(h)$ contains a finite subgroup F with $[F : E^\star(h)] \leq 2$ so that $E(h) = \langle h\rangle F$ and, for any $g \in E(h) \setminus F$, we have $\ell_X(g) > L \cdot d(o, bo)$.

Let $N_0 > 0$ be given by Lemma 2.12 satisfying $\sharp E^\star(h) \leq N_0$, and let

$$D = D(\lambda, c, \delta) > 0$$

be given by Lemma 2.9.

Lemma 4.4. *For any $L > 0$, there exists $n_1 = n_1(\lambda, c, \theta, L, \delta) \geq n_0$ such that the loxodromic element $h = fb^n$ for any $n \geq n_1$ has injective radius L. Precisely, there exists a subgroup F of $E(h)$ such that the following hold.*

(1) *$\sharp F \leq 2N_0$ and $\ell_z(F) \leq 2D$ for any $z \in \beta$.*

(2) *For any $g \in E(h) \setminus F$, we have $\ell_X(g) > L \cdot d(o, bo)$.*

(3) *For any $g \in E(h)$, there exist $i \in \mathbb{Z}$ and $t \in F$ such that $g = h^i t$.*

Proof. We keep the same notation as in the proofs of Lemmas 4.1 and 4.2. For any $h = fb^n$ with $n \geq n_0$, the path β in (4.3) is a (λ_1, c_1)-quasi-geodesic.

Recall that $x = b^{m-n}o$, $y = b^{-m}o$. Denote by $\beta_0 = [x, y]_\alpha[y, hx]$ the fundamental domain for the action of $\langle h\rangle$ on β. Let $g \in E(h)$. By the Morse Lemma, the finite Hausdorff distance $d_H(\beta, g\beta) < \infty$ implies the existence of a constant $R = R(\lambda_1, c_1, \delta) > 0$ such that $d_H(\beta, g\beta) \leq R$. Since $d(o, bo) > C$ is assumed, the constant θ_1 defined as follows depends on λ, c:

$$\theta_1 := \frac{R}{d(o, bo)} \leq \frac{R}{C}.$$

Let $N = N(\lambda, c, \delta, \theta_1) > 0$ be given by Lemma 2.11. By the linear growth of $d(o, b^n o)$ in (4.1), there exists $n_1 > 0$ depending only on $\theta, \lambda, c, \delta$ such that the second inequality

$$\begin{aligned} d(x, y) - d(y, hx) &\geq d(o, b^{n-2m}o) - d(o, fo) - 2d(o, b^m o) \\ &> 2Nd(o, bo) + 2R \end{aligned} \tag{4.4}$$

holds for any given $h := fb^n$ with $n > n_1$.

Since $h^i y$ lies on the (λ_1, c_1)-quasi-geodesic β, we have, for any $i \neq 0$,

$$\begin{aligned} \lambda_1 d(y, h^i y) &\geq d(y, hy) - c_1 \\ &\geq d(o, b^n o) - 2md(o, bo) - d(o, fo) - c_1. \end{aligned}$$

Hence, by (4.1), for any $L > 0$, there exists a constant still denoted by

$$n_1 = n_1(\lambda, c, \theta, L, \delta)$$

such that

$$d(y, h^i y) > 2D + L \cdot d(o, bo) \quad \text{for all } i \neq 0 \in \mathbb{Z}, \tag{4.5}$$

where $h = fb^n$ for any $n \geq n_1$.

By construction, the path β in (4.3) is contained in the union $\bigcup_{i\in\mathbb{Z}} h^i\alpha$ and $\bigcup_{i\in\mathbb{Z}} h^i[y, hx]$. By the inequality (4.4) and $d_H(\beta, g\beta) \leq R$, the path $[x, y]_\alpha$ contains a subpath α_0 of diameter at least $Nd(o, bo)$ such that $g\alpha_0$ lies in the R-neighborhood $h^i\alpha$ for some $i \in \mathbb{Z}$. Recalling that α is a (λ, c)-quasi-axis for b, Lemma 2.11 thus implies $g^{-1}h^i \in E(b)$.

We claim that $t := g^{-1}h^i \in E(h) \cap E(b)$ is of finite order. If this is not the case, then $E(h) \cap E(b)$ is an infinite subgroup. Thus, $E(h) \cap E(b)$ act co-compactly on the quasi-axis of both h and b, so the (λ_1, c_1)-quasi-axis β of h is preserved by b up to finite Hausdorff distance. By definition of $E(b)$, we obtain $h = fb^n \in E(b)$ and then $f \in E(b)$. This contradicts the choice of f.

To define the finite subgroup F, we consider two cases. If $d(y, gy) > D$ for all $g \in E^-(h)$, set $F := E^\star(h)$ and then $\ell_X(F) \le D$ by Lemma 2.9. Otherwise, there exists $r \in E^-(h)$ so that $d(y, ry) \le D$. In this case, $F := \langle E^\star(h), r\rangle$ has order at most $2N_0$ and $\ell_y(F) \le \ell_y(E^\star(h)) + d(y, ry) \le 2D$.

We prove statements (2) and (3) first for the elements $g \in E^+(h)$. Then the finite order element $t \in E^\star(h)$ preserves the (λ_1, c_1)-quasi-axis β of h. By Lemma 2.9, we have $d(z, tz) < D$ for any $z \in \beta$. Thus, any $g \in E^+(h)$ can be written as $g = h^i t$, where $\ell_z(t) < D$. Then statement (3) follows.

If an element $g = h^i t \in E^+(h)$ does not belong to F, then we have $i \neq 0$ and $d(y, gy) > L \cdot d(o, bo)$ from (4.5), so statement (2) holds for $g \in E^+(h)$.

To complete the proof of (2) and (3), it remains to consider elements $g \in E^-(h)$. Since $E^+(h)$ is of index 2 in $E(h)$ and the element $r \in E^-(h)$ chosen as above reverses the orientation of β, we have $g \cdot r \in E^+(h)$. Applying the previous argument to gr, we can write $g = h^i(t \cdot r^{-1})$ for some $t \in E^\star(h)$ and $t \cdot r^{-1} \in F$. If $i \neq 0$, one deduces again from (4.5) that

$$d(y, gy) \ge d(y, h^i y) - d(y, ty) - d(y, ry) \ge L \cdot d(o, bo).$$

By Lemma 2.7, we have $\ell_X(g) \ge d(y, gy) - C_1$ for a uniform $C_1 > 0$. Since $d(o, bo) > C$, there is a constant $L_1 = L_1(C, C_1) > 0$ such that

$$\ell_X(g) \ge L \cdot d(o, bo) - C_1 > L_1 d(o, bo).$$

Therefore, all the statements are proved. □

5 Proof of Theorem 1.1

We keep the notation from Section 4. Recall that S is a finite symmetric generating set for a non-elementary subgroup H (see Section 2.2). By Lemma 3.1, there exists some $n_0 = n_0(28\delta)$ such that $\ell_X(S^{\le n_0}) > 28\delta$.

Choose a basepoint $o \in X$ such that $|\ell_o(S) - \ell_X(S)| \le \delta$. Let $\lambda_0 = \lambda_0(\delta)$, $c_0 = c_0(\delta)$, $C_0 = C_0(\delta) \ge \delta$ be given by Lemma 3.2. Thus, $S^{\le 2n_0}$ contains a loxodromic element $b \in H$ such that $d(o, bo) \ge \ell_X(S) - C_0$ and the path

$$\alpha := \bigcup_{n\in\mathbb{Z}} b^n[o, bo]$$

is a (λ_0, c_0)-quasi-axis for b.

Since $\sharp(S^{\le 2n_0}) \ge \sharp S$, it suffices to prove Theorem 1.1 assuming the generating set S with $\ell_X(S) > \max\{28\delta, 3C_0\}$. Note that

$$d(o, bo) \ge \ell_X(S) - C_0 \ge 2C_0. \tag{5.1}$$

Since H is not virtually cyclic, S contains an element f such that $f \notin E(b)$. Indeed, if not, any $f \in S$ would fix the set of fixed points of b, so it follows from $H = \langle S \rangle$ that the limit set of H consists of two points. By the subgroup classification in (the convergence action of) G, we obtain that H would be virtually cyclic. This is a contradiction.

Since $f \in S$, we have

$$d(o, fo) \leq \ell_o(S) \leq d(o, bo) + 2C_0 \leq 2d(o, bo)$$

by (5.1). Set $\theta = 2$ and $\lambda = \lambda_0$, $c = c_0$ all depending only on δ. We apply the results of Section 4 to the elements b and f.

Let $m_1 = m_1(\delta) \geq 2$, $n_1 = n_1(\delta)$ be as given by Lemma 4.2, and make the reference point at $y = b^{-m_1}o$ on the quasi-axis β in (4.3). Keep in mind that $d(o, bo) = d(y, by)$.

Set $L := 4(m_1 + 6) \geq 10$. For any $s \in S$, we have

$$d(o, so) \leq \ell_o(S) \leq \ell_X(S) + \delta \leq 2C_0 + d(o, bo)$$

by (5.1). Thus,

$$\begin{aligned} d(y, sy) = d(b^{-m_1}o, sb^{-m_1}o) &\leq (2m_1 + 1)d(o, bo) + 2C_0 \\ &< \frac{L}{2} d(y, by) \end{aligned} \tag{5.2}$$

For any $t := s^{-1}s'$ with $s \neq s' \in S$, this yields

$$d(y, ty) < L \cdot d(y, by) = L \cdot d(o, bo). \tag{5.3}$$

Choose $n_2 = n_2(L, \delta) > n_1$ and $F \leqslant E(h)$ according to Lemma 4.4. The following result holds for $h = fb^n$ with any integer $n \geq n_2$.

Lemma 5.1. *Choose a largest subset S_0 of S such that*

$$sF \neq s'F \quad \textit{for any } s \neq s' \in S_0.$$

Then, for any $s \neq s' \in S_0$, $s^{-1}s' \notin E(h)$.

Proof. By item (2) of Lemma 4.4, inequality (5.3) implies that $s^{-1}s'$ must be contained in F so $sF = s'F$. This contradicts the choice of S_0. □

Let $C_1 = C_1(\lambda_1, c_1, \delta) \geq \delta$ such that any quadrilateral with (λ_1, c_1)-quasi-geodesic sides is C_1-thin. Recall $t = s^{-1}s'$ for given $s \neq s' \in S_0$. To apply the results of Section 4 (in particular, Lemma 4.1) to the elements $t \cdot h^n$ for $n \gg 0$, we have to verify conditions (i), (ii), (iii) on t and h with the basepoint y.

Namely, consider the basepoint $y \in X$ and $\theta = L$, $\lambda = \lambda_1$, $c = c_1$. The element $h = fb^n$ with any integer $n \ge n_2$ admits the (λ_1, c_1)-quasi-axis β of the form (4.3). This verifies condition (i).

As mentioned earlier, $2d(o, bo) \ge d(o, fo)$, and the constants C_0, m_1 depend only on δ. Note that $d(y, hy) = d(y, fb^n y) = d(b^{-m_1} o, fb^{n-m_1} o)$. So we can choose the least integer $n_3 \ge n_2$ depending only on δ such that the last inequality in

$$\begin{aligned} d(y, h^i y) &\ge \lambda_1^{-1}(d(y, hy) - c_1) \\ &\ge \lambda_1^{-1}(d(o, b^n o) - 2d(o, b^{m_1} o) - d(o, fo) - c_1) \\ &> \max\{LC_1, L^2 d(o, bo)\} \end{aligned} \tag{5.4}$$

holds for any $n \ge n_3$ and $i \ne 0 \in \mathbb{Z}$. Hence, $d(y, hy) > C_1$, $d(y, hy) > Ld(y, ty)$ from (5.3) fulfill the requested conditions (ii) and (iii) on t, h.

Construct the free bases. Let us now fix $h = fb^{n_3}$ throughout the proof. By the above discussion on t and h, we again apply the results of Section 4: let $m_2 = m(\lambda_1, c_1, L, \delta)$, $k = n(\lambda_1, c_1, L, \delta)$ be given by Lemma 4.1.

We define the free base as follows:

$$T = \{sh^k s^{-1} : s \in S_0\}.$$

If we set $\kappa := 2 + k(n_3 + 1)$, then $d_S(1, sh^k s^{-1}) \le \kappa$ and $T \subset S^{\le \kappa}$. Recall that $\sharp F \le 2N_0$ from Lemma 4.4, so $\sharp T = \sharp S_0 \ge \sharp S / N_0$.

The goal is the following.

Lemma 5.2. *The set T generates a free subgroup of rank $\sharp T$ in H.*

Remark. If the group H is torsion-free, then F is trivial and $S_0 = S$ can be chosen in Lemma 5.1. In this case, we have $\sharp T = \sharp S$.

Proof. Let W be a non-empty reduced word over $T \cup T^{-1}$ written as follows:

$$\begin{aligned} W &= (s_1 \cdot h^{i_1 k} \cdot s_1^{-1})(s_2 \cdot h^{i_2 k} \cdot s_2^{-1}) \cdots (s_l \cdot h^{i_l k} \cdot s_l^{-1}) \\ &= s_1 \cdot (h^{i_1 k} \cdot t_1 \cdot h^{i_2 k} \cdot t_2 \cdots t_{l-1} h^{i_l k}) \cdot s_l^{-1}, \end{aligned}$$

where $l > 1$ and $t_i := s_i^{-1} s_{i+1}$ for $1 \le i \le l - 1$.

First of all, let β_j be the subpath of β starting from y to $h^{i_j k} y$ consisting of $i_j \cdot k$ copies of $[y, hy]_\beta$. Let $p_j = [y, t_j y]$ be a geodesic labeled by t_j.

We choose z_j, w_j on β_j so that the initial subpath of β_j until z_j contains exactly m_2 copies of $[y, hy]_\beta$, and the terminal path starting at w_j contains exactly m_2 copies of $[y, hy]_\beta$. To be precise, set $z_j = h^{m_2 k} y$, $w_j = h^{i_j - m_2} y$.

Furthermore, if $j = 1$, we let z_1 be the initial point of β_1; if $j = l$, let w_l be the initial point of β_l.

We now properly translate β_j and p_j for $1 \le j \le l$ so that β_1 originates at y, and then the terminal points of β_j followed by the initial points of p_j in a way produces the following concatenated path:

$$\gamma = \beta_1 \cdot p_1 \cdot \beta_2 \cdot p_2 \cdot \beta_3 \cdots p_{l-1} \cdot \beta_l.$$

(We refer the reader to Figure 1 for a similar illustration of cutting out quadrilaterals, where x, y, hx, hy, h^2x, h^2y should be marked as $z_1, w_1, z_2, w_2, z_3, w_3$, etc.)

By abuse of language, after translation, the corresponding points of z_j, w_j on β_j are still denoted by z_j, w_j, so we have a sequence of points $z_1, w_1, z_2, w_2, \ldots, z_l, w_l$ on γ. By the choice of z_1, w_l, the path γ starts at z_1 and ends at w_l, labeled by the word $s_1^{-1} W s_l$.

The key construction is then to cut quadrilaterals off γ along $[w_j, z_{j+1}]$ and verify that $\{z_1, w_1, z_2, w_2, \ldots, z_l, w_l\}$ is a quasi-geodesic.

To truncate the quadrilaterals, we apply Lemma 4.1 to $\beta_j, c_j, \beta_{j+1}, c_{j+1}$ in order for $1 \le j \le l$. For concreteness, set $j = 1$. Lemma 4.1 gives

$$\langle z_1, z_2 \rangle_{w_2}, \langle w_1, w_2 \rangle_{z_2} \le C_1 \quad \text{and} \quad d(w_1, z_2) \ge L d(y, hy) \ge 10C_1.$$

Since $C_1 \ge \delta$ is assumed, we then derive

$$\langle z_1, z_2 \rangle_{w_2}, \langle w_1, w_2 \rangle_{z_2} \le \frac{1}{4} d(w_1, z_2) - \delta. \tag{5.5}$$

By inequality (5.4), we have $d(z_2, w_2) = d(y, h^{i_2 k - 2m_2} y) \ge LC_1 \ge 10C_1$, so

$$\langle w_1, w_2 \rangle_{z_2}, \langle z_2, z_3 \rangle_{w_2} \le \frac{1}{4} d(z_2, w_2) - \delta \tag{5.6}$$

In conclusion, inequalities (5.5) and (5.6) verifying the assumption of Corollary 2.3 hold for every four consecutive points in $z_1, w_1, z_2, w_2, \ldots, z_l, w_l$. Thus,

$$d(z_1, w_l) \ge \frac{1}{2} \sum_{1 \le j \le l} d(z_j, w_j) \ge L d(y, by)$$

By (5.2), we have $d(y, s_1 y) + d(y, s_l y) < L d(y, by)$. Thus,

$$d(o, Wo) = d(z_1, w_1) - d(y, s_1 y) - d(y, s_l y) > 0.$$

Hence, any non-empty reduced word W is mapped to a nontrivial isometry, so T generates a free subgroup of rank $\sharp T$. □

We now finish the proof of Theorem 1.1. Summarizing the above discussion, for each generating set S of H, we constructed a finite set $T \subset S^{\leq \kappa}$ satisfying

$$\sharp T \geq \frac{1}{2N_0} \sharp S$$

so that $\langle T \rangle$ is a free group of rank $\sharp T$. Thus,

$$\sharp(S^{\leq n\kappa}) \geq (2\sharp T - 1)^n \geq \left(\frac{\sharp S - N_0}{N_0} \right)^n, \tag{5.7}$$

and there exists $c_0 > 0$ such that $\omega(H, S) \geq c_0$ for any finite symmetric set S.

Choose the least integer $M = M(N_0) > 0$ such that $\sharp S / N_0 \geq 1 + \sqrt{\sharp S}$ holds for any S with $\log \sharp S > M$. In this case, we thus obtain

$$\omega(H, S) \geq \frac{1}{2\kappa} \log(\sharp S).$$

Otherwise, $\log \sharp S \leq M$, and we have

$$\omega(H, S) \geq c_0 \geq \frac{c_0}{M} \log(\sharp S).$$

The proof of Theorem 1.1 is finished.

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