J. Group Theory **26** (2023), 1067–1093
DOI 10.1515/jgth-2021-0095

The Higman operations and embeddings of recursive groups

Vahagn H. Mikaelian*

Communicated by Dessislava Kochloukova

Abstract. In the context of Higman embeddings of recursive groups into finitely presented groups, we suggest an approach, termed the H-machine, which for certain wide classes of groups allows constructive Higman embeddings of recursive groups into finitely presented groups. The approach is based on Higman operations, and it explicitly constructs some specific recursively enumerable sets of integer sequences arising during the embeddings. Specific auxiliary operations are introduced to make the work with Higman operations a simpler and more intuitive procedure. Also, an automated mechanism of constructive embeddings of countable groups into 2-generator groups preserving certain "patterns" is mentioned.

1 Introduction

1.1 Higman's embedding theorem

In 1961, Higman proved that *a finitely generated group can be embedded in a finitely presented group if and only if it is recursively presented* [7] (see definitions and notation in Section 2.1). In his work, Higman extensively uses specific recursively enumerable sets of integer sequences which in some sense "code" defining relations of groups. Our objective is to suggest an algorithm for construction of such integer sequences for certain types of groups. This allows us to list wide classes of groups for which Higman's famous embedding construction can be constructive and effective.

The approach of [7] will be very briefly outlined in Section 3 below. For now, let us just mention the main steps of Higman's construction to distinguish those parts to which our new algorithm concerns. A finitely generated group

$$G = \langle A \mid R \rangle = \langle a_1, a_2, \dots \mid r_1, r_2, \dots \rangle$$

with recursively enumerable relations $r_1, r_2, \dots$ can be constructively embedded into a 2-generator group $T = \langle b, c \mid r'_1, r'_2, \dots \rangle$ where the relations $r'_1 = r'_1(b, c)$,

The current work is supported by the 21T-1A213 grant of SCS MES RA.

$r'_2 = r'_2(b, c), \dots$ are certain words on letters b, c, and they also are recursively enumerable. Then, for each r'_s, $s = 1, 2, \dots,$ a unique sequence f_s of integers is compiled (see details in Section 3.2) so that the set $\{r'_1, r'_2, \dots\}$ of relations is "coded" by means of a set $\mathcal{B} = \{f_1, f_2, \dots\}$ of such sequences. Since the transaction from relations set R to sequences set $\mathcal{B}$ is done via just a few constructive steps, the set $\mathcal{B}$ also is recursively enumerable.

The tedious part of [7] is to show that $\mathcal{B}$ is recursively enumerable if and only if $\mathcal{B}$ can be constructed by some chain of special operators (H). And parallel to application of those operations, a respective *benign subgroup* is being constructed in the free group $F_3 = \langle a, b, c\rangle$ of rank three (see Section 2.5). As this process ends up on construction of $\mathcal{B}$, the respective benign subgroup $A_{\mathcal{B}}$ is obtained inside F_3.

In the final short step, the benign subgroup $A_{\mathcal{B}}$ is used to get another benign subgroup in the free group $F_2 = \langle b, c\rangle$ of rank two. Then that new subgroup is used to embed the group T (and hence also G) into a finitely presented group via the "The Higman Rope Trick" (see [12, p. 219] and [22]).

1.2 Our algorithm for construction of $\mathcal{B}$

In [7], Higman just relies on *theoretical possibility* for construction of $\mathcal{B}$ via operations (H), without any examples of such construction for particular groups. This is understandable as the objective of the fundamental article [7] is much deeper, and for its purposes, it is sufficient to know that such a construction of $\mathcal{B}$ is possible, provided that the set of Gödel numbers specifically constructed for the set R is equal to the range of a certain partial recursive function described by Kleene's characterization (see references in Section 2.1 below).

However, it is rather strange that, after Higman's result, there was no attempt to explicitly find constructions of $\mathcal{B}$ by operations (H) for particular groups (at least, we were unable to find them in the literature). Investigating the topic, we noticed that such construction may be a *doable* task for many classes of groups, such as the free abelian, metabelian, soluble, nilpotent groups, the additive group of rational numbers $\mathbb{Q}$, the quasicyclic group $\mathbb{C}_{p^\infty}$, divisible abelian groups, etc. (see examples in Section 3.3).

We suggest an *H-machine* algorithm with some generic tools that allow to explicitly construct $\mathcal{B}$ by operations (H) without any usage of Kleene's characterization, at all. In Example 4.11, we show how simple it is to apply the algorithm (see Remark 4.12).

The advantage of this approach is that construction of the benign subgroup $A_{\mathcal{B}}$ and, thus, of the explicit embedding of the given G into a finitely presented group becomes a manageable procedure.

To shorten the routine of work with basic Higman operations, we introduce a few *auxiliary operations* which make the proofs not only shorter but also, we hope, more intuitive to understand (see notation in Section 2.4).

Another embedding aspect we touch upon is the manner by which the initial group G is constructively embedded into a 2-generator group T, and how the relations of T can be obtained from those of G. In the literature, there is no shortage in constructive embeddings of this type (in fact, the original method of [8] already allows that). However, for our purposes, we need a method which not only makes deduction of the relations of T from the relations of G a trivial automated task, but also *preserves certain "patterns"* in the relations (for illustration of the "patterns", see [16] and also examples in Section 3.3 below).

2 Definitions, references, preliminary constructions

2.1 Basic notation and references

For general group theory information, we refer to textbooks [10, 19, 21]. For background on free constructions, such as free products, free products with amalgamated subgroups, HNN-extension, we refer to [4, 12, 21]. See also the recent note [15], where we apply some methods related to free constructions. We use them here without restating the notation again. Information on varieties of groups can be found in Hanna Neumann's monograph [18].

We will study recursive groups in the language of Higman operations (H). Recall that a recursive (or recursively presented) group G is that possessing a presentation

$$G = \langle X \mid R \rangle = \langle x_1, \dots, x_n, \dots \mid r_1, r_2, \dots \rangle$$

with finite or countable set of generators X and with a *recursively enumerable* set of defining relations R. That is to say, to each relation $r_i \in R$, one can assign a *Gödel number* (see [7, Section 2] or [12, p. 218]) to interpret R via a set of respective Gödel numbers, and then that set turns out to be the range of a partial recursive function. By Kleene's characterization, a *partial recursive function* is that obtained from the zero function, the successor function and the identity function using the operations of composition, primitive recursion and minimization (see [5, 6, 20] for details).

Although Higman's theorem is for *finitely* generated recursive groups, its analog holds for embeddings of *countably* generated recursive groups into finitely presented groups. For, a countably generated recursive group can first be effectively embedded into a finitely generated recursive group (see the remark proceeding the corollary in [7, p. 456]). Thus, in embedding procedures, we will not take care of the number of generators, as long as the relations are recursively enumerated.

2.2 Sets of integer-valued functions and sequences of integers

Denote by $\mathcal{E}$ the set of all functions $f : \mathbb{Z} \to \mathbb{Z}$ with finite support

$$\sup(f) = \{i \in \mathbb{Z} \mid f(i) \neq 0\}.$$

When m is any positive integer such that $\sup(f) \subseteq \{0, 1, \ldots, m-1\}$, then we can interpret f as a *sequence* $f = (n_0, \ldots, n_{m-1})$ of length m, assuming $f(i) = n_i$ for each index $i = 0, \ldots, m-1$. The value $f(i)$ is called the i-th coordinate of f, or the coordinate of f at the index i. Say, $f = (0, 0, 7, -8, 5, 5, 5, 5)$ means that $f(2) = 7$, $f(3) = -8$, $f(i) = 5$ for $i = 4, \ldots, 7$, and $f(i) = 0$ for any $i \leq 1$ or $i \geq 8$. Here the initial 0 is the 0th coordinate, and 7 is the 2nd coordinate.

Depending on the situation, we may interpret the same function by sequences of different length by adding some zeros to it. Say, the above function f can be interpreted as the sequence $f = (0, 0, 7, -8, 5, 5, 5, 5, 0, 0, 0)$ with three "new" coordinates $f(8) = f(9) = f(10) = 0$. And the constant zero function $f(i) = 0$ may equally well be interpreted as $f = (0)$ or, say, as $f = (0, 0, 0, 0)$.

2.3 The Higman operations

Start by two specific subsets of $\mathcal{E}$,

$$\mathcal{Z} = \{(0)\}, \quad \mathcal{S} = \{(n, n+1) \mid n \in \mathbb{Z}\}.$$

We are going to extensively use the following operations from [7]:

$$\iota, \upsilon; \quad \rho, \sigma, \tau, \theta, \zeta, \pi, \omega_m \quad (\text{for each } m = 1, 2, \ldots), \tag{H}$$

which we call *Higman operations* on subsets of $\mathcal{E}$. The first two operations are binary functions, and for any subsets $\mathcal{A}, \mathcal{B}$ of $\mathcal{E}$, they are defined as just the intersection $\iota(\mathcal{A}, \mathcal{B}) = \mathcal{A} \cap \mathcal{B}$ and the union $\upsilon(\mathcal{A}, \mathcal{B}) = \mathcal{A} \cup \mathcal{B}$ of those sets. This notation differs a little from the original notation $\iota\mathcal{A}\mathcal{B}$ and $\upsilon\mathcal{A}\mathcal{B}$ of [7] which in our case would cause confusion when used in long formulas together with other operations.

The other Higman operations are unary functions defined on any subset $\mathcal{A}$ of $\mathcal{E}$ as follows.

- $\rho(\mathcal{A})$ consists of all $f \in \mathcal{E}$ for which there is a $g \in \mathcal{A}$ such that $f(i) = g(-i)$.
- $\sigma(\mathcal{A})$ consists of all $f \in \mathcal{E}$ for which there is a $g \in \mathcal{A}$ such that $f(i) = g(i-1)$.
- $\tau(\mathcal{A})$ consists of all $f \in \mathcal{E}$ for which there is a $g \in \mathcal{A}$ such that $f(0) = g(1)$, $f(1) = g(0)$ and $f(i) = g(i)$ for $i \neq 0, 1$.
- $\theta(\mathcal{A})$ consists of all $f \in \mathcal{E}$ for which there is a $g \in \mathcal{A}$ such that $f(i) = g(2i)$.

- $\zeta(\mathcal{A})$ consists of all $f \in \mathcal{E}$ for which there is a $g \in \mathcal{A}$ such that $f(i) = g(i)$ for $i \neq 0$.
- $\pi(\mathcal{A})$ consists of all $f \in \mathcal{E}$ for which there is a $g \in \mathcal{A}$ such that $f(i) = g(i)$ for $i \leq 0$.
- ζ and π are called *liberations* of $\mathcal{A}$ in the sense that ζ *liberates* the sequences on 0: for every $g \in \mathcal{A}$, it adds to our set $\mathcal{A}$ all the functions f which accept *any* value at 0, but which coincide with g elsewhere. And π *liberates* the sequences on positive integers: for every $g \in \mathcal{A}$, the operation π adds to $\mathcal{A}$ all the functions which accept *any* values at positive indices, but which coincide with g on zero and on all negative indices.

For a fixed $m = 1, 2, \ldots$, the set $\omega_m(\mathcal{A})$ consists of all $f \in \mathcal{E}$ for which, for every $i \in \mathbb{Z}$, there is a $g = t_i = (f(mi), f(mi+1), \ldots, f(mi+m-1)) \in \mathcal{A}$. This operation is called *sequence building* as it constructs the functions f by means of some subsequences g of length m chosen from $\mathcal{A}$. Since $\sup(f)$ is finite, either $\mathcal{A}$ contains the zero function, or $\omega_m(\mathcal{A}) = \emptyset$.

We may agree to apply the unary Higman operations to individual functions also: a set $\mathcal{A} = \{f\}$ may consist of a single function f only, so notations like $\rho f, \sigma f, \tau f$, etc., should cause no confusion.

To get familiar with these operations, the reader may check examples and basic lemmas in [7, Section 2] or in [15, Section 2.2].

Following Higman [7], we denote by $\mathscr{S}$ the set of all subsets of $\mathcal{E}$ which can be obtained from $\mathcal{Z}$ and $\mathcal{S}$ by any series of operations (H). The elements in $\mathscr{S}$ play a key role in the study of recursively presented groups. One of our main tasks below is going to be the discovery of many "natural" generic types of subsets of $\mathcal{E}$ inside $\mathscr{S}$.

2.4 Extra auxiliary operations

Our proofs will be much simplified by some auxiliary operations, each of which is a combination of a few Higman operations on subsets $\mathcal{A}$ of $\mathcal{E}$.

For a positive integer i, naturally denote by $\sigma^i \mathcal{A} = \sigma \cdots \sigma \mathcal{A}$ the result of application of σ for i times. Set the inverse $\sigma^{-1} = \rho\sigma\rho$ as follows: $f \in \sigma^{-1}(\mathcal{A})$ when there is a $g \in \mathcal{A}$ such that $f(i) = g(i+1)$. This allows to define the negative powers of σ. Setting $\sigma^0 \mathcal{A} = \mathcal{A}$, we have the powers σ^i for *any* integer $i \in \mathbb{Z}$. Clearly, σ^i just "shifts" a sequence $g \in \mathcal{A}$ by $|i|$ steps to the right or to the left depending on the sign of i.

It is easy to verify that $\sigma^i \zeta \sigma^{-i} \mathcal{A}$ consists of all functions $f \in \mathcal{A}$ in which the i-th coordinate is liberated. For briefness, denote $\zeta_i = \sigma^i \zeta \sigma^{-i}$. Moreover, for a finite subset $S = \{i_1, \ldots, i_m\} \subseteq \mathbb{Z}$, denote the result of application of $\zeta_{i_1} \cdots \zeta_{i_m}$ by

$\zeta_{i_1,\dots,i_m}$ or by ζ_S. That is, $\zeta_S \mathcal{A}$ is the set of all those functions $f \in \mathcal{E}$ for which there is some $g \in \mathcal{A}$ such that $f(i) = g(i)$ for each $i \notin S$.

Denote by $\pi' \mathcal{A} = \rho\pi\rho\mathcal{A}$ the liberation of $\mathcal{A}$ on all *negative* coordinates, i.e., the set of all functions $f \in \mathcal{E}$ for which there is some $g \in \mathcal{A}$ such that $f(i) = g(i)$ for each $i \geq 0$. Denote by $\pi_i \mathcal{A} = \sigma^i \pi \sigma^{-i} \mathcal{A}$ the liberation of $\mathcal{A}$ on all coordinates *after* the i-th coordinate and, similarly, denote by $\pi'_i \mathcal{A} = \sigma^{-i} \pi' \sigma^i \mathcal{A}$ the liberation of $\mathcal{A}$ on all coordinates *before* the i-th coordinate. In this notation, the original Higman operation π is nothing but π_0.

For any integers $k < l$, set $s = l - k - 1$. It is not hard to verify that the set $\tau_{k,l} \mathcal{A} = \sigma^k (\tau\sigma)^s \tau (\sigma^{-1}\tau)^{-s} \sigma^{-k} \mathcal{A}$ consists of all modified functions of $\mathcal{A}$ with k-th and l-th coordinates "swapped". More precisely, $\tau_{k,l} \mathcal{A}$ is the set of all functions $f \in \mathcal{E}$ for which there is some $g \in \mathcal{A}$ such that $f(k) = g(l)$, $f(l) = g(k)$ and $f(i) = g(i)$ for each $i \neq k, l$. In this notation, the Higman operation τ is nothing but $\tau_{0,1}$.

Furthermore, since any permutation α of a finite set S has a transposition decomposition $\alpha = (k_1 l_1) \cdots (k_m l_m)$, we may introduce the set

$$\alpha \mathcal{A} = \tau_{k_1,l_1} \cdots \tau_{k_m,l_m} \mathcal{A}$$

which can be obtained from $\mathcal{A}$ by respective permutation of coordinates for all $g \in \mathcal{A}$. Clearly, $\alpha\mathcal{A}$ is the set of all functions f for which there is a $g \in \mathcal{A}$ such that $f(i) = g(\alpha^{-1}(i))$ for any $i \in \mathbb{Z}$.

For any finite set of indices $S = \{i_1, \dots, i_m\}$ and for any $\mathcal{A} \subseteq \mathcal{E}$, define the *extract* $\epsilon_S \mathcal{A} = \epsilon_{i_1,\dots,i_m} \mathcal{A}$ to be the m-tuples set $\{(g(i_1), \dots, g(i_m)) \mid g \in \mathcal{A}\}$. This operation can be constructed using (H) as follows. Assume $i_1 < \cdots < i_m$ for clarity, and denote $S' = \{i_1, i_1 + 1, \dots, i_m - 1, i_m\} \backslash S$ (the set of all integers from i_1 to i_m except those in S). The set $\mathcal{A}_1 = \zeta_{S'} \pi'_{i_1} \pi_{i_m} \mathcal{A}$ consists of functions from $\mathcal{A}$ with *all* coordinates outside S liberated. And the set $\mathcal{A}_2 = \zeta_S \mathcal{Z}$ consists of all functions which accept any integer values on $S = \{i_1, \dots, i_m\}$ and which are zero elsewhere. The intersection $\mathcal{A}_3 = \iota(\mathcal{A}_1, \mathcal{A}_2)$ consists of those functions f for which there is some $g \in \mathcal{A}$ such that $f(i) = g(i)$ when $i \in S$, and $f(i) = 0$ elsewhere. To get $\epsilon_S \mathcal{A}$ from $\mathcal{A}_3$, it remains to apply the appropriate permutation α that re-distributes the coordinates of $f \in \mathcal{A}_3$ at the indices $i_1, \dots, i_m$ on the $0, 1, \dots, m-1$ (here α is a permutation of the union $\{i_1, \dots, i_m; 0, 1, \dots, m-1\}$).

For any subset $\mathcal{A} \subseteq \mathcal{E}$, denote $\sup(\mathcal{A}) = \bigcup\{\sup(f) \mid f \in \mathcal{A}\}$. The point-wise sum $f + g$ of any functions $f, g \in \mathcal{E}$ is defined as

$$(f + g)(n) = f(n) + g(n), \quad n \in \mathbb{Z}.$$

For any subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{E}$, their sum $\mathcal{A} + \mathcal{B}$ is the set $\{f + g \mid f \in \mathcal{A}, g \in \mathcal{B}\}$. The sum of three or more sets is defined in the same manner. We are going to use this operation for cases when $\sup(\mathcal{A})$ and $\sup(\mathcal{B})$ are disjoint finite sets.

The intersection of any three or more subsets, such as $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{E}$, can be expressed by Higman operations as $\iota(\iota(\mathcal{A}, \mathcal{B}), \mathcal{C})$. To have shorter notation, record this as $\iota_3(\mathcal{A}, \mathcal{B}, \mathcal{C})$. Similarly, define intersections ι_n and unions υ_n. In [7], Higman denotes the same by $\iota^2 \mathcal{A}\mathcal{B}\mathcal{C}$, but in our case, this would create confusion in long formulas with many operations.

Investigating the subsets of $\mathcal{E}$ in $\mathscr{S}$, in addition to standard operations (H), we may often use the introduced auxiliary operations. This will shorten the proofs without changing the actual set $\mathscr{S}$, for we above have representation of each auxiliary operation via (H).

2.5 The benign subgroups

The concept of *benign subgroups* is the key group-theoretical notion used in [7] to connect the sets in $\mathscr{S}$ with subgroups in free groups, needed in construction of embeddings into finitely presented groups. A subgroup H in a finitely generated group G is called a benign subgroup in G if G can be embedded in a finitely presented group K with a finitely generated subgroup $L \leq K$ such that $G \cap L = H$. For basic properties and examples of benign subgroups, we refer to [7, Section 3] or to [15, Sections 3.1 and 3.2].

Below, we reserve the letters K, L for these specific groups only. In particular, if we have K, L for an "old" group and then we construct a "new" group with a respective finitely presented overgroup and its finitely generated subgroup, we may again denote them by the same letters K and L. Also, if we have two benign subgroups, say, H_1 and H_2, we will denote the respective groups by K_1, K_2 and L_1, L_2. The context will tell us *for which* benign subgroups they are being considered, and no misunderstanding will occur.

3 The main steps of embeddings of recursive groups

3.1 Embedding with "universal" words in a free group of rank 2

Any countable group G is embeddable into a 2-generator group T (see [8]). Higman's embedding construction [7] starts by some *effective* embedding of G into an appropriate T. In the recent note [16], we suggested a method of effective embedding of any countable group G into a 2-generator group T such that the defining relations of T are straightforward to deduce from relations of G. In fact, the very first embedding construction [8] (based on free constructions) and some other embedding constructions (based on wreath products, group extensions, etc.) already allow finding the relations of T. However, we need a method that not only makes deduction of the relations of T from those of G an *automated task*, but also *preserves certain pattern* in them, as we will see a little later; see Remark 3.8.

Let a countable group G be given as

$$G = F/\bar{R} = \langle A \mid R\rangle = \langle a_1, a_2, \dots \mid r_1, r_2, \dots\rangle,$$

where F is a free group on a countable alphabet A and where $\bar{R} = \langle r_1, r_2, \dots\rangle^F$ is the normal closure of the set of all defining relations

$$r_s(a_{i_{s,1}}, \dots, a_{i_{s,k_s}}), \quad s = 1, 2, \dots,$$

in F.

In the free group $F_2 = \langle b, c\rangle$ of rank 2, choose the words

$$a_i(b, c) = c^{(bc^i)^2 b^{-1}} (c^{-1})^b = bc^{-i} b^{-1} c^{-i} b^{-1} cbc^i bc^i b^{-2} c^{-1} b, \tag{3.1}$$

$i = 1, 2, \dots$. The map $\gamma\colon a_i \to a_i(b, c)$ defines a correspondence

$$r_s(a_{i_{s,1}}, \dots, a_{i_{s,k_s}}) \to r'_s(b, c) = r_s\big(a_{i_{s,1}}(b, c), \dots, a_{i_{s,k_s}}(b, c)\big)$$

obtained by replacing each $a_{i_{s,j}}$ in r_s by the word $a_{i_{s,j}}(b, c)$, $j = 1, \dots, k_s$. In fact, γ defines an embedding of G into the 2-generator group

$$T = \langle b, c \mid r'_1(b, c), r'_2(b, c), \dots\rangle$$

given by the relations $r'_s(b, c)$, $s = 1, 2, \dots$, on letters b, c (see [16, Theorem 1.1]). If R is recursively enumerable, then the set R' of all above relations $r'_s(b, c)$ also is recursively enumerable. That is, T is a recursive group, in case G is.

And when G is a torsion-free group, then (3.1) can be replaced by shorter words

$$\bar{a}_i(b, c) = c^{(bc^i)^2 b^{-1}} = bc^{-i} b^{-1} c^{-i} b^{-1} cbc^i bc^i b^{-1}. \tag{3.2}$$

Inserting these $\bar{a}_i(b, c)$ in r_s, we get shorter words $r''_s(b, c)$. Then $\gamma\colon a_i \to \bar{a}_i(b, c)$ defines an embedding of G into the 2-generator group

$$T = \langle b, c \mid r''_1(b, c), r''_2(b, c), \dots\rangle$$

(see [16, Theorem 3.2]).

Example 3.1. Let $G = \langle a_1, a_2, \dots \mid [a_k, a_l],\ k, l = 1, 2, \dots\rangle$ be the free abelian group $\mathbb{Z}^\infty$ of countable rank with relations $r_s = r_{k,l} = [a_k, a_l]$. Since G is torsion-free, we can use the shorter formula (3.2) to map each a_i respectively to $\bar{a}_i(b, c)$ in order to get the embedding of G into the 2-generator recursive group

$$T = \langle b, c \mid [c^{(bc^k)^2 b^{-1}}, c^{(bc^l)^2 b^{-1}}],\ k, l = 1, 2, \dots\rangle.$$

3.2 The main construction of the Higman embedding

For free generators a, b, c, fix the free group $F_3 = \langle a, b, c\rangle$ in addition to the above mentioned $F_2 = \langle b, c\rangle$. Denote by b_i the conjugate b^{c^i} for any $i \in \mathbb{Z}$. Then, for each function $f \in \mathcal{E}$ define the product

$$b_f = \cdots b_{-1}^{f(-1)} b_0^{f(0)} b_1^{f(1)} \cdots$$

and the conjugate $a_f = a^{b_f}$. Say, for $f = (5, 2, -1)$, we have

$$a_f = a^{b_0^5 b_1^2 b_2^{-1}} = c^{-2} b c b^{-2} c b^{-5} \cdot a \cdot b^5 c^{-1} b^2 c^{-1} b^{-1} c^2.$$

For any subset $\mathcal{B}$ of $\mathcal{E}$, introduce the subgroup $A_{\mathcal{B}} = \langle a_f \mid f \in \mathcal{B}\rangle$ in F_3. In particular, for the zero set $\mathcal{B} = \mathcal{Z}$, we get the subgroup

$$A_{\mathcal{Z}} = \langle a_f \mid f = (0)\rangle = \langle a\rangle,$$

and for the set $\mathcal{B} = \mathcal{S}$, we get the subgroup

$$A_{\mathcal{S}} = \langle a_f \mid f \in \mathcal{S}\rangle = \langle c^{-1} b^{-(n+1)} c b^{-n} \cdot a \cdot b^n c^{-1} b^{n+1} c \mid n \in \mathbb{Z}\rangle.$$

As is verified in [7, Lemma 4.4], $A_{\mathcal{Z}}$ and $A_{\mathcal{S}}$ are benign in F_3, and the respective K, L (check notation in Section 2.5) can easily be constructed for each of them.

The most part of [7] is occupied by proofs for [7, Theorems 3 and 4] which set up the environment in which recursion is studied by group-theoretical means. By [7, Theorem 4], a set $\mathcal{B}$ is recursively enumerable in $\mathcal{E}$ if and only if $A_{\mathcal{B}}$ is benign in F_3, and by [7, Theorem 3], $\mathcal{B}$ is recursively enumerable in $\mathcal{E}$ if and only if it belongs to $\mathscr{S}$, i.e., it can be constructed from the basic sets $\mathcal{Z}$ and $\mathcal{S}$ using the Higman operations (H). This means we can start from benign subgroups $A_{\mathcal{Z}}$ and $A_{\mathcal{S}}$, and as $\mathcal{B}$ is being built from $\mathcal{Z}$ and $\mathcal{S}$ by some series of operations (H), the benign subgroup $A_{\mathcal{B}}$ is being constructed step by step. Note that, after Section 2.4, we are free to also use the new auxiliary operations we suggested there.

Each relation r'_s we constructed in Section 3.1 for our recursive 2-generator group $T = \langle b, c \mid R'\rangle$ can be written as

$$r'_s(b, c) = b^{n_0} c^{n_1} \cdots b^{n_{2m}} c^{n_{2m+1}} \quad \text{for some } m = m(s),$$

and this presentation will be unique if we also require $n_1, \dots, n_{2m} \neq 0$. Thus, r'_s can be "coded" by the sequence of exponents $f_s = (n_0, n_1, \dots, n_{2m+1})$, and the elements $b_f = b_{f_s}$ and $a_f = a_{f_s}$ can be defined for these particular $f = f_s$. Say, for $b^3 c b^{-1} c^2$, we have $f = (3, 1, -1, 2)$ and

$$b_f = b_0^3 b_1 b_2^{-1} b_3^2 \quad \text{with} \quad a_f = a^{b_0^3 b_1 b_2^{-1} b_3^2}.$$

The set $\mathcal{B} = \{f_1, f_2, \ldots\}$ of all such sequences clearly is a subset of $\mathcal{E}$, and we can define the respective subgroup $A_{\mathcal{B}} = \langle a_f \mid f \in \mathcal{B}\rangle = \langle a_{f_s} \mid s = 1, 2, \ldots\rangle$ in F_3. As we mentioned above, $A_{\mathcal{B}}$ is benign in F_3 if and only if $\mathcal{B}$ can be constructed from the basic sets $\mathcal{Z}$ and $\mathcal{S}$ using the Higman operations. This launches the following massive procedure in [7]: the set $\mathcal{B}$ is written as an output of a series of operations (H) started from $\mathcal{Z}$ and $\mathcal{S}$. For each step, one of the following actions may be taken.

(1) $\mathcal{B}_1, \mathcal{B}_2$ are already given, and $\mathcal{B}_3$ is obtained from them by any of the *binary* Higman operations ι, υ. Also given are the respective benign subgroups $A_{\mathcal{B}_1}, A_{\mathcal{B}_2}$ in F_3, together with the respective groups K_1, K_2 and L_1, L_2 (see the remark about notation in Section 2.5). Then $A_{\mathcal{B}_3}$ is also benign and can construct the respective K_3 and L_3.

(2) $\mathcal{B}_1$ is already given, and $\mathcal{B}_2$ is obtained from it by any of the *unary* Higman operations $\rho, \sigma, \tau, \theta, \zeta, \pi, \omega_m$ for $m = 1, 2, \ldots$. Also given are the respective benign subgroup $A_{\mathcal{B}_1}$ in F_3, together with the respective groups K_1 and L_1. Then $A_{\mathcal{B}_2}$ is also benign, and we have a mechanism allowing us to construct the K_2 and L_2.

This procedure eventually outputs our sequences set $\mathcal{B}$ together with $A_{\mathcal{B}}$, with the respective group K and its subgroup L.

If, for a group G (or for groups of a given generic type), we are able to explicitly write the set $\mathcal{B}$ and are able to tell how $\mathcal{B}$ can be extracted from $\mathcal{Z}$ and $\mathcal{S}$ by operations (H), then we have an embedding of F_3 into a finitely presented group K with a finitely generated group L such that $F_3 \cap L = A_{\mathcal{B}}$.

The final part of the Higman embedding is far shorter. By the proofs of [7, Lemmas 5.1 and 5.2], the normal closure $\bar{R} = \langle R'\rangle^{F_2}$ is benign in F_2 if and only if $A_{\mathcal{B}}$ is benign in F_3. The proofs of these lemmas also provide the finitely presented group K with a finitely generated subgroup L such that K embeds F_2, and also $F_2 \cap L = \bar{R}$. Then “the Higman Rope Trick” (see the end of [7, Section 5], [12, p. 219] or [22]) uses these K and L to embed $T = \langle b, c \mid R'\rangle$, and thus also G, into a finitely presented group using a free product with amalgamation and an HNN-extension.

That is, *if we are able to explicitly write* $\mathcal{B}$ *by the operations* (H)*, then we can construct the explicit embedding of G into a finitely presented group*. This is what we are going to do in the rest of this article.

3.3 Examples, the structure of sequences in $\mathcal{B}$

Let us continue Example 3.1 by applying the constructions from the previous section for the group $\mathbb{Z}^\infty$.

Example 3.2. The group T in Example 3.1 has the relations

$$r''_s(b,c) = r''_{k,l}(b,c) = [c^{(bc^k)^2 b^{-1}}, c^{(bc^l)^2 b^{-1}}]$$
$$= bc^{-k}b^{-1}c^{-k}b^{-1}c^{-1}bc^k bc^{k-l}b^{-1}c^{-l}b^{-1}c^{-1}bc^l bc^{l-k}$$
$$b^{-1}c^{-k}b^{-1}cbc^k bc^{k-l}b^{-1}c^{-l}b^{-1}cbc^l bc^l b^{-1},$$

$k, l = 1, 2, \ldots$. The respective sequence in $\mathcal{E}$ is

$$f_s = f_{k,l} = (1, -k, -1, -k, -1, -1, 1, k, 1, k-l, -1, -l, -1, -1, 1, l, 1, l-k,$$
$$-1, -k, -1, 1, 1, k, 1, k-l, -1, -l, -1, 1, 1, l, 1, l, -1).$$

As we see, each $f_{k,l}$ is a sequence of length 35 mostly filled by eleven entries 1 and by eleven entries -1, with the following exceptions only: two entries are k; three entries are $-k$; three entries are l; two entries are $-l$; two entries are $k-l$; one entry is $l-k$ (the case with $k = l$ is not an exception, as a_k commutes with itself, and we will just have some coordinates $k - k = 0$ in the sequence above). Denote the respective set of sequences as $\mathcal{B} = \{f_{k,l}, k, l = 1, 2, \ldots\}$.

Can this $\mathcal{B}$ *be constructed by a series of operations* (H)*?* For now let us just simplify this question, postponing the full answer to Example 4.11. As we saw in Section 2.4, if α is any permutation of $\sup(\mathcal{B})$, then $\mathcal{B}$ belongs to $\mathcal{S}$ if and only if $\alpha\mathcal{B}$ belongs to $\mathcal{S}$. In our case, $\sup(\mathcal{B})$ is in the set $\{0, 1, \ldots, 34\}$ of all 35 indices. It is trivial to find the permutation

$$\alpha = (0)(1\,24\,7\,22\,6)(2\,11\,30\,9\,32\,10\,14\,3\,25\,33\,29\,8)$$
$$(4\,12\,15\,27\,31\,28\,20\,18\,17\,34\,21\,5\,13\,16)(19\,26)(23)$$

(we write the cycles of length 1 also) that reorders the indices so that all the similar coordinates in $f_{k,l}$ are grouped, i.e., $\alpha f_{k,l}$ starts by 1 repeated eleven times, followed by -1 repeated eleven times, then followed by two times k, etc.,

$$\alpha f_{k,l} = \big(11 \times 1, 11 \times -1, 2 \times k, 3 \times -k, 3 \times l, 2 \times -l, 2 \times (k-l), l-k\big), \quad (3.3)$$

where 11×1 naturally means 1 repeated eleven times, etc....

Remark 3.3. The above trick will be used below repeatedly: applying a permutation α, we may transform the set $\mathcal{B}$ to such an $\alpha\mathcal{B}$ in which coordinates are grouped in the manner of (3.3). It is simpler to work with such an *appropriately permuted* set $\alpha\mathcal{B}$ keeping in mind that $\mathcal{B}$ belongs to $\mathcal{S}$ if and only if $\alpha\mathcal{B}$ belongs to $\mathcal{S}$, as we saw in Section 2.4.

Example 3.4. Let

$$G = \big\langle a_1, a_2, \ldots \;\big|\; \big[[a_k, a_l], [a_u, a_v]\big],\ k, l, u, v = 1, 2, \ldots \big\rangle$$

be the free metabelian group $F_\infty(\mathfrak{M})$ of countable rank in the variety of all metabelian groups $\mathfrak{M}$. It is easy to deduce that this torsion-free group can be embedded into the 2-generator group

$$T = \big\langle b, c \;\big|\; \big[[c^{(bc^k)^2 b^{-1}}, c^{(bc^l)^2 b^{-1}}], [c^{(bc^u)^2 b^{-1}}, c^{(bc^v)^2 b^{-1}}]\big],$$
$$k, l, u, v = 1, 2, \ldots \big\rangle.$$

Then the respective appropriately permuted sequences (see the previous remark) will be

$$\alpha f_{k,l,u,v} = (40 \times 1, 49 \times -1, 6 \times k, 7 \times -k, 6 \times l,$$
$$6 \times -l, 7 \times u, 6 \times -u, 6 \times v, 6 \times -v,$$
$$l-k, l-u, v-u, v-l, k-l, k-v, u-v)$$

(we omit the routine calculations).

Example 3.5. The additive group of rational numbers $\mathbb{Q}$ has a presentation

$$\langle a_1, a_2, \ldots \mid a_s^s = a_{s-1},\ s = 2, 3, \ldots \rangle,$$

where a generator a_i corresponds to the fraction $\frac{1}{i!}$ with $i = 2, 3, \ldots$ (see [9]). In [16, Example 3.5], we gave the embedding of $\mathbb{Q}$ into the 2-generator group

$$T = \langle b, c \mid (c^s)^{(bc^s)^2 b^{-1}} (c^{-1})^{(bc^{s-1})^2 b^{-1}},\ s = 2, 3, \ldots \rangle$$
$$= \langle b, c \mid (c^s)^{bc^s bc} (c^{-1})^{bc^{s-1} b},\ s = 2, 3, \ldots \rangle.$$

The respective appropriately permuted sequences (see Remark 3.3) then are

$$\alpha f_s = \big(6 \times 1, 6 \times -1, 2 \times s, 2 \times -s, 1-s, 2 \times (s-1)\big),$$

or a little but shorter variant

$$\alpha f_s = (5 \times 1, 6 \times -1, 2 \times s, -s, 1-s, s-1),$$

$s = 2, 3, \ldots$ (the omitted calculations are easy to verify).

Remark 3.6. In 1999, Bridson and de la Harpe posed in the Kourovka notebook [11, Problem 14.10] in which they grouped a few questions as a "well-known problem". The questions mainly concern explicit embeddings of some countable groups into finitely generated or finitely presented groups. In particular, one of the points of [11, Problem 14.10 (a)] asks to find an explicit embedding of $\mathbb{Q}$ into a "natural" finitely presented group.

As the main steps outlined in Section 3 show, we are able to explicitly embed a recursive group G into a finitely presented group, as soon as we have the explicit embedding of G into the respective 2-generator group $T = T_G$, have the set $\mathcal{B}$ of integer sequences corresponding to defining relations of T_G and also are able to construct $\mathcal{B}$ from the sets $\mathcal{Z}$ and from $\mathcal{S}$ using the Higman operations (H). Example 3.5 directly provides T for $\mathbb{Q}$, and it gives $\mathcal{B}$ by means of αf_s.

The H-machine of Section 4 shows how to easily write down the operations (H) if we know $\mathcal{B}$. That is, a group answering [11, Problem 14.10] of Bridson and de la Harpe can be constructed by a series of free constructions matching to the series of Higman operations. Of course, the question is if that group can be called a "natural" finitely presented group....

Recently, a direct solution to the problem of Bridson and de la Harpe was found by Belk, Hyde and Matucci in [3]. Moreover, one of the remarkable finitely presented groups constructed by them is the group $T\mathcal{A}$ which is 2-generator and also simple [3].

Example 3.7. The quasicyclic Prüfer p-group $G = \mathbb{C}_{p^\infty}$ can be presented as

$$G = \langle a_1, a_2, \ldots \mid a_1^p, a_{s+1}^p = a_s,\ s = 1, 2, \ldots \rangle,$$

where each a_i corresponds to the primitive (p^i)-th root ε_i of unity [10]. As we found in [16, Example 3.6], this group can be embedded into the 2-generator group

$$T = \big\langle b, c \;\big|\; \big(c^{(bc)^2 b^{-1}} (c^{-1})^b\big)^p,\\ \big(c^{(bc^{s+1})^2 b^{-1}} (c^{-1})^b\big)^p c^b (c^{-1})^{(bc^s)^2 b^{-1}},\ s = 1, 2, \ldots \big\rangle.$$

From the first single relation, we get the appropriately permuted sequence (see Remark 3.3)

$$\alpha f_0 = \big((5p+2) \times 1, 5p \times -1, (p-1) \times 2, p \times -2\big).$$

And from the remaining relations, we get the respective appropriately permuted sequences

$$\alpha' f_s = \big((3p+4) \times 1, (3p+3) \times -1, s, -s, p \times (s+1), p \times (-s-1)\big),$$

$s = 2, 3, \ldots$ (the calculations are omitted). Clearly, $\alpha' \neq \alpha$.

Examples similar to Example 3.2 and Example 3.4 are easy to construct for free *soluble* groups, for free *nilpotent* groups and, more generally, for other types of groups defined by commutator-based identities.

Further, since any *divisible* abelian group is a direct product of copies of $\mathbb{Q}$ and of some $\mathbb{C}_{p^\infty}$, it is not hard to use Example 3.5 and Example 3.7 to get sequences of similar formats for them also. Moreover, every abelian group is a subgroup in an abelian divisible group, so we get similar sequences for embeddings of any countable abelian group (provided that its embedding into a countable divisible abelian group is constructively, effectively given).

Remark 3.8. We could continue to collect examples with the same features, but it already seems to be clear that there are numerous groups for which the respective sequence sets have a similar "format".

(1) Some coordinates in them have a *fixed* value (or one of pre-given fixed values). Say, the initial 0th coordinate is equal to 1 in each sequence $\alpha f_{k,l}$ in Example 3.2.

(2) Some coordinates can accept *any* integer values k, like the 22nd coordinate k in the sequence $\alpha f_{k,l}$.

(3) Some coordinates are *duplicates* of certain other coordinates. Say, the 1st, 2nd, ..., 10th coordinates in $\alpha f_{k,l}$ all are the duplicates of the 0th coordinate 1. And also the 23rd coordinate k is the duplicate of the 22nd coordinate k.

(4) Some coordinates are the *opposites* of certain other coordinates. Say, the 11th coordinate -1 in $\alpha f_{k,l}$ is the opposite of the 10th coordinate 1. Also, the 27th coordinate $-k$ is the opposite of the 26th coordinate k.

(5) And some coordinates are obtained from other coordinates by *arithmetical operations*. Say, the 33rd coordinate $k-l$ in $\alpha f_{k,l}$ is the difference of the 22nd coordinate k and of the 28th coordinate l.

As we see now, construction of a set $\mathcal{B} \in \mathcal{S}$ by operations (H) in many cases can be reduced to the following question: can we build a "machine" which *constructs $\mathcal{B}$ by performing the five operations listed above*, i.e., by assigning fixed pre-given values to some coordinates, then copying those values to other coordinates, then assigning the opposites, the sums or differences of those values to some other coordinates? *If yes, then constructive Higman embeddings are available for the considered types of groups.*

In the next section, we will step by step collect a positive answer to this question. The reader may skip to Example 4.11 to see an application of the method.

4 The H-machine

This is the main section of this note, and its objective is to show that $\mathscr{S}$ contains some general kinds of subsets of $\mathcal{E}$ which can be constructed by generic operations outlined in Remark 3.8. The reader not interested in the routine of proofs may skip the details below.

4.1 Construction of sum of subsets with disjoint supports

For definition of the sum of subsets from $\mathscr{S}$ and of other auxiliary operations, we refer to Section 2.4.

Lemma 4.1. *If the sets $\mathcal{B}_k$, $k = 1, \ldots, m$, all belong to $\mathscr{S}$ and their supports $S_k = \sup(\mathcal{B}_k)$ are finite pairwise disjoint sets, then the sum $\mathcal{B}_1 + \cdots + \mathcal{B}_m$ also belongs to $\mathscr{S}$.*

Proof. The set $\mathcal{B}_1^* = (\zeta_{S_2} \cdots \zeta_{S_m})\mathcal{B}_1$ clearly consists of all functions from $\mathcal{B}_1$ with all coordinates from $S_2, \ldots, S_m$ liberated. This can be achieved by applying some operations ζ_i for finitely many times, so $\mathcal{B}_1^*$ belongs to $\mathscr{S}$. In a similar way, we define the sets $\mathcal{B}_2^*, \ldots, \mathcal{B}_m^*$ in $\mathscr{S}$. It is easy to see that

$$\mathcal{B}_1 + \cdots + \mathcal{B}_m = \iota_m(\mathcal{B}_1^*, \ldots, \mathcal{B}_m^*). \qquad \square$$

The analog of this lemma could be proved for the case of infinite supports, but we restrict to this case for simplicity.

4.2 Construction of (n) with restrictions on n

Denote by $\mathcal{B}_+ = \{(n) \mid n = 1, 2, \ldots\}$ the set of all functions with a single positive coordinate, and by $\mathcal{B}_- = \{(n) \mid n = -1, -2, \ldots\}$ the set of all functions with a single negative coordinate. Their union $\mathcal{B}_\pm = \{(n) \mid n \in \mathbb{Z}\backslash\{0\}\}$ is the set of all functions with a single non-zero coordinate.

Lemma 4.2. *The sets $\mathcal{B}_+$, $\mathcal{B}_-$ and $\mathcal{B}_\pm$ belong to $\mathscr{S}$.*

Proof. $\mathcal{A}_1 = \omega_2 \upsilon(\zeta_1 \mathcal{Z}, \tau \mathcal{S})$ clearly consists of functions g in which, for every $i \in \mathbb{Z}$, the subsequence $t_i = (g(2i), g(2i+1))$ is either of type $(0, n)$ or of type $(n, n-1)$, with $n \in \mathbb{Z}$. For any *even* $n = 2, 4, \ldots$, we can apply ω_2 to the pairs

$$(n, n-1), (n-2, n-3), \ldots, (2, 1) \in \tau\mathcal{S}$$

to construct in $\mathcal{A}_1$ the sequence $g = (n, n-1, \ldots, 1)$. The sequence

$$g' = (0, n, n-1, \ldots, 1, 0)$$

can be built by the pair $(0, n) \in \zeta_1 \mathcal{Z}$ and the pairs

$$(n-1, n-2), \ldots, (3,2), (1,0) \in \tau \mathcal{S}.$$

Clearly, $\sigma^{-1} g' = g$, and so $g \in \mathcal{A}_2 = \iota(\mathcal{A}_1, \sigma^{-1}\mathcal{A}_1)$. In a similar manner, we discover in $\mathcal{A}_2$ all the functions $g = (n, n-1, \ldots, 1)$ for *odd* $n = 1, 3, \ldots$. This time, g is constructed by the pairs $(n, n-1), (n-2, n-3), \ldots, (1,0) \in \tau\mathcal{S}$, and g' can be built by $(0, n) \in \zeta_1 \mathcal{Z}$ with $(n-1, n-2), \ldots, (2,1) \in \tau\mathcal{S}$. Thus, for any $n = 1, 2, \ldots$, the set $\mathcal{A}_2$ contains a function g with the property $g(0) = n$.

Let us show that $g(i) < 0$ is impossible for any $g \in \mathcal{A}_2$. Assuming the contrary, suppose the *least* coordinate $g(k) < 0$ of g is achieved at some index k. If k is *even*, then the pair $(g(k), g(k+1))$ in $g \in \mathcal{A}_1$ has to be either of type $(n, n-1) \in \tau\mathcal{S}$ (which is impossible as $g(k-1) \nless g(k)$) or of type $(0, n) \in \zeta_1 \mathcal{Z}$ (which is impossible as $g(k) = 0 \nless 0$). If k is *odd*, then the pair $(g(k), g(k+1))$ in $g \in \sigma^{-1}\mathcal{A}_1$ has to be either of type $\sigma^{-1}(n, n-1)$ or of type $\sigma^{-1}(0, n)$ (which both again are impossible).

Next let us exclude those functions $g \in \mathcal{A}_2$ for which $g(0) = 0$. Clearly, we have that $\mathcal{A}_3 = \pi_1 \pi' \tau \mathcal{S}$ is the set of all those functions from $\mathcal{E}$ which coincide with $(n, n-1)$ on indices $0, 1$ and which may have any coordinates elsewhere. Then $g(0) > 0$ for each $g \in \iota(\mathcal{A}_2, \mathcal{A}_3)$, and the extract $\epsilon_0 \iota(\mathcal{A}_2, \mathcal{A}_3)$ is the set $\mathcal{B}_+$.

In a similar way, we can construct $\mathcal{B}_-$. And the union of the above is

$$\mathcal{B}_\pm = \upsilon(\mathcal{B}_+, \mathcal{B}_-). \qquad \square$$

The reader may compare the above proof with the argument of [7, Lemma 2.1].

Let $\mathcal{B}_{k+}$ or $\mathcal{B}_{k-}$ denote the set of all (n) for which $n > k$ or $n < k$, respectively.

Lemma 4.3. *For any integers $k \in \mathbb{Z}$ the sets $\mathcal{B}_{k+}$ and $\mathcal{B}_{k-}$ belong to $\mathcal{S}$.*

Proof. The set $\mathcal{D}_1 = \iota(\tau\mathcal{S}, \zeta\sigma\mathcal{B}_+)$ consists of all pairs $(n+1, n)$ for $n = 1, 2, \ldots$. Then $\mathcal{B}_{1+}$ is the extract $\epsilon_0 \mathcal{D}_1$. By induction, we construct in $\mathcal{S}$ the set

$$\mathcal{D}_k = \iota(\tau\mathcal{S}, \zeta\sigma\mathcal{B}_{(k-1)+}),$$

and the extract $\mathcal{B}_{k+} = \epsilon_0 \mathcal{D}_k$. The case of $\mathcal{B}_{k-}$ is discussed analogously. $\square$

For an integer $n \in \mathbb{Z}$, denote by $\mathcal{N}_n$ the set $\{(n)\}$ consisting of a single sequence (n) of length 1. More generally, denote $\mathcal{N}_{n_1,\ldots,n_k} = \{(n_1), \ldots, (n_k)\}$ the set consisting of k functions of the above type.

Lemma 4.4. *For any fixed integers $n_1, n_2, \ldots, n_k \in \mathbb{Z}$, the set $\mathcal{N}_{n_1,\ldots,n_k}$ belongs to $\mathcal{S}$.*

Proof. It is clear that $\mathcal{N}_1 = \{(1)\}$ is in $\mathscr{S}$, for $\tau\mathcal{S} = \{(n+1, n) \mid n \in \mathbb{Z}\}$, and so $\mathcal{N}_1 = \iota(\tau\mathcal{S}, \zeta\mathcal{Z})$ consists of $(0+1, 0) = (1, 0) = (1)$ only. Similarly, we have that $\mathcal{N}_2 = \{(2)\}$ is in $\mathscr{S}$ because $\mathcal{N}_2 = \epsilon_0\iota(\tau\mathcal{S}, \zeta\sigma\mathcal{N}_1)$. By induction, we construct all the $\mathcal{N}_3, \mathcal{N}_4, \ldots$. The sets $\mathcal{N}_{-1}, \mathcal{N}_{-2}, \ldots$ can be obtained in a similar way. Finally, $\mathcal{N}_0 = \{(0)\} = \mathcal{Z}$. Taking the union of the required one-element sets $\{(n_1)\}, \ldots, \{(n_k)\}$, we finish the proof. □

4.3 Duplication of the last term

Let $\mathcal{B} \in \mathscr{S}$ be any set of functions g which are zero after the k-th coordinate, i.e., $g(i) = 0$ for each $i > k$. Then, by Higman operations, we can "duplicate" the k-th coordinate in all $g \in \mathcal{B}$. More precisely, for each $g \in \mathcal{B}$, let g' be defined as $g'(i) = g(i)$ for all $i \neq k+1$, $g'(k+1) = g(k)$. In this notation, define $\mathcal{B}_{k,k} = \{g' \mid g \in \mathcal{B}\}$.

Lemma 4.5. *Let $\mathcal{B} \in \mathscr{S}$ be a set of functions g which are zero after the k-th coordinate. Then the set $\mathcal{B}_{k,k}$ also belongs to $\mathscr{S}$.*

Proof. For simpler notation, assume $k = 0$ as the general case can be reduced to this by shifting $\mathcal{B}$ by σ^{-k}, and then shifting back by σ^k after duplication of the 0th coordinate.

Denote by $\mathcal{F}_1$ the set of all functions from $\mathcal{B}$ with the 1st and 2nd coordinates liberated, i.e., $\mathcal{F}_1 = \zeta_{1,2}\mathcal{B}$. Let $\mathcal{F}_2$ be the set of all functions $g \in \mathcal{E}$ in which $g(1) = g(0) + 1$, and the 2nd coordinate together with *all* the negative coordinates are liberated, i.e., $\mathcal{F}_2 = \pi'\zeta_2\mathcal{S}$. Let $\mathcal{F}_3$ be the set of all functions g in which $g(2) = g(1) - 1$, and the 0th coordinate together with *all* the negative coordinates are liberated, i.e., $\mathcal{F}_3 = \pi'\zeta\sigma\tau\mathcal{S} = \pi'_1\sigma\tau\mathcal{S}$. Then the 2nd and 0th coordinates of each function from $\mathcal{F}_4 = \iota_3(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ are equal.

To get the duplicated set $\mathcal{B}_{0,0}$ it remains to swap the 2nd and 1st coordinates in $\mathcal{F}_4$, and then to erase the new 2nd coordinates. Namely, set $\mathcal{F}_5 = \zeta_2\tau_{1,2}\mathcal{F}_4$ and $\mathcal{F}_6 = \pi'_2\mathcal{Z}$, and take the intersection $\mathcal{B}_{0,0} = \iota(\mathcal{F}_5, \mathcal{F}_6)$. □

4.4 Construction of the pairs $(n, -n)$

Denote by $\mathcal{B}_{+,-} = \{(n, -n) \mid n = 1, 2, \ldots\}$ the set of all couples $(n, -n)$ with $n = 1, 2, \ldots$. The objective of this section is to prove the following lemma.

Lemma 4.6. *The set $\mathcal{B}_{+-}$ belongs to $\mathscr{S}$.*

Our proof will follow from a series of steps, cases, examples below.

The set $\mathcal{L}_1 = \zeta_{2,3}\mathcal{Z}$ can be interpreted as the set of all 4-tuples

$$(0, 0, m, n) \tag{4.1}$$

with $m, n \in \mathbb{Z}$. Next $\mathcal{C}_1 = \tau_{1,2}\tau\mathcal{S}$ can be interpreted as the set of all 4-tuples

$$(m, 0, m-1, 0),$$

while $\mathcal{C}_2 = \tau_{1,2}\sigma^2\mathcal{S}$ can be interpreted as the set of all 4-tuples

$$(0, n, 0, n+1).$$

Then the sum $\mathcal{L}_2 = \mathcal{C}_1 + \mathcal{C}_2$ is the set of all 4-tuples

$$(m, n, m-1, n+1). \tag{4.2}$$

The set $\mathcal{L}_3 = \omega_4 \upsilon(\mathcal{L}_1, \mathcal{L}_2)$ consists of $g \in \mathcal{E}$ in which, for every $i \in \mathbb{Z}$, the subsequence

$$t_i = \big(g(4i), g(4i+1), g(4i+2), g(4i+3)\big) \tag{4.3}$$

is of type (4.1) or of type (4.2) (not ruling out the zero 4-tuple which is of type (4.1) for $m = n = 0$). Define a set $\mathcal{M} = \iota(\mathcal{L}_3, \sigma^{-2}\mathcal{L}_3)$.

Step 1. Start by showing that if $g \in \mathcal{M}$, then $g(k) \geq 0$ for any $k = 4i, 4i+2$ with $i \in \mathbb{Z}$. Assume the contrary: $m = g(k) < 0$ for some k of one of the above types.

If $k = 4i$, i.e., m is the initial term of the 4-tuple t_i in (4.3), then t_i is of type (4.2) because the tuples of type (4.1) have to start by a zero. Thus we have $g(4i+2) = m - 1 < 0$.

Next assume $k = 4i + 2$. As $g \in \sigma^{-2}\mathcal{L}_3$, there exists a $g' \in \mathcal{L}_3$ such that $\sigma^{-2}g' = g$. Then $g'(4(i+1)) = g'(4i+2+2) = g(4i+2) = m$, i.e., the next 4-tuple of g' also starts by negative number m and has to be of type (4.2). But then $g'(4(i+1)+2) = m-1$, and so $g(4i+2+2) = g(4(i+1)) = m-1 < 0$, i.e., the $(i+1)$-th sequence t_{i+1} in g starts by $m-1$.

We got that, for any $k = 4i$ and $k = 4i+2$, from $g(k) < 0$, it follows that $g(k+2) < 0, g(k+4) < 0$, etc. because $g(k+2) = m-1, g(k+4) = m-2$, etc. This leads to a contradiction as $g \in \mathcal{E}$ cannot have infinitely many non-zero coordinates.

In a similar way, we show that $g(k) \leq 0$ for any $k = 4i+1, 4i+3$ for $i \in \mathbb{Z}$.

Example 4.7. Consider two functions $g \in \mathcal{M}$ of above types. First, the function

$$g = (4, -4, 3, -3; 2, -2, 1, -1) \tag{4.4}$$

is constructed by two 4-tuples of type (4.2), and it can be presented as $g = \sigma^{-2}g'$ for

$$g' = (0, 0, 4, -4; 3, -3, 2, -2; 1, -1, 0, 0)$$

which is constructed by one 4-tuple of type (4.1) and two 4-tuples of type (4.2).

Next consider another function

$$g = (0, 0, 4, -4; 3, -3, 2, -2; 1, -1, 0, 0) \tag{4.5}$$

constructed by one 4-tuple of type (4.1) and two 4-tuples of type (4.2), and this g can be presented as $g = \sigma^{-2}g'$ for the function

$$g' = (0, 0, 0, 0; 4, -4, 3, -3; 2, -2, 1, -1)$$

which is constructed by two 4-tuples of type (4.2) (and the zero 4-tuples of course).

Observe that, in these two functions, we took 4 and -4 to be the *opposites* of each other, ignoring the case of a tuple, say, $(4, 0, -5, 0)$. We will cover that issue later.

Step 2. We see that, from any positive $g(k)$, a *chain* of positive, descending coordinates $g(k), g(k+2), g(k+4), \dots$ starts for $k = 4i$ or $k = 4i + 2$. How may this chain end?

Case 2.1. The chain achieves 1 (its *last* positive coordinate) at some index $4j$, i.e., in the first half of some 4-tuple t_j, like in (4.5); then the term $g(4j + 2)$ automatically is $1 - 1 = 0$. Starting from the term $g(4j + 4)$ in t_{j+1}, we may have either zeros, or a new chain may begin from there.

Case 2.2. The chain achieves 1 at some index $4j + 2$, i.e., in the second half of some 4-tuple t_j, like in (4.4). Then the next term $g(4j + 2 + 2) = g(4(j + 1))$ (which is 0 and which lies in the next tuple t_{j+1}) may have two potential ways to occur: either the next tuple is of type (4.1), i.e., it starts by two zeros, and after them, we may have either zeros, or a new chain may begin there; or the next tuple is of type (4.2) with an initial term $g(4(j + 1)) = 0$. But then the $(4(j + 1) + 2)$-th term of that tuple has to be $0 - 1 = -1$. Since negative values are ruled out for such coordinates, that is impossible.

The analogs of these arguments hold for negative, ascending chains

$$g(k), g(k+2), g(k+4), \dots$$

starting at some $g(k)$ for a $k = 4i + 1$ or $k = 4i + 3$. Namely, the *last* negative term -1 of such a term is achieved.

Case 2.3. Either at some $4j + 1$, i.e., in the first half of some 4-tuple, like in (4.4); then the next term $g(4j + 3)$ automatically is $1 - 1 = 0$.

Case 2.4. Or -1 is achieved at some index $4j + 3$, i.e., in the second half of some 4-tuple, like in (4.5). Then the next tuple may be of type (4.1) only, i.e., it starts by two zeros.

Step 3. The key feature of this construction is that the two chains we discuss (the ascending and the descending chains residing inside some consecutive 4-tuples) *terminate simultaneously*, i.e., the last 4-tuple t_j either ends by $(1, -1)$ (i.e., $t_j = (2, -2, 1, -1)$ or $t_j = (0, 0, 1, -1)$), or $t_j = (1, -1, 0, 0)$. Assume the contrary, and arrive to contradiction in all cases occurring.

Case 3.1. Assume t_j ends by $(m, -1)$ for an $m \geq 2$. Then, by Case 2.2 above, the next 4-tuple t_{j+1} need start with two zeros. We get a contradiction because $m - 1 \neq 0$.

Case 3.2. Assume t_j ends by $(0, -1)$ (that is, $m = 0$ in terms of the previous case). Since $g = \sigma^{-2} g' \in \mathcal{L}_3$, the $(j + 1)$-th 4-tuple in g' starts by $(0, -1)$. Then that 4-tuple in g' is of type (4.2), i.e., it ends by $(0 - 1, -1 + 1) = (-1, 0)$. So t_{j+1} in g starts by $(-1, 0)$, which is a contradiction as $g(4(j + 1)) = -1$ cannot be negative.

Case 3.3. Assume the last 4-tuple t_j is $(m, -1, 0, 0)$ with $m \geq 2$ or $m = 0$. Since $-1 \neq 0$, then t_j is of type (4.2). Then its 2nd term is $m - 1$, which is impossible as $m - 1 \neq 0$.

We get that, whenever a $g \in \mathcal{M}$ contains a couple (m, n) with a positive m and a negative n, we have $n = -m$ (see the remark at the end of Example 4.7). In particular, if, for some $g \in \mathcal{M}$, we have $g(0) > 0$ and $g(2) < 0$, then $g(2) = -g(2)$. Clearly, for any positive n, we can build a $g \in \mathcal{M}$ with $g(0) = n$ and $g(2) = -n$.

Step 4. Denote by $\mathcal{C}_3 = \pi_2 \pi' \mathcal{M}$ the set of all functions $g \in \mathcal{E}$ which coincide with some $(n, -n)$ with $n = 0, 1, \ldots$ and which have any coordinates elsewhere. Then $\mathcal{C}_4 = \mathcal{B}_+ + \sigma \mathcal{B}_-$ can be interpreted as the set of all couples (m, n) with positive m and negative n, and $\mathcal{B}_{+,-} = \iota(\mathcal{C}_3, \mathcal{C}_4)$ is the set of all couples $(n, -n)$ with $n = 1, 2, \ldots$.

Thus, Lemma 4.6 is fully argued.

If needed, we can easily get the analogs of Lemma 4.6 not only for the couples $(n, -n)$ for *all* $n = 1, 2, \ldots$ but for, say, $n = k, k + 1, \ldots$, or for n from a given finite set only.

4.5 Construction of the triples $(p, q, p - q)$ and $(p, q, p + q)$

Assume the set $\mathcal{P}$ consists of some pairs (p, q). In this section, we show that if $\mathcal{P}$ belongs to $\mathcal{S}$, then the set consisting of all triples $(p, q, p - q)$ and the set consisting of all triples $(p, q, p + q)$ also belong to $\mathcal{S}$.

For a fixed pair (p, q), denote by $\mathscr{P}_1$ the set of 8-tuples of the following types:

$$(p, q, q, 0, 0, 0, 0, 0), \tag{4.6}$$

$$(0, 0, 0, 0, p, q, p, n), \tag{4.7}$$

$$(p, q, m, n, p, q, m-1, n-1), \tag{4.8}$$

$$(p, q, m, n, p, q, m+1, n+1) \tag{4.9}$$

for any $m, n \in \mathbb{Z}$, together with the zero function which we can interpret as the 8-tuple $(0, 0, 0, 0, 0, 0, 0, 0)$.

The set of 8-tuples of type (4.6) is in $\mathscr{S}$ since we can apply Lemma 4.5 to couples (p, q) to duplicate the coordinate q. Similarly, the set of 8-tuples of type (4.7) is also in $\mathscr{S}$. The 8-tuples of type (4.8) can be obtained as follows: the set of all 8-tuples of type $(0, 0, m, n, 0, 0, m-1, n-1)$ can be obtained using a permutation α of the set $\sigma^2\tau\mathcal{S} + \sigma^3\tau\mathcal{S}$. Then we take the sum of that set and the set of all 8-tuples $(p, q, 0, 0, p, q, 0, 0)$. The case of 8-tuples of types (4.9) is covered in a similar way. This means the combined set $\mathscr{P}_1$ of *all* 8-tuples of types (4.6)–(4.9) is in $\mathscr{S}$.

Thus, the intersection $\mathscr{P}_2 = \iota(\mathscr{P}_1, \sigma^{-4}\mathscr{P}_1)$ also is in $\mathscr{S}$. Using Higman operations on $\mathscr{P}_2$, we can construct $(p, q, p-q)$. Let us first explain the idea by simple examples.

Example 4.8. Let $p = 6$ and $q = 2$. The sequence

$$g = (6, 2, 6, 4, 6, 2, 5, 3; 6, 2, 4, 2, 6, 2, 3, 1; 6, 2, 2, 0, 0, 0, 0, 0)$$

is constructed by two 8-tuples of type (4.8) and by one 8-tuple of type (4.6). And g can be presented as $g = \sigma^{-4}g'$ for

$$g' = (0, 0, 0, 0, 6, 2, 6, 4; 6, 2, 5, 3, 6, 2, 4, 2; 6, 2, 3, 1, 6, 2, 2, 0)$$

which is constructed by one 8-tuple type (4.7) and two 8-tuples of type (4.8). Note that the 3rd coordinate in g is $p - q = 6 - 2 = 4$.

Yet another function

$$g = (0, 0, 0, 0, 6, 2, 6, 4; 6, 2, 5, 3, 6, 2, 4, 2; 6, 2, 3, 1, 6, 2, 2, 0)$$

is constructed by one 8-tuple of type (4.7) and by two 8-tuples of type (4.8). And g can be presented as $g = \sigma^{-4}g'$ for

$$g' = (0, 0, 0, 0, 0, 0, 0, 0; 6, 2, 6, 4, 6, 2, 5, 3;\\ 6, 2, 4, 2, 6, 2, 3, 1; 6, 2, 2, 0, 0, 0, 0, 0)$$

which is constructed by two 8-tuples of type (4.8) and one 8-tuple of type (4.6) (and the zero sequences of course). Note that the 11th coordinate in g is

$$p - q = 6 - 2 = 4.$$

As we see, using Higman operations, it is easy to obtain the triple $(6, 2, 4)$ from any of the functions g constructed above. Using some loose wording, we could say that we "mimic" the arithmetical operation $6 - 2 = 4$ by means of Higman operations (in the sense that we were able to build the triple $(6, 2, 6 - 2)$). We did this using some descending chains of coordinates (at indices $4k + 2, 4k + 6, \ldots$) starting by 6 and ending by 2.

The purpose of the numbers 6, 2 standing at some indices $8k$, $8k + 1$ or $8k + 4$, $8k + 5$ is the following. Besides the pair $(6, 2)$, our set $\mathcal{P}$ may also contain another pair, say, $(9, 1)$. We want to construct in $\mathcal{P}$ the triple $(6, 2, 4)$ *without* adding the unnecessary triple $(6, 1, 5)$ into $\mathcal{P}$. That is, we need a descending chain starting by 6 and ending by 2 (but *not* by 1). So those numbers 6, 2 guarantee that we concatenate 8-tuple corresponding to the *same* pair $(6, 2)$ only.

Observe that $p \geq q$ in each of above examples. When $p < q$, then we could build *ascending* chains using 8-tuples of type 4.9. Say, if $p = 3$ and $q = 9$, the sequence

$$g = (3, 9, 3, -6, 3, 9, 4, -5; 3, 9, 5, -4, 3, 9, 6, -3;\\ 3, 9, 7, -2, 3, 9, 8, -1; 3, 9, 9, 0, 0, 0, 0, 0)$$

is constructed by three 8-tuples of type (4.9) and by one 6-tuple of type (4.6). And g can be presented as $g = \sigma^{-4} g'$ for the function

$$g' = (0, 0, 0, 0, 3, 9, 3, -6; 3, 9, 4, -5, 3, 9, 5, -4;\\ 3, 9, 6, -3, 3, 9, 7, -2; 3, 9, 8, -1, 3, 9, 9, 0)$$

which is constructed by one 8-tuple of type (4.7) and three 8-tuples of type (4.9). Note that the 3rd coordinate in g is $p - q = 3 - 9 = -6$.

After these examples, the formal proofs are simpler to understand. Assume a pair (p, q) is chosen and g is any non-zero function in $\mathcal{P}_2$. Since $g \in \mathcal{E}$, there is a *first* 8-tuple

$$t_i = \big(g(8i), g(8i + 1), g(8i + 2), g(8i + 3),\\ g(8i + 4), g(8i + 5), g(8i + 6), g(8i + 7)\big)$$

in which g has its first non-zero coordinate. Using arguments similar to those in Section 4.4, we show that if t_i is of type (4.8), then a *descending* chain of coordinates $g(8i+2), g(8i+6), g(8i+10), \ldots$ starts from t_i. If $p < q$, then this chain never ends, which is a contradiction to the fact that $g \in \mathcal{E}$ has finitely many non-zero coordinates. If $p \geq q$, then this chain ends either by the 8-tuple $(p, q, q, 0, 0, 0, 0, 0)$ of type (4.6), or by the 8-tuple $(p, q, q+1, 1, p, q, q, 0)$ of type (4.8). This means that $g(8i+2)$ is equal to $p - q$.

If t_i is of type (4.9), then an *ascending* chain of coordinates starts from t_i. If $p > q$, we get a contradiction, and if $p \geq q$, we again get that $g(8i+2)$ is equal to $p - q$.

Finally, if t_i is of type (4.7), we get a descending or ascending chain, and then $g(8i+6)$ is equal to $p - q$.

The "extremal" case when $t_i = (p, q, q, 0, 0, 0, 0, 0)$ is of type (4.6) for a g is possible only if the respective g' either starts by $(0, 0, 0, 0, p, q, p, n)$ of type (4.7) (i.e., $p = q$ and $n = 0$, that is, we again have the equality $p - q = p - p = n = 0$), or g' starts by $(p, q, m, n, p, q, m-1, n-1)$ of type (4.8) (i.e., $p = q = m = n = 0$, which leads to a contradiction, as then t_i is a zero 8-tuple). The case when t_i is of type (4.9) is excluded in a similar way.

We see that a sequence $g \in \mathcal{P}_2$ can consist of a few 8-tuples (holding a chain of the above types) only. Now we need extract the required fragments $(p, q, p - q)$.

Clearly, $\pi_1 \mathcal{P}$ consists of sequences of type

$$h = (p, q, n_2, n_3, \ldots) \quad \text{with } (p, q) \in \mathcal{P},$$

and with only finitely many of the coordinates $n_2, n_3, \ldots$ being non-zero.

Denote $\mathcal{P}_3 = \iota(\mathcal{P}_2, \pi_1 \mathcal{P})$ and choose any $g \in \mathcal{P}_3$. Since $g \in \mathcal{P}_2$, it is constructed by some 8-tuples of one of the types (4.6)–(4.9). Since also $g \in \pi_1 \mathcal{P}$, its *first* non-zero 8-tuple occupies indices 0–7 and is of types (4.7) or (4.8) with one of p, q being non-zero. Then, by our construction, g starts by the triple $(p, q, p - q)$.

The case when $\mathcal{P}$ does contain the couple $(0, 0)$ also is covered by our construction because, in that case, the 8-tuple of type (4.6) with $p = q = 0$ is in $\mathcal{P}_2$, and to $\mathcal{P}_3$ contains a sequence starting by $(0, 0, 0)$.

The extract set $\epsilon_{0,1,2} \mathcal{P}_3$ is the set of triples

$$\mathcal{P}_{1,2,1-2} = \{(p, q, p - q) \mid (p, q) \in \mathcal{P}\}.$$

The other set $\mathcal{P}_{1,2,1+2}$ can now be obtained in two ways. Either we can modify the constructions above to adapt it for the triples $(p, q, p + q)$. Or we can use the construction of Section 4.4 to build the set of $\mathcal{P}_4$ of triples $(p, q, -q)$. Then the extract $\epsilon_{0,2} \mathcal{P}_4$ is the set $\{(p, p - q) \mid (p, q) \in \mathcal{P}\}$. So we can directly apply the already constructed proof to get the triples $(p, -q, p - (-q)) = (p, -q, p + q)$ and finally replace $-q$ by q.

We proved the following lemma.

Lemma 4.9. *If the sets* $\mathcal{P}$ *belongs to* $\mathcal{S}$*, then the sets* $\mathcal{P}_{1,2,1-2}$ *and* $\mathcal{P}_{1,2,1+2}$ *both belong to* $\mathcal{S}$*.*

Combining Lemma 4.9 with Lemma 4.5 and Lemma 4.6, we get that if $\mathcal{Q}$ is a set of some (q), then $\mathcal{Q} \in \mathcal{S}$ implies that $\mathcal{S}$ contains the set of all couples (q, q), the set of all triples $(q, q, 2q)$ and the set of all couples $(q, 2q)$ (which is obtained from the set of previous triples via the extract $\epsilon_{0,2}$). Repeating this, we get the set of all couples $(q, s \cdot q)$ for any pre-given integer s, and $q \in \mathcal{Q}$.

Lemma 4.9 can also be generalized by taking any distinct indices instead of $0, 1, 2$. Let $\mathcal{B}$ be any subset of $\mathcal{E}$, and let $p = g(i)$ and $q = g(j)$ be the i-th and j-th coordinates of generic $g \in \mathcal{B}$. For an index k, different from i, j, denote by $\mathcal{P}_{i,j,i-j,k}$ the set of all functions $f \in \mathcal{E}$ for which there is a $g \in \mathcal{B}$ such that f coincides with g on all coordinates except the k-th, and $f(k) = p - q$. In other words, we replace the k-th coordinate in each $g \in \mathcal{B}$ by $p - q = g(i) - g(j)$. We can similarly define the set $\mathcal{P}_{i,j,i+j,k}$.

Lemma 4.10. *If the set* $\mathcal{B}$ *belongs to* $\mathcal{S}$ *and, for fixed* i, j*, the set*

$$\mathcal{P} = \{(p, q) \mid g \in \mathcal{B},\ p = g(i),\ q = g(j)\}$$

also belongs to $\mathcal{S}$*, then the sets* $\mathcal{P}_{i,j,i-j,k}$ *and* $\mathcal{P}_{i,j,i+j,k}$ *both belong to* $\mathcal{S}$*.*

Proof. Applying the appropriate permutation α, we can reorder the coordinates of each $g \in \mathcal{B}$ so that $\alpha g(0) = p = g(i)$, $\alpha g(1) = q = g(j)$, $\alpha g(2) = g(k)$. Then $\mathcal{R}_1 = \zeta_2 \alpha \mathcal{B}$ consists of all those reordered sequences with the 2nd coordinate liberated.

Applying Lemma 4.9 to $\mathcal{P}$, the set

$$\mathcal{P}_{1,2,1-2} = \{(p, q, p-q) \mid (p, q) \in \mathcal{P}\}$$

is in $\mathcal{S}$. Then $\mathcal{R}_2 = \pi' \pi_2 \mathcal{P}_{1,2,1-2}$ consists of all $g \in \mathcal{E}$ which coincide with $(p, q, p-q)$ on indices $0, 1, 2$ and which may have arbitrary coordinates elsewhere. Then the intersection $\mathcal{R}_3 = \iota(\mathcal{R}_1, \mathcal{R}_2)$ consists of all sequences g from $\mathcal{R}_1$ in which the 2nd coordinate is replaced by $p - q = g(0) - g(1)$. It remains to apply the permutation α^{-1} to get $\mathcal{P}_{i,j,i-j,k} = \alpha^{-1} \mathcal{R}_3$.

The proof for $P_{i,j,i+j,k}$ is similar. □

4.6 An application of the H-machine

Now we are in position to launch the H-machine to construct the series of sets $\mathcal{B}$ mentioned in examples in Section 3.3 by Higman operations (H). Here we do that for the group $\mathbb{Z}^{\infty}$.

Example 4.11. For the free abelian group $\mathbb{Z}^\infty$ in Example 3.2, we have the set $\mathcal{B}$ of sequences $f_{k,l}$ of length 35 constructed in Example 3.2. The following algorithm constructs $\mathcal{B}$ by operations (H).

(1) Using the single permutation α constructed in Example 3.2, bring the sequences $f_{k,l}$ to simpler form $\alpha f_{k,l}$.

(2) Using Lemma 4.4, obtain the set $\{(1)\}$.

(3) Using Lemma 4.5 ten times, duplicate the 0th coordinate 1 to get the set $\mathcal{A}_1$ consisting of one 11-tuple $(1, \dots, 1)$.

(4) Using Lemma 4.4, obtain the set $\{(-1)\}$.

(5) Using Lemma 4.5 ten times, duplicate the 0th coordinate -1 to get the set $\mathcal{A}_2$ consisting of one 11-tuple $(-1, \dots, -1)$.

(6) By Lemma 4.6, the set $\mathcal{B}_{+,-}$ of all couples $(k, -k)$, $k = 1, 2, \dots,$ is in $\mathcal{S}$. Construct the set $\mathcal{A}_3$ of all 5-tuples $(k, k, -k, -k, -k)$, $k = 1, 2, \dots,$ i.e., duplicate in $\mathcal{B}_{+-}$ the coordinate $-k$ twice by Lemma 4.5, then rotate the resulting set by ρ, duplicate the 0th coordinate k, and then rotate back by ρ and shift by σ.

(7) Similarly, construct the set $\mathcal{A}_4$ of all 5-tuples $(l, l, l, -l, -l)$, $l = 1, 2, \dots$.

(8) The sum $\mathcal{A}_5 = \mathcal{A}_1 + \sigma^{11}\mathcal{A}_2 + \sigma^{22}\mathcal{A}_3 + \sigma^{27}\mathcal{A}_4$ consists of 32-tuples (indexed by $0, 1, \dots, 31$): the above 11-tuples $(1, \dots, 1)$, followed by 11-tuples $(-1, \dots, -1)$, followed by 5-tuple $(k, k, -k, -k, -k)$ and then followed by 5-tuple $(l, l, l, -l, -l)$ with any $k, l = 1, 2, \dots$.

(9) Using Lemma 4.10 on the set $\mathcal{A}_5$ for $i = 22$, $j = 27$, $k = 32$, we adjoin a new 32nd entry (equal to the respective $k - l$) to sequences from A_5. Repeating this step for $k = 33$, adjoin a 33rd entry $k - l$. Call the new set $\mathcal{A}_6$.

(10) Using Lemma 4.10 on $\mathcal{A}_6$ for $i = 27$, $j = 22$, $k = 34$, we adjoin a new 34th entry $l - k$ to all sequences from $\mathcal{A}_6$. That is, we get the set of all sequences $\alpha\mathcal{B}$.

(11) Apply the inverse α^{-1} of the permutation α used in Example 3.2 to get the set $\mathcal{B}$ of all $f_{k,l}$.

(12) As a last step, use the definition in Section 2.4 to replace by Higman operations (H) each of the auxiliary operations

$$\sigma^i,\ \zeta_i,\ \zeta_S,\ \pi_i,\ \pi'_i,\ \tau_{k,l},\ \alpha = \tau_{k_1,l_1} \cdots \tau_{k_m,l_m},\ \epsilon_S,\ \text{addition } +,\ \iota_n$$

that we used in previous steps.

Remark 4.12. Comparing the very similarly structured sequences of Example 3.2, Example 3.4, Example 3.5 or Example 3.7, the reader can see how easy it would be to adapt the above algorithm for free metabelian, soluble, nilpotent groups, for $\mathbb{Q}$ (for [17]), for $\mathbb{C}_{p^\infty}$, or for their direct products including divisible abelian groups, and for any other constructively given subgroups therein.

Acknowledgments. Application of the methods that we present here allows to build a group answering a question of Bridson and de la Harpe on embedding of $\mathbb{Q}$ into a finitely presented group mentioned in the Kourovka notebook [11, Problem 14.10 (a)]. Recently, a direct solution to that problem was found by Belk, Hyde and Matucci in [3]; see Remark 3.6 for details. Another explicit embedding will be given in [17]. It is a pleasure to me to thank the referee of the Journal of Group Theory for careful work and for very helpful and encouraging remarks.

Bibliography

[1] S. Aanderaa, A proof of Higman's embedding theorem using Britton extensions of groups, in: *Word Problems: Decision Problems and the Burnside Problem in Group Theory*, Stud. Logic Found. Math. 71, North-Holland, Amsterdam (1973), 1–18.

[2] S. I. Adyan and V. S. Atabekyan, n-torsion groups, *J. Contemp. Math. Anal.* **54** (2019), no. 6, 319–327.

[3] J. Belk, J. Hyde and F. Matucci, Embedding $\mathbb{Q}$ into a finitely presented group, *Bull. Amer. Math. Soc. (N. S.)* **59** (2022), no. 4, 561–567.

[4] O. Bogopolski, *Introduction to Group Theory*, EMS Textbk. Math., European Mathematical Society, Zürich, 2008.

[5] G. Boolos, J. Burgess and R. Jeffrey, *Computability and Logic*, 4th ed., Cambridge University, Cambridge, 2002.

[6] M. Davis, *Computability and Unsolvability*, McGraw-Hill, New York, 1958.

[7] G. Higman, Subgroups of finitely presented groups, *Proc. Roy. Soc. Lond. Ser. A* **262** (1961), 455–475.

[8] G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups, *J. Lond. Math. Soc.* **24** (1949), 247–254.

[9] D. L. Johnson, *Presentations of Groups*, 2nd ed., London Math. Soc. Stud. Texts 15, Cambridge University, Cambridge, 1997.

[10] M. I. Kargapolov and J. I. Merzljakov, *Fundamentals of the Theory of Groups*, Grad. Texts in Math. 62, Springer, New York, 1979.

[11] E. I. Khukhro and V. D. Mazurov, *The Kourovka Notebook*, 19th ed., Sobolev Institute of Mathematics, Novosibirsk, 2018.

[12] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Ergeb. Math. Grenzgeb. (3) 89, Springer, Berlin, 1977.

[13] V. H. Mikaelian, Metabelian varieties of groups and wreath products of abelian groups, *J. Algebra* **313** (2007), no. 2, 455–485.

[14] V. H. Mikaelian, Subvariety structures in certain product varieties of groups, *J. Group Theory* **21** (2018), no. 5, 865–884.

[15] V. H. Mikaelian, A modified proof for Higman's embedding theorem, preprint (2019), https://arxiv.org/abs/1908.10153.

[16] V. H. Mikaelian, Embeddings determined by universal words in the rank 2 free group, *Sib. Math. J.* **62** (2021), no. 1, 123–130.

[17] V. H. Mikaelian, An explicit embedding of $\mathbb{Q}$ into a finitely presented group, in preparation.

[18] H. Neumann, *Varieties of Groups*, Springer, New York, 1967.

[19] D. J. S. Robinson, *A Course in the Theory of Groups*, 2nd ed., Grad. Texts in Math. 80, Springer, New York, 1996.

[20] H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York, 1967.

[21] J. J. Rotman, *An Introduction to the Theory of Groups*, 4th ed., Grad. Texts in Math. 148, Springer, New York, 1995.

[22] Why is "The Higman Rope Trick" thus named?, https://mathoverflow.net/questions/195011/why-is-the-higman-rope-trick-thus-named.

Received June 15, 2021; revised April 14, 2023

Author information

Corresponding author:
Vahagn H. Mikaelian, Informatics & Applied Mathematics Department,
Yerevan State University, Alex Manoogian 1, Yerevan 0025, Armenia.
E-mail: v.mikaelian@gmail.com, vmikaelian@ysu.am