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Free coarse groups

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Abstract. A coarse group is a group endowed with a coarse structure so that the group multiplication and inversion are coarse mappings. Let $(X, \mathcal{E})$ be a coarse space, and let $\mathfrak{M}$ be a variety of groups different from the variety of singletons. We prove that there is a coarse group $F_{\mathfrak{M}}(X, \mathcal{E}) \in \mathfrak{M}$ such that $(X, \mathcal{E})$ is a subspace of $F_{\mathfrak{M}}(X, \mathcal{E})$, X generates $F_{\mathfrak{M}}(X, \mathcal{E})$ and every coarse mapping $(X, \mathcal{E}) \to (G, \mathcal{E}')$, where $G \in \mathfrak{M}$, $(G, \mathcal{E}')$ is a coarse group, can be extended to coarse homomorphism $F_{\mathfrak{M}}(X, \mathcal{E}) \to (G, \mathcal{E}')$. If $\mathfrak{M}$ is the variety of all groups, the groups $F_{\mathfrak{M}}(X, \mathcal{E})$ are asymptotic counterparts of Markov free topological groups over Tikhonov spaces.

In [4], A. A. Markov proved that, for every Tikhonov space $(X, \mathcal{T})$, there exists a group topology $\mathcal{T}'$ on the free group $F(X)$ in the alphabet X such that $(X, \mathcal{T})$ is a closed subset of $(F(X), \mathcal{T}')$ and every continuous mapping from $(X, \mathcal{T})$ to a topological group G can be extended to continuous homomorphism $(F(X), \mathcal{T}') \to G$. In particular, every Tikhonov space can be embedded as a closed subset into some topological group.

Our purpose is to construct the natural counterparts of Markov free topological groups in the category of coarse groups and coarse homomorphisms. A coarse group is a group endowed with a coarse structure in such a way that the group multiplication and inversion are coarse mappings. All necessary facts about coarse spaces and coarse groups are in Sections 1 and 2, and the construction of free coarse groups is in Section 3.

1 Coarse structures

Following [10], we say that a family $\mathcal{E}$ of subsets of $X \times X$ is a *coarse structure* on a set X if

- each $\varepsilon \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) : x \in X\}$,
- if $\varepsilon, \delta \in \mathcal{E}$, then $\varepsilon \circ \delta \in \mathcal{E}$ and $\varepsilon^{-1} \in \mathcal{E}$, where
$$\varepsilon \circ \delta = \{(x, y) : \text{there exists } z((x, z) \in \varepsilon,\ (z, y) \in \delta)\},$$
$$\varepsilon^{-1} = \{(y, x) : (x, y) \in \varepsilon\},$$
- if $\varepsilon \in \mathcal{E}$ and $\triangle_X \subseteq \varepsilon' \subseteq \varepsilon$, then $\varepsilon' \in \mathcal{E}$.

Each $\varepsilon \in \mathcal{E}$ is called an *entourage* of the diagonal. A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $\varepsilon \in \mathcal{E}$, there exists $\varepsilon' \in \mathcal{E}'$ such that $\varepsilon \subseteq \varepsilon'$.

The pair $(X, \mathcal{E})$ is called a *coarse space*. For $x \in X$ and $\varepsilon \in \mathcal{E}$, we denote $B(x, \varepsilon) = \{y \in X : (x, y) \in \varepsilon\}$ and say that $B(x, \varepsilon)$ is a *ball of radius ε around x*. We note that a coarse space can be considered an asymptotic counterpart of a uniform topological space and could be defined in terms of balls; see [7, 9]. In this case, a coarse space is called a *ballean*.

A coarse space $(X, \mathcal{E})$ is called *connected* if, for any $x, y \in X$, there exists $\varepsilon \in \mathcal{E}$ such that $y \in B(x, \varepsilon)$. A subset Y of X is called *bounded* if there exist $x \in X$ and $\varepsilon \in \mathcal{E}$ such that $Y \subseteq B(x, \varepsilon)$. The coarse structure

$$\mathcal{E} = \{\varepsilon \in X \times X : \triangle_X \subseteq \varepsilon\}$$

is the unique coarse structure such that $(X, \mathcal{E})$ is connected and bounded.

In what follows, all coarse spaces under consideration are supposed to be *connected*.

Given a coarse space $(X, \mathcal{E})$, each subset $Y \subseteq X$ has the natural coarse structure $\mathcal{E}|_Y = \{\varepsilon \cap (Y \times Y) : \varepsilon \in \mathcal{E}\}$, and $(Y, \mathcal{E}|_Y)$ is called a *subspace* of $(X, \mathcal{E})$. A subset Y of X is called *large* (or *coarsely dense*) if there exists $\varepsilon \in \mathcal{E}$ such that $X = B(Y, \varepsilon)$, where $B(Y, \varepsilon) = \bigcup_{y \in Y} B(y, \varepsilon)$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \to X'$ is called *coarse* (or *bornologous* in the terminology of [10]) if, for every $\varepsilon \in \mathcal{E}$, there exists $\varepsilon' \in \mathcal{E}'$ such that, for every $x \in X$, we have $f(B(x, \varepsilon)) \subseteq B(f(x), \varepsilon')$. If f is surjective and coarse, then $(X', \mathcal{E}')$ is called a *coarse image* of $(X, \mathcal{E})$. If f is a bijection such that f and f^{-1} are coarse mappings, then f is called an *asymorphism*. The coarse spaces $(X, \mathcal{E})$, $(X', \mathcal{E}')$ are called *coarsely equivalent* if there exist large subsets $Y \subseteq X$, $Y' \subseteq X'$ such that $(Y, \mathcal{E}|_Y)$ and $(Y', \mathcal{E}'|_{Y'})$ are asymorphic.

To conclude the coarse vocabulary, we take a family $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha < \kappa\}$ of coarse spaces, and we define the *product* $P_{\alpha<\kappa}(X_\alpha, \mathcal{E}_\alpha)$ as the Cartesian product $P_{\alpha<\kappa} X_\alpha$ endowed with the coarse structure with the base $P_{\alpha<\kappa}\mathcal{E}_\alpha$. If $\varepsilon_\alpha \in \mathcal{E}_\alpha$, $\alpha < \kappa$ and $x, y \in P_{\alpha<\kappa} X_\alpha$, $x = (x_\alpha)_{\alpha<\kappa}$, $y = (y_\alpha)_{\alpha<\kappa}$, then $(x, y) \in (\varepsilon_\alpha)_{\alpha<\kappa}$ if and only if $(x_\alpha, y_\alpha) \in \varepsilon_\alpha$ for every $\alpha < \kappa$.

2 Coarse groups

Let G be a group with the identity e. For a cardinal κ, $[G]^{<\kappa}$ denotes the set $\{Y \subseteq G : |Y| < \kappa\}$.

A family $\mathcal{J}$ of subsets of G is called a *group ideal* if $\mathcal{J}$ is closed under formation of subsets and finite unions, $[G]^{<\omega} \subseteq \mathcal{J}$ and $AB^{-1} \in \mathcal{J}$ for all $A, B \in \mathcal{J}$.

A group ideal $\mathcal{J}$ is called *invariant* if $\bigcup_{g \in G} g^{-1}Ag \in \mathcal{J}$ for each $A \in \mathcal{J}$. For example, $[G]^{<\kappa}$ is a group ideal for any infinite cardinal κ. If $\kappa > |G|$, we get the ideal $\mathcal{P}_G$ of all subsets of G. We note also that $[G]^{<\omega}$ is invariant if and only if the set $\{x^{-1}gx : x \in G\}$ is finite for each $g \in G$. By [6], for every countable group G, there are $2^{2^{\omega}}$ distinct group ideals on G.

Let X be a G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$. We assume that G acts on X transitively, take a group ideal $\mathcal{I}$ on G and consider the coarse structure $\mathcal{E}(G, \mathcal{I}, X)$ on X with the base

$$\{\varepsilon_A : A \in \mathcal{I},\ e \in A\}, \quad \varepsilon_A = \{(x, gx) : x \in X, g \in A\}.$$

Then $(x, y) \in \varepsilon_A$ if and only if $y \in Ax$, so $B(x, \varepsilon) = Ax$, $Ax = \{gx : g \in A\}$.

By [5, Theorem 1], for every coarse structure $\mathcal{E}$ on X, there exist a group G of permutations of X and a group ideal $\mathcal{I}$ on G such that $\mathcal{E} = \mathcal{E}(G, \mathcal{I}, X)$.

Now let $X = G$, where G acts on X by left shifts. We denote $\mathcal{E}_{\mathcal{I}} = \mathcal{E}(G, \mathcal{I}, G)$. Thus every group ideal $\mathcal{I}$ on G turns G into the coarse space $(G, \mathcal{E}_{\mathcal{I}})$. We note that a subset A of G is bounded in $(G, \mathcal{E}_{\mathcal{I}})$ if and only if $A \in \mathcal{I}$.

For finitely generated groups, the right coarse groups $(G, \mathcal{E}_{[G]^{<\omega}})$ in metric form play a significant role in *Geometrical Group Theory*; see [1, Chapter 4].

A group G endowed with a coarse structure $\mathcal{E}$ is called a *left (right) coarse group* if, for every $\varepsilon \in \mathcal{E}$, there exists $\varepsilon' \in \mathcal{E}$ such that

$$gB(x, \varepsilon) \subseteq B(gx, \varepsilon') \quad (B(x, \varepsilon)g \subseteq B(xg, \varepsilon')) \quad \text{for all } x, g \in G.$$

A group G endowed with a coarse structure $\mathcal{E}$ is called a *coarse group* if the group multiplication $(G, \mathcal{E}) \times (G, \mathcal{E}) \to (G, \mathcal{E})$, $(x, y) \mapsto xy$ and the inversion $(G, \mathcal{E}) \to (G, \mathcal{E})$, $x \mapsto x^{-1}$ are coarse mappings. In this case, $\mathcal{E}$ is called a *group coarse structure.*

For proofs of the following two statements, see [8] or [9, Section 6].

Proposition 1. *A group G endowed with a coarse structure $\mathcal{E}$ is a right coarse group if and only if there exists a group ideal $\mathcal{I}$ on G such that $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$.*

Proposition 2. *For a group G endowed with a coarse structure $\mathcal{E}$, the following conditions are equivalent:*

(i) $(G, \mathcal{E})$ *is a coarse group.*

(ii) $(G, \mathcal{E})$ *is left and right coarse group.*

(iii) *There exists an invariant group ideal $\mathcal{I}$ on G such that $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$.*

Proposition 3. *Every group coarse structure $\mathcal{E}$ on a subgroup H of an Abelian group G can be extended to a group coarse structure $\mathcal{E}'$ on G.*

Proof. We take a group ideal $\mathcal{I}$ on G such that $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$, denote by $\mathcal{I}'$ the group ideal on G with the base $A + B$, $A \in [G]^{<\omega}$, $B \in \mathcal{I}$, and put $\mathcal{E}' = \mathcal{E}_{\mathcal{I}'}$. □

Example 1. We construct a group G with a normal Abelian subgroup H of index $|G : H| = 2$ such that some group coarse structure $\mathcal{E}$ on H cannot be extended to a right group coarse structure on G. Let $H = \bigoplus_{n\in\mathbb{Z}} C_n$, $C_n \simeq \mathbb{Z}_2$. Every element $a \in H$ can be written as $a = (a_n)_{n\in\mathbb{Z}}$ with $a_n \in C_n$ and $a_n = 0$ for all but finitely many n. We define an automorphism φ of order 2 of H by $\varphi(a_n)_{n\in\mathbb{Z}} = (c_n)_{n\in\mathbb{Z}}$, $c_n = a_{-n}$ for each $n \in \mathbb{Z}$. We put $\langle\varphi\rangle = \{\varphi, id\}$ and consider the semidirect product $G = H \leftthreetimes \langle\varphi\rangle$. If $(h_1, \varphi_1), (h_2, \varphi_2) \in G$, then

$$(h_1, \varphi_1)(h_2, \varphi_2) = (h_1\varphi_1(h_2), \varphi_1\varphi_2).$$

For each $m \in \mathbb{Z}$, we set $H_m = \bigoplus_{n\geq m} H_n$. Then the family $\{H_m : m \in \mathbb{Z}\}$ is a base for some group ideal $\mathcal{I}$ on H. We put $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$ and take an arbitrary invariant group ideal $\mathcal{J}$ on G such that $\mathcal{I} \subset \mathcal{J}$. Since $\varphi H_0 \varphi \cup H_0 = H$, we see that $H \in \mathcal{J}$. It follows that the coarse structure $\mathcal{E}_{\mathcal{J}}|_H$ is bounded, so $\mathcal{E}_{\mathcal{J}}|_H \neq \mathcal{E}$.

Example 2. Let G be an infinite group with only two classes of conjugated elements; see [3]. Then there is only one group coarse structure $\mathcal{E}$ on G, namely, $\mathcal{E} = \mathcal{E}_{\mathcal{P}(G)}$.

3 Free coarse groups

A class $\mathfrak{M}$ of groups is called a *variety* if $\mathfrak{M}$ is closed under formation of subgroups, homomorphic images and products. We assume that $\mathfrak{M}$ is non-trivial (i.e., there exists $G \in \mathfrak{M}$ such that $|G| > 1$) and recall that the *free group* $F_{\mathfrak{M}}(X)$ is defined by the following conditions: $F_{\mathfrak{M}}(X) \in \mathfrak{M}$, $X \subset F_{\mathfrak{M}}(X)$, X generates $F_{\mathfrak{M}}(X)$ and every mapping $X \to G$, $G \in \mathfrak{M}$, can be extended to homomorphism $F_{\mathfrak{M}}(X) \to G$.

Let $(X, \mathcal{E})$ be a coarse space. We assume that $(F_{\mathfrak{M}}(X), \mathcal{E}')$ is a coarse group such that $(X, \mathcal{E})$ is a subspace of $(F_{\mathfrak{M}}(X), \mathcal{E}')$ and every coarse mapping

$$(X, \mathcal{E}) \to (G, \mathcal{E}''), \quad G \in \mathfrak{M},\ (G, \mathcal{E}'') \text{ is a coarse group},$$

can be extended to coarse homomorphism $(F_{\mathfrak{M}}(X), \mathcal{E}') \to (G, \mathcal{E}'')$. We observe that this $\mathcal{E}'$ is unique, denote $F_{\mathfrak{M}}(X, \mathcal{E}) = (F_{\mathfrak{M}}(X), \mathcal{E}')$ and say that $F_{\mathfrak{M}}(X, \mathcal{E})$ is a *free coarse group* over $(X, \mathcal{E})$ in the variety $\mathfrak{M}$.

Our goal is to prove the existence of $F_{\mathfrak{M}}(X, \mathcal{E})$ for every coarse space $(X, \mathcal{E})$ and every non-trivial variety $\mathfrak{M}$.

Lemma 1. *Let $(X, \mathcal{E})$ be a coarse space. If there is a group coarse structure $\mathcal{E}'$ on $F_{\mathfrak{M}}(X)$ such that $\mathcal{E}'|_X = \mathcal{E}$, then there exists $F_{\mathfrak{M}}(X, \mathcal{E})$.*

Proof. We denote

$$\mathfrak{F} = \{\mathcal{T} : \mathcal{T} \text{ is a group coarse structure on } F_{\mathfrak{M}}(X) \text{ such that } \mathcal{T}|_X = \mathcal{E}\}.$$

By the assumption, $\mathcal{E}' \in \mathfrak{F}$. We take the minimal by inclusion group coarse structure $\mathcal{T}'$ on $F_{\mathfrak{M}}(X)$ containing all coarse structures from $\mathfrak{F}$. Let $G \in \mathfrak{M}$, $(G, \mathcal{E}'')$ be a coarse group, $f\colon (X, \mathcal{E}) \to (G, \mathcal{E}'')$ be a coarse mapping. We extend f to homomorphism $f\colon F_{\mathfrak{M}}(X) \to G$. Then the coarse structure on $F_{\mathfrak{M}}(X)$ with the base $\{f^{-1} \times f^{-1}(\varepsilon'') : \varepsilon'' \in \mathcal{E}''\}$ is in $\mathfrak{F}$. It follows that the homomorphism

$$f\colon (F_{\mathfrak{M}}(X), \mathcal{T}') \to (G, \mathcal{E}'')$$

is coarse. Hence $(F_{\mathfrak{M}}(X), \mathcal{T}') = F_{\mathfrak{M}}(X, \mathcal{E})$. □

Lemma 2. *For every coarse space $(X, \mathcal{E})$ and every non-trivial variety $\mathfrak{M}$ of groups, there exists a group coarse structure $\mathcal{E}'$ on $F_{\mathfrak{M}}(X)$ such that $\mathcal{E}'|_X = \mathcal{E}$.*

Proof. For some prime number p, $\mathfrak{M}$ contains the variety $\mathcal{A}_p$ of all Abelian groups of exponent p. We prove the theorem for $\mathcal{A}_p$ and then for $\mathfrak{M}$.

We take the free group $A(X)$ over X in A_p. Every non-zero element $a \in A(X)$ has the unique (up to permutation of items) representation

$$m_1x_1 + m_2x_2 + \cdots + m_kx_k, \quad x_i \in X,\ m_i \in \mathbb{Z}_p \setminus \{0\},\ i \in \{1, \dots, k\}. \tag{3.1}$$

For every $\varepsilon \in \mathcal{E}$, $\varepsilon = \varepsilon^{-1}$, we denote $Y_\varepsilon = \{x - y : x, y \in X,\ (x, y) \in \varepsilon\}$ and by $Y_{n,\varepsilon}$ the sum on n copies of Y_ε. We take $z \in X$ and consider the ideal $\mathcal{J}$ on $A(X)$ with the base

$$Y_{n,\varepsilon} + \{0, z, 2z, \dots, (p-1)z\}, \quad n < \omega.$$

Note that $Y_{n,\varepsilon} - Y_{n',\varepsilon'} \subseteq Y_{n+n',\varepsilon\circ\varepsilon'}$. It follows that $B - C \in \mathcal{J}$ for all $B, C \in \mathcal{J}$. To show that $[F_{\mathfrak{M}}(X)]^{<\omega} \subseteq \mathcal{J}$, we take $x \in X$ and find $\varepsilon \in \mathcal{E}$ such that $(x, z) \in \varepsilon$. Then $x - z \in Y_\varepsilon$ and $x \in Y_\varepsilon + z$. Hence $\mathcal{J}$ is a group ideal. We put $\mathcal{E}' = \mathcal{E}_{\mathcal{J}}$ and show that $\mathcal{E}'|_X = \mathcal{E}$.

If $\varepsilon \in \mathcal{E}$, $\varepsilon = \varepsilon^{-1}$ and $(x, y) \in \varepsilon$, then $x - y \in Y_\varepsilon$, so $\mathcal{E} \subseteq \mathcal{E}'$. To prove the inverse inclusion, we take $Y_{n,\varepsilon} + \{0, z, \dots, (p-1)z\}$, assume that

$$x - y \in Y_{n,\varepsilon} + \{0, z, \dots, (p-1)z\}$$

and consider two cases.

Case: $x - y \in Y_{n,\varepsilon} + iz$, $i \neq 0$. We denote by H the subgroup of all $a \in A(X)$ such that $m_1 + \cdots + m_k = 0 \pmod p$ in the canonical representation (3.1). Then $x - y \in H$, $Y_{n,\varepsilon} \subseteq H$, but $iz \notin H$, so this case is impossible.

Case: $x - y \in Y_{n,\varepsilon}$. We show that $(x, y) \in \varepsilon^n$. We write $x - y$ as

$$(x_1 - y_1) + \cdots + (x_n - y_n), \quad x_i, y_i \in Y_\varepsilon,$$

so $(x_i, y_i) \in \varepsilon$. Assume that there exists $k \in \{1, \dots, n-1\}$ such that

$$\{x_1, y_1, \dots, x_k, y_k\} \cap \{x_{k+1}, y_{k+1}, \dots, x_n, y_n\} = \emptyset.$$

Then

$$\text{either} \qquad (x_1 - y_1) + \dots + (x_k - y_k) = 0,$$

$$\text{or} \qquad (x_{k+1} - y_{k+1}) + \dots + (x_n - y_n) = 0.$$

Otherwise, $x - y$ in representation (3.1) has more than two items. It follows that there is a representation

$$x - y = (x'_1 - y'_1) + \dots + (x'_k - y'_k), \quad x'_i, y'_i \in Y_\varepsilon,\ i \in \{1, \dots, k\},\ k \leq n,$$

such that $\{x'_{i+1}, y'_{i+1}\} \cap \{x'_1, y'_1, \dots, x'_i, y'_i\} \neq \emptyset$ for each $i \in \{1, \dots, k-1\}$. If $(x', y') \in \varepsilon^i$ for all $x', y' \in \{x'_1, y'_1, \dots, x'_i, y'_i\}$ then

$$(x', y') \in \varepsilon^{i+1} \quad \text{for all } x', y' \in \{x'_1, y'_1, \dots, x'_{i+1}, y'_{i+1}\}.$$

After n steps, we get $x - y \in \varepsilon^n$.

To conclude the proof, we extend the mapping id: $X \to X$ to a homomorphism $f\colon F_{\mathfrak{M}}(X) \to A(X)$. Then $\{f^{-1}(Y) : Y \in \mathcal{I}\}$ is a base for some invariant group ideal $\mathcal{J}$ on $F_{\mathfrak{M}}(X)$. Then $(F_{\mathfrak{M}}(X), \mathcal{E}_{\mathcal{J}})$ is a coarse group. Since $f|_X = \text{id}$, we have $\mathcal{E}_{\mathcal{J}}|_X = \mathcal{E}$. □

Theorem. *For every coarse space $(X, \mathcal{E})$ and non-trivial variety $\mathfrak{M}$ of groups, there exists the free coarse group $F_{\mathfrak{M}}(X, \mathcal{E})$.*

Proof. Apply Lemma 2 and Lemma 1. □

Remark 1. To describe the coarse structure $\mathcal{E}^*$ of $F_{\mathfrak{M}}(X, \mathcal{E})$ explicitly, for every $\varepsilon \in \mathcal{E}$, we put $\mathcal{D}_\varepsilon = \{xy^{-1} : x, y \in X, (x, y) \in \varepsilon\}$, take $z \in X$ and denote by $P_{n,\varepsilon}$ the product on n copies of the set

$$\bigcup_{g \in F_{\mathfrak{M}}(X)} g^{-1}(\mathcal{D}_\varepsilon \bigcup \mathcal{D}_\varepsilon z)g.$$

Then $\{P_{n,\varepsilon} : \varepsilon \in \mathcal{E}, n < \omega\}$ is a base for some invariant group ideal $\mathcal{I}^*$ on $F_{\mathfrak{M}}(X)$. Each subset $A \in \mathcal{I}^*$ is bounded in $F(X, \mathcal{E})$, so $\mathcal{E}_{\mathcal{I}^*} \subseteq \mathcal{E}^*$. To see that $\mathcal{E}^* \subseteq \mathcal{E}_{\mathcal{I}^*}$, the reader can repeat the arguments concluding the proof of Lemma 2. Hence $\mathcal{E}^* = \mathcal{E}_{\mathcal{I}^*}$.

Remark 2. Each metric space (X, d) defines the coarse structure $\mathcal{E}_d$ on X with the base $\{(x, y) : d(x, y) < n\}$, $n < \omega$. By [9, Theorem 2.1.1], a coarse structure $\mathcal{E}$ is metrizable if and only if $\mathcal{E}$ has a countable base. If $\mathcal{E}$ is metrizable then, in view of Remark 1, the coarse structure of $F_{\mathfrak{M}}(X, \mathcal{E})$ is metrizable.

Remark 3. If the coarse spaces $(X, \mathcal{E})$, $(X, \mathcal{E}')$ are asymorphic, then evidently $F_{\mathfrak{M}}(X, \mathcal{E})$, $F_{\mathfrak{M}}(X', \mathcal{E}')$ are asymorphic, but this is not true with coarse equivalences in place of asymorphisms.

Let $\mathfrak{M} = \mathcal{A}_p$, and let X be an infinite set endowed with the bounded coarse structure $\mathcal{E}$. We take $X', |X'| = 1$ and denote by $\mathcal{E}'$ the unique coarse structure on X'. Clearly, $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ are coarsely equivalent, and $F_{\mathfrak{M}}(X', \mathcal{E}')$ is a cyclic group of order p with bounded coarse structure. To see that $F_{\mathfrak{M}}(X)$ is unbounded, we take the subset $Y_{n,\varepsilon}$ (see the proof of Lemma 2) and note that the length of any element from $Y_{n,\varepsilon}$ in representation (3.1) does not exceed $2n$, but $F_{\mathfrak{M}}(X)$ has elements of any length.

Remark 4. Let X be a Tikhonov space with distinguished point x_0. M. I. Graev [2] defined a group topology on $F(X \setminus \{x_0\})$ in such a way that X is a closed subset of $F(X \setminus \{x_0\})$, $x_0 = e$, and every continuous mapping $f\colon (X) \to G$, $f(x_0) = e$, G is a topological group, can be extended to continuous homomorphism

$$F(X \setminus \{x_0\}) \to G.$$

Let $(X, \mathcal{E})$ be a coarse space with distinguished point x_0, $Y = X \setminus \{x_0\}$ and $\mathcal{E}' = \mathcal{E}|_Y$. We take the free coarse group $F(Y, \mathcal{E}')$ and note that $\{e\} \cup Y$ is asymorphic to $(X, \mathcal{E})$ via the mapping $h(y) = y$, $y \in Y$ and $h(e) = x_0$. Hence it does not make sense to define the coarse counterparts of the Graev free topological groups.

Bibliography

[1] P. de la Harpe, *Topics in Geometric Group Theory*, University Chicago, Chicago, 2000.

[2] M. I. Graev, Free topological groups, *Izv. Acad. Nauk SSSR* **12** (1948), 279–324; translation in *Amer. Math. Soc. Transl.* **35** (1951).

[3] G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups, *J. Lond. Math. Soc.* **24** (1949), 247–254.

[4] A. A. Markov, On free topological groups, *Izv. Acad. Nauk SSSR* **9** (1945), 3–64; translation in *Amer. Math. Soc. Transl.* **30** (1950), 11–88.

[5] O. V. Petrenko and I. V. Protasov, Balleans and G-spaces, *Ukrainian Math. J.* **64** (2012), 387–393.

[6] I. V. Protasov, Counting Ω-ideals, *Algebra Universalis* **62** (2010), 339–343.

[7] I. V. Protasov and T. Banakh, *Ball Structures and Colorings of Groups and Graphs*, Math. Stud. Monogr. Ser. 11, VNTL, Lviv, 2003.

[8] I. V. Protasov and O. I. Protasova, Sketch of group balleans, *Math. Stud.* **22** (2004), 10–20.

[9] I. V. Protasov and M. Zarichnyi, *General Asymptopogy*, Math. Stud. Monogr. Ser. 12, VNTL, Lviv, 2007.

[10] J. Roe, *Lectures on Coarse Geometry*, Univ. Lecture Ser. 31, American Mathematical Society, Providence, 2003.

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