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Some generalized characters associated to a transitive permutation group

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Abstract. Let G be a finite transitive permutation group of degree n, with point stabilizer $H \neq 1$ and permutation character π. For every positive integer t, we consider the *generalized* character $\psi_t = \rho_G - t(\pi - 1_G)$, where ρ_G is the regular character of G and 1_G the 1-character. We give necessary and sufficient conditions on t (and G) which guarantee that ψ_t is a character of G. A necessary condition is that $t \leq \min\{n-1, |H|\}$, and it turns out that ψ_t is a character of G for $t = n-1$ resp. $t = |H|$ precisely when G is 2-transitive resp. a Frobenius group.

Suppose G is a transitive permutation group of finite degree $n \geq 2$, with associated permutation character $\pi = \pi(G)$ (which counts fixed points). Let H be a point stabilizer (so that $|G : H| = \pi(1) = n$), and let $\rho = \rho_G$ and 1_G denote the regular and 1-character of G, respectively. For every positive integer t, we consider the generalized character

$$\psi_t = \rho - t(\pi - 1_G)$$

of G. Since $\pi = \mathrm{Ind}_H^G(1_H)$ is part of $\rho = \mathrm{Ind}_1^G(1)$ by transitivity of character induction, for $t = 1$, this $\psi_t = \rho - \pi + 1_G$ is an (ordinary) character of G. We are going to examine when this happens in general.

This work is motivated by the note [6] of the authors on Frobenius groups (where $t = |H|$). Unfortunately, there is a misprint in that note (due to typesetting). One of the objectives of the present article is to correct this and to make the corresponding result somewhat more precise. One may ask whether G can be found such that ψ_t is a character of G for some $t > |H|$, or when $t = n$. Both questions will be answered in the negative.

If G is regular ($H = 1$), then $\rho = \pi$ and $\psi_t = t1_G - (t-1)\rho$ is a character only when $t = 1$ (as $n \geq 2$ and so $\pi \neq 1_G$). Excluding this trivial case, we assume throughout that $|H| > 1$. We let $\mathrm{Irr}(G)$ denote the set of (complex) irreducible characters of G. Recall that the cardinality $|\mathrm{Irr}(G)| = k(G)$ is the class number of G.

Since G is transitive, the multiplicity (scalar product) $\langle 1_G, \pi\rangle = 1$ (Huppert [3, Satz V.20.2]). There is an irreducible character $\xi \neq 1_G$ of G which occurs in π, and we let Ξ be the (non-empty) set of these irreducible characters of G. Of course, $\Xi = \Xi(G)$ is determined by the permutation group G, as is

$$t^* = t^*(G) = \min_{\xi \in \Xi} \left\lfloor \frac{\xi(1)}{\langle \xi, \pi \rangle} \right\rfloor.$$

We have $\langle \xi, \pi \rangle \leq \langle \xi, \rho \rangle = \xi(1)$ for all $\xi \in \Xi$. Hence t^* is a positive integer.

Note. By definition, $\Xi = \Xi(G)$ and $t^* = t^*(G)$ actually depend only on the permutation character $\pi = \pi(G)$. Observe that G is determined by π as a linear group but in general not as a permutation group. Indeed, Wielandt's example in [3, V.20.10] shows that it can happen that $\pi = \mathrm{Ind}_K^G(1_K)$ for some subgroup K of G which is not G-conjugate to H. When $n = p$ is a prime, all such possibilities are listed by Feit [1] (in view of the classification theorem; see also [4, XII.10.10] and [4, XII.10.11]). The case $n = p^2$ has been treated by Guralnick [2]. When G is a Frobenius group to H, π determines G as a permutation group since any complement in G to the Frobenius kernel is G-conjugate to H by the Schur–Zassenhaus theorem (see below).

Lemma. *For $\xi \in \Xi$, we have $\langle \xi, \psi_t \rangle = \xi(1) - t\langle \xi, \pi \rangle$, whereas $\langle \chi, \psi_t \rangle = \chi(1)$ whenever $\chi \in \mathrm{Irr}(G) \setminus \Xi$. Hence ψ_t is a character of G if and only if $t \leq t^*$.*

Proof. Let $\chi \in \mathrm{Irr}(G)$ be an irreducible character of G. Since the scalar product is hermitian, and since $\langle \chi, \rho \rangle = \chi(1)$ and $\langle 1_G, \pi \rangle = 1$, we have $\langle 1_G, \psi_t \rangle = 1$ and

$$\langle \chi, \psi_t \rangle = \chi(1) - t\langle \chi, \pi \rangle$$

when $\chi \neq 1_G$. By definition, $\langle \chi, \pi \rangle = 0$ if and only if $\chi \notin \Xi$ and $\chi \neq 1_G$, in which case $\langle \chi, \psi_t \rangle = \chi(1)$. It follows that ψ_t is a character of G precisely when $t\langle \xi, \pi \rangle \leq \xi(1)$ for all $\xi \in \Xi$, that is, when the positive integer $t \leq t^*$. □

Let us write $\psi^* = \psi_t$ for $t = t^*$. By the lemma, this $\psi^* = \psi^*(G)$ is a character of G; we shall describe that explicitly in two extremal cases.

Theorem 1. *We have $t^* \leq n - 1$, and $t^* = n - 1$ if and only if G is 2-transitive. In this extremal case ($t^* = n - 1$), the following hold:*

(i) *We have $\Xi = \{\xi\}$ for some unique irreducible character $\xi \neq 1_G$ of G, with $\xi(1) = n - 1$ dividing $|H|$.*

(ii) $\psi^* = \sum_{\chi \in \mathrm{Irr}(G) \setminus \Xi} \chi(1)\chi$.

Proof. The integer $t^* = t^*(G)$ is as large as possible just when $\pi = 1_G + \xi$ for some irreducible character $\xi \neq 1_G$ of G. Then $t^* = \xi(1) = n - 1$ and G is 2-transitive [3, Satz V.20.2]) so that t^* is a divisor of $|H|$. In general, $t^* \leq n - 1$, and if $t = n - 1$, then $\psi_t = \rho - (n - 1)\xi = \sum_{\chi \neq \xi} \chi(1)\chi$ is a character of G. Hence, then $t = t^*$ and $\psi^* = \psi_t$ is as claimed. □

One can have $t^* = \lfloor \frac{n-1}{2} \rfloor$ when G is a suitable rank 3 permutation group, but it cannot happen that $\lfloor \frac{n-1}{2} \rfloor < t^* < n-1$. If $n = p$ is a prime, by a celebrated theorem of Burnside, G is 2-transitive or a (solvable) Frobenius group (for proofs, see [3, Satz V.21.3] and [4, Theorem XII.10.8]). In this case, $t^* = p - 1$ or $t^* = |H|$, with $|H|$ dividing $p - 1$ (see below).

Let $D = \mathrm{Der}(G) = G \setminus \bigcup_{g \in G} H^g$ denote the (normal) subset of *derangements* in G, the elements acting fixed-point-freely ($H^g = g^{-1}Hg$). Thus

$$D = \{x \in G \mid \pi(x) = 0\},$$

and if $\pi = \mathrm{Ind}_K^G(1_K)$ for some subgroup K of G, then $\bigcup_{g \in G} K^g = \bigcup_{g \in G} H^g$ by definition of induced characters. It is known and easy to see that $|D| \geq n - 1$ (Jordan) and that $|D| \geq |H|$ (Cameron–Cohen); see for instance [6, Theorems 1 and 2].

The transitive permutation group G is a Frobenius group (to $H \neq 1$) provided $|D| = n - 1$, i.e. if $H \cap H^g = 1$ for each $g \in G \setminus H$. The famous theorem of Frobenius [3, Hauptsatz V.7.6] tells us that then $D \cup \{1\}$ is a (normal) subgroup of G (Frobenius kernel). The Frobenius kernel has order n, whereas the complement H has order dividing $n - 1$ since all nontrivial H-orbits have size $|H|$.

Theorem 2. *We have $t^* \leq |H|$, and $t^* = |H|$ if and only if G is a Frobenius group. In this extremal case ($t^* = |H|$), one has $k(G) = k(H) + k_G(D)$, where $k_G(D)$ is the number of G-conjugacy classes in D, and the following hold:*

(i) $|\Xi| = k_G(D)$ *and* $\frac{\xi(1)}{\langle \xi, \pi \rangle} = |H|$ *for each* $\xi \in \Xi$.

(ii) $\psi^* = \sum_{\chi \in \mathrm{Irr}(G) \setminus \Xi} \chi(1)\chi$ *and* $\mathrm{Ker}(\psi^*) = D \cup \{1\}$ *is the Frobenius kernel of G. Moreover,* $\mathrm{Res}_H^G(\psi^*) = \rho_H$ *is the regular character of H, and* $|\mathrm{Irr}(G) \setminus \Xi| = k(H)$ *is equal to the class number of H.*

Proof. Assume that $t^* > |H|$. Then, by the lemma, there is an integer $r \geq 1$ such that ψ_t is a character of G for $t = |H| + r$. Using that $|G| = |H|n$, we then have

$$\psi_t(1) = |H|n - (|H| + r)(n - 1) > 0.$$

It follows that $|H| + r < |H| \cdot \frac{n}{n-1}$, and this yields the estimate $r(n - 1) < |H|$. On the other hand, we know from Theorem 1 that $|H| + r = t \leq n - 1$. We obtain that $rn = r(n - 1) + r < |H| + r \leq n - 1$, which is impossible.

Suppose that $t^* = |H|$. Recall that $H \neq 1$ by assumption. We know from the lemma that $\psi^* = \psi^*(G)$ is a character of G ($\psi^* = \psi_t$ for $t = t^*$). Hence the kernel

$$F = \mathrm{Ker}(\psi^*) = \{x \in G \mid \psi^*(x) = \psi^*(1)\}$$

is a normal subgroup of G. We have

$$\psi^*(1) = |G| - |H|(n - 1) = |H|$$

(as $|G| = |H|n$), and $\psi^*(x) = |H|$ if and only if $x = 1$ or $\pi(x) = 0$ (as $\rho(x) = 0$ for $x \neq 1$). Hence $F = D \cup \{1\}$. It follows that $F \cap H = 1$. Since

$$|D| \geq n - 1 = |G : H| - 1,$$

we get $G = FH$ and $|D| = n - 1$. Thus G is a Frobenius group, with Frobenius kernel F.

From now on, we suppose that G is a Frobenius group to H, and we write $F = D \cup \{1\}$ (an n-set) and $\psi = \psi_t$ for $t = |H|$. In order to show that $t^* = |H|$ and $\psi = \psi^*$, we have to verify that ψ is a character of G. As before, $\psi(1) = |H|$ and $\psi(x) = |H|$ if and only if $x \in F$. Since by hypothesis no element $\neq 1$ of G has more than one fixed point, by definition,

$$\psi(x) = \begin{cases} |H| & \text{if } x \in F, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the restriction $\mathrm{Res}^G_H(\psi) = \rho_H$ is the regular character of H. We claim that the assignments $h^H \mapsto h^G$ for $h \in H$ give rise to a bijection from the conjugacy classes of H to those of G which do not meet D. Indeed, if $h^G = h_0^G$ for nontrivial elements h, h_0 in H, then $h = h_0^g$ for some $g \in G$, and this forces that $g \in H$ and $h^H = h_0^H$. This shows that $k(G) = k(H) + k_G(D)$. For any irreducible character χ of G, by the above,

$$s_\chi := \langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{x \in F} \chi(x)|H| = \frac{1}{n} \sum_{x \in F} \chi(x).$$

By the lemma, $\langle 1_G, \psi \rangle = 1$. So let $\chi \neq 1_G$. It has been shown in the course of the proof for [6, Theorem 3], by elementary means based on the Cauchy–Schwarz inequality, that $s_\chi \neq 0$ implies that $\chi(x) = \chi(1)$ for all $x \in F$. (The misprint mentioned above appears on page 397, line 5, where parentheses are not put correctly.) Thus $s_\chi \neq 0$ implies that $\mathrm{Ker}(\chi) \supseteq F$ and that $s_\chi = \chi(1)$. Consequently, ψ is a character of G, and $F = \mathrm{Ker}(\psi)$ is a normal subgroup of G (Frobenius kernel).

Let again $\chi \neq 1_G$ be an irreducible character of G. We know that from $s_\chi \neq 0$ it follows that $s_\chi = \chi(1)$, and

$$\chi(1) = s_\chi = \langle \chi, \rho \rangle - |H|\langle \chi, \pi \rangle = \chi(1) - |H|\langle \chi, \pi \rangle$$

implies that $\langle \chi, \pi \rangle = 0$ and so $\chi \notin \Xi$. Conversely, if $\langle \chi, \pi \rangle = 0$, then $s_\chi = \chi(1)$. So Ξ consists precisely of the irreducible characters ξ of G satisfying $\langle \xi, \psi \rangle = 0$, which implies that $\xi(1) = |H|\langle \xi, \pi \rangle$ for all $\xi \in \Xi$.

The number $|\mathrm{Irr}(G) \setminus \Xi| = k(G) - |\Xi|$ of irreducible characters of G occurring in ψ is just the number of irreducible characters having F in its kernel. Using that $G/F \cong H$, we obtain that $k(G) - |\Xi| = |\mathrm{Irr}(H)| = k(H)$, which in turn yields that $|\Xi| = k_G(D)$ (see [3, Satz 16.13] for a corresponding result). The proof is complete. $\square$

Remark. Let G be a Frobenius group (to H), and let $F = \mathrm{Der}(G) \cup \{1\}$ (an n-set). We know that $\psi = \rho - |H|(\pi - 1_G)$ is a character of G with $\mathrm{Ker}(\psi) = F$ and $\mathrm{Res}_H^G(\psi) = \rho_H$. Note that $\mathrm{Ind}_H^G(\rho_H) = \rho = \rho_G$. Associate to every character θ of H the generalized character

$$\widehat{\theta} = \mathrm{Ind}_H^G(\theta) - \theta(1)(\pi - 1_G)$$

of G. This the unique class function of G extending θ and satisfying $\widehat{\theta}(x) = \theta(1)$ for all $x \in F$. Using that $G/F \cong H$, we see that $\widehat{\theta}$ is a character of G. In particular, $\widehat{\theta} = \psi$ for $\theta = \rho_H$, and if χ is an irreducible character of G occurring in ψ, then $\mathrm{Ker}(\chi) \supseteq F$ and $\mathrm{Res}_H^G(\chi) = \theta$ is irreducible, and we have $\chi = \widehat{\theta}$ and $\langle \chi, \psi \rangle = \chi(1) = \theta(1) = \langle \theta, \rho_H \rangle$.

Of course, one can establish the theorem of Frobenius by showing directly that, for any $\theta \in \mathrm{Irr}(H)$, the class function $\widehat{\theta}$ is an irreducible character of G, and this is the approach given in the proof for [5, Theorem 7.2].

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