Isomorphisms of subcategories of fusion systems of blocks and Clifford theory

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Abstract. Let k be an algebraically closed field of prime characteristic p. Let G be a finite group, let N be a normal subgroup of G, and let c be a G-stable block of kN so that (kN)c is a p-permutation G-algebra. As in Section 8.6 of [M. Linckelmann, The Block Theory of finite Group Algebras: Volume 2, London Math. Soc. Stud. Texts 92, Cambridge University, Cambridge, 2018], a (G,N,c)-Brauer pair (R,f_R) consists of a p-subgroup R of G and a block f_R of $(kC_N(R))$. If G is a defect group of G and G and

Our notation and terminology are standard and tend to follow the notation and terminology of [2–4].

Let k be an algebraically closed field of prime characteristic p. Let G be a finite group, let $N \subseteq G$, and let $c \in B\ell(kN)^G$. Thus we are in the situation of fusion systems of blocks and Clifford theory of [3, Section 8.6]. This section is devoted to a study of fusion systems arising from the G-algebra (kN)c and the results of [2, Section 5.9].

Thus, by [3, Definition 8.6.1], a (G, N, c)-Brauer pair is a pair (R, f_R) , where R is a p-subgroup of G and $f_R \in B\ell(kC_N(R)Br_R(c))$. The results of [3, Section 8.6] imply that the set of (G, N, c)-Brauer pairs is a (finite) poset under a defined inclusion. Moreover, by [3, Definition 8.6.2 and Theorem 8.6.3], if (P, f_P) is a maximal (G, N, c)-Brauer pair, then $\mathcal{F}_{(P, f_P)}(G, N, c)$ is a fusion system on P and $(PN)/N \in Syl_p(G/N)$. Note that all maximal (G, N, c)-Brauer pairs are G-conjugate.

Let Q be a defect group of $c \in B\ell(kN)$. Thus $Br(c) \in B\ell(kN_N(Q))$ with defect group Q and, [4, Theorem 5.5.15], $Br_Q(c)$ corresponds to an orbit Γ of $N_H(Q)$ on $B\ell(kC_N(Q))$ such that Γ is the set of blocks of $kC_N(Q)$ that are

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covered by $B\ell_Q(c)$ and $Br_Q(c) = \sum_{\gamma \in \Gamma} \gamma$ is an orthogonal sum of blocks of $kC_N(Q)$. Thus $\{(Q,\gamma) \mid \gamma \in \Gamma\}$ is the set of (G,N,c)-Brauer pairs with first component Q. Let $f_Q \in \Gamma$.

Let (R, f_R) be a maximal (G, N, c)-Brauer pair. Then, as in [3, Section 8.6], $R \cap N$ is a defect group of $c \in B\ell(kN)^G$. Then there is an $x \in N$ such that ${}^x(R, f_R) = ({}^xR, {}^xf_R)$ is maximal (G, N, c)-Brauer pair that contains (Q, f_O) .

Let (P, f_P) be a maximal (G, N, c)-Brauer pair that contains (Q, f_Q) . Thus $P \cap N = Q$ and $Q \leq P \leq N_G(P) \leq N_G(Q)$.

Set

$$H = N_G(Q, f_Q)$$

so that Q extleq H, $C_N(Q) extleq H$ and (Q, f_Q) is an $(H, C_N(Q), f_Q)$ -Brauer pair. Thus if (U, f_U) is a maximal $(H, C_N(Q), f_Q)$ -Brauer pair that contains (Q, f_Q) , then $\mathcal{F}_{(U, f_U)}(H, C_N(Q), f_Q)$ is a fusion system such that

$$(UC_N(Q))/C_N(Q) \in Syl_p(H/C_N(Q)).$$

Set

$$\Phi(G, N, c \mid (Q, f_Q)) = \{(R, f_R) \mid (R, f_R) \text{ is a } (G, N, c)\text{-Brauer pair such that } (Q, f_Q) \leq (R, f_R)$$
 as (G, N, c) -Brauer pairs}.

Clearly, $(Q, f_Q) \in \Phi(G, N, c \mid (Q, f_Q))$. Similarly, $\Phi(H, C_N(Q), f_Q \mid (Q, f_Q))$ is defined and

$$(Q,f_Q)\in\Phi(H,C_N(c),f_Q\mid (Q,f_Q)).$$

Remark 1. If $(R, f_R) \in \Phi(G, N, c \mid (Q, f_Q))$, then we may choose a maximal (G, N, c)-Brauer pair containing (R, f_R) so that $R \cap N = Q$. Thus

$$Q \le R \le N_G(R) \le N_G(Q)$$
.

In fact, $N_G(R, f_R) \leq N_G(Q, f_Q)$ by [2, Proposition 5.9.9]. A similar result holds for $\Phi(H, C_N(Q), f_Q \mid (Q, f_Q))$.

As suggested by [3, Lemma 8.6.4]:

Lemma 2. Let S be a p-subgroup of G such that $Q \leq S$, and let $e \in B\ell(kC_N(S))$. The following two conditions are equivalent:

(a)
$$(S, e) \in \Phi(G, N, c \mid (Q, f_Q));$$

(b)
$$(S, e) \in \Phi(H, C_N(Q), f_Q \mid (Q, f_Q)).$$

Proof. Assume (a). Then $Q \subseteq S$, and f_Q is the unique S-stable block of $kC_N(Q)$ such that $Br_S(f_Q)e = e$. Since $C_N(S) = C_{C_N(Q)}(S)$, (b) holds. The proof of the converse is similar.

Let S, T be p-subgroups of G such that $Q \leq S \leq T$, and let $f_S \in B\ell(kC_N(S))$ and $f_T \in B\ell(kC_N(T))$ be such that

$$(S, f_S) \in \Phi(G, N, c \mid (Q, f_Q)),$$

 $(T, f_T) \in \Phi(G, N, c \mid (Q, f_Q)).$

Thus

$$(S, f_S) \in \Phi(H, C_N(Q), f_Q \mid (Q, f_Q)),$$

 $(T, f_T) \in \Phi(H, C_N(Q), f_Q \mid (Q, f_Q)).$

Lemma 3. The following two conditions are equivalent:

- (a) $(S, f_S) \leq (T, f_T)$ as (G, N, c)-Brauer pairs;
- (b) $(S, f_S) \leq (T, f_T)$ as $(H, C_N(Q), f_Q)$ -Brauer pairs.

Proof. Clearly, we may assume that Q < S < T. We proceed by induction on |T:Q|. Thus we may assume that T is a minimal subgroup of H such that there is a subgroup S of T and a (G, N, c)-Brauer pair (S, f_S) such that (a) and (b) are not equivalent for (S, f_S) and (T, f_T) .

Clearly, there is a subgroup R of T such that $Q < S \le R \le T$ with R < T. Let $f_R \in B\ell(kC_N(R))$ be the unique (G,N,c)-Brauer pair so that $(R,f_R) \le (T,f_T)$ as (G,N,c)-Brauer pairs. Thus f_R is the unique T-stable block of $kC_N(R)$ such that $Br_T(f_R)f_T = f_T$. Hence (R,f_R) is an $(H,C_N(Q),f_Q)$ -Brauer pair such that $(R,f_R) \le (T,f_T)$ as $(H,C_N(Q),f_Q)$ -Brauer pairs.

Assume (a). Thus $(S, f_S) \leq (R, f_R)$ as (G, N, c)-Brauer pairs. Since we have |R:Q| < |T:Q|, we conclude that $(S, f_S) \leq (R, f_R)$ as $(H, C_N(Q), f_Q)$ -Brauer pairs. Thus (b) holds.

Assume (b). Then the same argument shows that (a) holds. This contradiction establishes the lemma.

Thus we have:

Corollary 4. Let
$$(U, f_U) \in \Phi(G, N, c \mid (Q, f_Q) \text{ so that}$$

 $(U, f_U) \in \Phi(H, C_N(Q), f_Q) \mid (Q, f_Q)).$

The following two condition are equivalent:

- (a) (U, f_U) is a maximal (G, N, c)-Brauer pair;
- (b) (U, f_U) is a maximal $(H, C_N(Q), f_Q)$ -Brauer pair.

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Let (U, f_U) satisfy (a) and (b) of Corollary 4. Then $\mathcal{F}_{(U, f_U)}(G, N, c)$ and $\mathcal{F}_{(U, f_U)}(H, C_N(Q), f_Q)$ are fusion systems by [3, Theorem 8.6.3].

Proposition 5. Let

$$(S, f_S) \in \Phi(G, N, c \mid (Q, f_Q))$$

be such that $(S, f_S) \leq (U, f_U)$ as (G, N, c)-Brauer pairs. Also let (T, f_T) be a (G, N, c)-Brauer pair such that $(T, f_T) \leq (U, f_U)$ as (G, N, c)-Brauer pairs, and let $x \in G$ be such that ${}^x(S, f_S) \leq (T, f_T)$ as (G, N, c)-Brauer pairs. Then

- (a) $x \in H \ and \ (T, f_T) \in \Phi(G, N, c \mid (Q, f_Q));$
- (b) $^{x}(S, f_S) \leq (T, f_T)$ as $(H, C_N(Q), f_O)$ -Brauer pairs.

Proof. Here ${}^x(Q, f_Q) \le (T, f_T) \le (U, f_U)$ and $(Q, f_Q) \le (U, f_U)$. Since we have ${}^xQ \le U \cap N = Q$, both (a) and (b) hold.

Our main result is: Let \mathcal{C} denote the full subcategory of $\mathcal{F}_{(U,f_U)}(G,N,c)$ of objects in $\Phi(G,N,c\mid (Q,f_O))$, and let \mathcal{D} denote the full subcategory of

$$\mathcal{F}_{(U,f_U)}(H,C_N(Q),f_Q\mid (Q,f_Q))$$

of objects in $\Phi(H, C_N(Q), f_O \mid (Q, f_O))$. Thus we have:

Theorem 6. The identity functor $I: \mathcal{D} \to \mathcal{C}$ is an isomorphism of categories.

Our last results relate to blocks.

Lemma 7. Let $B \in B\ell((kG)c)$ so that Bc = B. Then there is a defect group P of B such that $P \cap N = Q$ and a block f_P of $kC_N(P)$ such that

$$(P, f_P) \in \Phi(G, N, c \mid (Q, f_Q)).$$

Proof. We may choose a defect group R of B such that $R \cap N = Q$. Hence $N_G(R) \leq N_G(Q)$, $C_G(R) \leq C_G(Q)$ and $C_N(R) \leq C_N(Q)$. Here we have that $Br_R(B) \in B\ell(kN_G(R))$ has defect group R and $C_N(R) \leq N_G(R)$. Thus there is an orbit Δ of $N_G(R)$ on $B\ell(kC_N(R))$ such that Δ consists of the blocks of $kC_N(R)$ that are covered by $Br_R(B)$ and

$$Br_R(B) = Br_R(B) \Big(\sum_{\delta \in \Delta} \delta \Big) = Br_R(B) B_R(c) = B_R(B) \Big(\sum_{\delta \in \Delta} \delta \Big) \Big(\sum_{\gamma \in \Gamma} B_R(\gamma) \Big).$$

Thus there is an $x \in N_N(Q)$ and a $\delta \in \Delta$ so that $Br_{R}(B)(^x\delta)Br_{R}(f_Q) \neq 0$. Here xR is a defect group of B,

$$({}^{x}R) \cap N = Q$$
, ${}^{x}\delta \in B\ell(kC_N({}^{x}R))$ and $({}^{x}\delta)Br_{R}(f_Q) \neq 0$.

Since $f_Q \in B\ell(kC_N(Q))$ and $C_N(^xR) \le C_N(Q)$, $(^x\delta) Br_x(f_Q) = ^x\delta$. Since xR fixes $Br_Q(c)$, it permutes Γ . Thus, since $Br_x(f_Q) \ne 0$, xR fixes f_Q . Thus $(^xR, ^x\delta)$ is a (G, N, c)-Brauer pair, $(Q, f_Q) \le (^xR, ^x\delta)$ and xR is a defect group of B. The proof is complete.

Let $B \in B\ell((kG)c)$. By [1, Theorem], there is a unique

$$\beta \in B\ell(kN_G(Q)Br_O(c))$$
 such that $\beta^G = B$.

Here $\beta \operatorname{Br}_Q(c) = \beta$ and $\operatorname{Br}_Q(c) = \sum_{\gamma \in \Gamma} \gamma$. Since $C_N(Q) \leq N_G(Q)$, we have $\Gamma \subseteq \operatorname{B\ell}(kC_N(Q))$, and so β covers f_Q . Since $H = N_G(Q, f_Q)$, there is a unique $b \in \operatorname{B\ell}(kH) f_Q$ such that $\operatorname{Tr}_H^{N_G(Q)}(b) = \beta$. Hence $(\operatorname{Tr}_H^{N_G(Q)}(b))^G = B$.

Here, by [4, Corollary 5.5.6], we have $f_Q \in B\ell(kQC_N(Q))^H$. Thus b is an $(H, QC_N(Q), f_Q)$ -Brauer pair since $bf_Q = b$. We assert that Q is a defect group of $f_Q \in B\ell(kQC_N(Q))$. Indeed, $Br_Q(c) \in B\ell(kN_N(Q))$ with defect group Q. Here $Br_Q(c) = \sum_{\gamma \in \Gamma} \gamma$ is an orthogonal decomposition of blocks of $kC_N(Q)$ such that $f_Q \in \Gamma$. Also if $\gamma \in \Gamma$, then $\gamma \in B\ell(kQC_N(Q))$ by [4, Corollary 5.5.6] and $QC_N(Q) \leq N_N(Q)$. Thus $Br_Q(c)$ covers γ for any $\gamma \in \Gamma$. Hence we have that $Q \cap (QC_N(Q)) = Q$ is a defect group of γ for any $\gamma \in \Gamma$. Thus Q is a defect group of $f_Q \in B\ell(kQC_N(Q))$, where $QC_N(Q) \leq H$, $f_Q \in B\ell(kQC_N(Q))^H$.

By Lemma 7, there is a defect group D of b such that $D \cap QC_N(Q) = Q$ and a block f_D of $C_{QC_N(Q)}(D)$ such that $(Q, f_Q) \leq (D, f_D)$ as $(H, QC_N(Q), f_Q)$ -Brauer pairs. Here $C_{QC_N(Q)}(D) \leq C_{QC_N(Q)}(Q) = C_N(Q)$, and so

$$C_{OC_N(O)}(D) = C_{C_N(O)}(D) = C_N(D).$$

Thus $Br_D(f_Q)f_D = f_D$ and (D, f_D) is an $(H, C_N(Q), f_Q)$ -Brauer pair such that $(Q, f_O) \leq (D, f_D)$ as $(H, C_N(Q), f_O)$ -Brauer pairs.

Here D is a defect group of b and B. Let $\mathcal{C}(D, f_D)$ denote the full subcategory of \mathcal{C} of objects (S, f_S) of \mathcal{C} such that $(S, f_S) \leq (D, f_D)$ as (G, N, c)-Brauer pairs, and let $\mathcal{D}(D, f_D)$ denote the full subcategory of \mathcal{D} of objects (T, f_T) of \mathcal{D} such that $(T, f_T) \leq (D, f_D)$ as $(H, C_N(Q), f_O)$ -Brauer pairs.

Recall that D is a defect group of b and B. Thus, by choosing a maximal $(H, C_N(Q), f_Q)$ -Brauer pair (U, f_U) such that

$$(D,,f_D)\leq U,f_U)$$

as $(H, C_N(Q), f_Q)$ -Brauer pairs and applying Theorem 6, we conclude:

Corollary 8. The identity functor $I: \mathcal{D}(D, f_D) \to \mathcal{C}(D, f_D)$ is an isomorphism of categories.

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