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The group of self-homotopy equivalences of A_n^2-polyhedra

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Abstract. Let X be a finite type A_n^2-polyhedron, $n \geq 2$. In this paper, we study the quotient group $\mathcal{E}(X)/\mathcal{E}_*(X)$, where $\mathcal{E}(X)$ is the group of self-homotopy equivalences of X and $\mathcal{E}_*(X)$ the subgroup of self-homotopy equivalences inducing the identity on the homology groups of X. We show that not every group can be realised as $\mathcal{E}(X)$ or $\mathcal{E}(X)/\mathcal{E}_*(X)$ for X an A_n^2-polyhedron, $n \geq 3$, and specific results are obtained for $n = 2$.

1 Introduction

Let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences of a space X, and let $\mathcal{E}_*(X)$ denote the normal subgroup of self-homotopy equivalences inducing the identity on the homology groups of X. Problems related to $\mathcal{E}(X)$ have been extensively studied, with Kahn's realisability problem deserving a special mention, having been placed first to solve in [2] (see also [1, 11, 12, 14]). It asks whether an arbitrary group can be realised as $\mathcal{E}(X)$ for some simply connected X, and though the general case remains an open question, it has recently been solved for finite groups [7]. As a way to approach Kahn's problem, in [9, Problem 19], the question of whether an arbitrary group can appear as the distinguished quotient $\mathcal{E}(X)/\mathcal{E}_*(X)$ is raised.

In this paper, we work with $(n-1)$-connected $(n+2)$-dimensional CW-complexes for $n \geq 2$, the so-called A_n^2-polyhedra. Homotopy types of these spaces have been classified by Baues in [4, Ch. I, § 8] using the long exact sequence of groups associated to simply connected spaces introduced by J. H. C. Whitehead in [15]. The author of [6] uses that classification to study the group of self-homotopy equivalences of an A_2^2-polyhedron X. He associates to X a group $\mathcal{B}^4(X)$ that is isomorphic to $\mathcal{E}(X)/\mathcal{E}_*(X)$ and asks if any group can be realised as such

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a quotient in this context, that is, if A^2_2-polyhedra provide an adequate framework to solve the realisability problem.

Here, in the general setting of an A^2_n-polyhedra X, $n \geq 2$, we also construct a group $\mathcal{B}^{n+2}(X)$ (see Definition 2.4) that is isomorphic to $\mathcal{E}(X)/\mathcal{E}_*(X)$ (see Proposition 2.5). We show that there exist many groups (for example $\mathbb{Z}/p$, p odd, Corollary 1.2) for which the question above does not admit a positive answer. This fact should illustrate that A^2_n-polyhedra might not be the right setting to answer [9, Problem 19].

We show, for instance, that under some restrictions on the homology groups of X, $\mathcal{B}^{n+2}(X)$ is infinite, which in particular implies that $\mathcal{E}(X)$ is infinite (see Proposition 3.6 and Proposition 3.9). Or for example, in many situations the existence of odd order elements in the homology groups of X implies the existence of involutions in $\mathcal{B}^{n+2}(X)$ (see Lemma 3.4 and Lemma 3.5).

In this paper, we prove the following result.

Theorem 1.1. *Let X be a finite type A^2_n-polyhedron, $n \geq 3$. Then $\mathcal{B}^{n+2}(X)$ is either the trivial group or it has elements of even order.*

As an immediate corollary, we obtain the following.

Corollary 1.2. *Let G be a non-trivial group with no elements of even order. Then G is not realisable as $\mathcal{B}^{n+2}(X)$ for X a finite type A^2_n-polyhedron, $n \geq 3$.*

The case $n = 2$ is more complicated. Detailed group-theoretical analysis shows that a finite type A^2_2-polyhedra might realise finite groups of odd order only under very restrictive conditions. Recall that, for a group G, $\operatorname{rank} G$ is the smallest cardinal of a set of generators for G [13, p. 91]. We have the following result.

Theorem 1.3. *Suppose that X is a finite type A^2_2-polyhedron with a non-trivial finite $\mathcal{B}^4(X)$ of odd order. Then the following holds:*

(1) $\operatorname{rank} H_4(X) \leq 1$,

(2) *$\pi_3(X)$ and $H_3(X)$ are 2-groups, $H_2(X)$ is an elementary abelian 2-group,*

(3) $\operatorname{rank} H_3(X) \leq \frac{1}{2} \operatorname{rank} H_2(X)(\operatorname{rank} H_2(X) + 1) - \operatorname{rank} H_4(X) \leq \operatorname{rank} \pi_3(X)$,

(4) *the natural action of $\mathcal{B}^4(X)$ on $H_2(X)$ induces a faithful representation*

$$\mathcal{B}^4(X) \leq \operatorname{Aut}(H_2(X)).$$

All our attempts to find a space satisfying the hypothesis of Theorem 1.3 were unsuccessful. We therefore make the following conjecture.

Conjecture 1.4. Let X be an A^2_2-polyhedron. If $\mathcal{B}^4(X)$ is a non-trivial finite group, then it necessarily has an element of even order.

This paper is organised as follows. In Section 2, we give a brief introduction to Whitehead's and Baues's results for the classification of homotopy types of A_n^2-polyhedra, or equivalently, isomorphism classes of certain long exact sequences of abelian groups (see Theorem 2.3). In Section 3, we study how restrictions on X affect the group $\mathcal{B}^{n+2}(X)$. Finally, Section 4 is devoted to the proof of our main results, Theorem 1.1 and Theorem 1.3.

2 The Γ-sequence of an A_n^2-polyhedron

Let Ab denote the category of abelian groups. In [15], J. H. C. Whitehead constructed a functor $\Gamma\colon \mathrm{Ab} \to \mathrm{Ab}$, known as Whitehead's universal quadratic functor, and an exact sequence, which are useful to our purposes and introduced in this section. The Γ-functor is defined as follows. Let A and B be abelian groups and $\eta\colon A \to B$ a map (of sets) between them. The map η is said to be quadratic if

(1) $\eta(a) = \eta(-a)$ for all $a \in A$,

(2) the map $A \times A \to B$ taking (a, a') to $\eta(a + a') - \eta(a) - \eta(a')$ is bilinear.

For an abelian group A, $\Gamma(A)$ is the only abelian group for which there exists a quadratic map $\gamma\colon A \to \Gamma(A)$ such that every other quadratic map $\eta\colon A \to B$ factors uniquely through γ. This means that there is a unique group homomorphism $\eta^\square\colon \Gamma(A) \to B$ such that $\eta = \eta^\square\gamma$. The quadratic map $\gamma\colon A \to \Gamma(A)$ is called the universal quadratic map of A.

The Γ-functor acts on morphisms as follows. Let $f\colon A \to B$ be a group homomorphism, and $\gamma\colon A \to \Gamma(A)$ and $\gamma\colon B \to \Gamma(B)$ the universal quadratic maps. Then $\gamma f\colon A \to \Gamma(B)$ is a quadratic map, so there exists a unique group homomorphism $(\gamma f)^\square\colon \Gamma(A) \to \Gamma(B)$ such that $(\gamma f)^\square\gamma = \gamma f$. Define $\Gamma(f) = (\gamma f)^\square$.

We now list some of its properties that will be used later in this paper.

Proposition 2.1 ([5, pp. 16–17]). *The Γ functor has the following properties:*

(1) $\Gamma(\mathbb{Z}) = \mathbb{Z}$.

(2) *$\Gamma(\mathbb{Z}_n)$ is $\mathbb{Z}_{2n}$ if n is even, or $\mathbb{Z}_n$ if n is odd.*

(3) *Let I be an ordered set and A_i an abelian group for each $i \in I$. Then*

$$\Gamma\Big(\bigoplus_I A_i\Big) = \Big(\bigoplus_I \Gamma(A_i)\Big) \oplus \Big(\bigoplus_{i<j} A_i \otimes A_j\Big).$$

Moreover, the groups $\Gamma(A_i)$ and $A_i \otimes A_j$ are respectively generated by elements $\gamma(a_i)$ and $a_i \otimes a_j$, with $a_i \in A_i$, $a_j \in A_j$, $i < j$, and

$$\gamma(a_i + a_j) = \gamma(a_i) + \gamma(a_j) + a_i \otimes a_j \quad \text{for } a_i \in A_i,\ a_j \in A_j,\ i < j$$

(see [15, § 5, § 7]*).*

We now introduce Whitehead's exact sequence. Let X be a simply connected CW-complex. For $n \geq 1$, the n-th Whitehead Γ-group of X is defined as

$$\Gamma_n(X) = \operatorname{Im}(i_*\colon \pi_n(X^{n-1}) \to \pi_n(X^n)).$$

Here, $i\colon X^{n-1} \to X^n$ is the inclusion of the $(n-1)$-skeleton of X into its n-skeleton. Then $\Gamma_n(X)$ is an abelian group for $n \geq 1$. This group can be embedded into a long exact sequence of abelian groups

$$\cdots \longrightarrow H_{n+1}(X) \xrightarrow{b_{n+1}} \Gamma_n(X) \xrightarrow{i_{n-1}} \pi_n(X) \xrightarrow{h_n} H_n(X) \longrightarrow \cdots, \quad (2.1)$$

where h_n is the Hurewicz homomorphism and b_{n+1} is a boundary representing the attaching maps.

For each $n \geq 2$, a functor $\Gamma_n^1\colon \mathrm{Ab} \to \mathrm{Ab}$ is defined as follows. Let $\Gamma_2^1 = \Gamma$ be the universal quadratic functor, and for $n \geq 3$, $\Gamma_n^1 = - \otimes \mathbb{Z}_2$. It turns out that if X is $(n-1)$-connected, then $\Gamma_n^1(H_n(X)) \cong \Gamma_{n+1}(X)$ (see [5, Theorem 2.1.22]). Thus the final part of the long exact sequence (2.1) can be written as

$$H_{n+2}(X) \xrightarrow{b_{n+2}} \Gamma_n^1(H_n(X)) \xrightarrow{i_n} \pi_{n+1}(X) \xrightarrow{h_{n+1}} H_{n+1}(X) \longrightarrow 0. \quad (2.2)$$

Now, for each $n \geq 2$, we define the category of A_n^2-polyhedra as the category whose objects are $(n+2)$-dimensional $(n-1)$-connected CW-complexes and whose morphisms are continuous maps between objects. Homotopy types of these spaces are classified through isomorphism classes in a category whose objects are sequences like (2.2) [4, Ch. I, § 8].

Definition 2.2 ([3, Ch. IX, § 4]). Let $n \geq 2$ be an integer. We define the category Γ-sequences^{n+2} as follows. Objects are exact sequences of abelian groups

$$H_{n+2} \to \Gamma_n^1(H_n) \to \pi_{n+1} \to H_{n+1} \to 0,$$

where H_{n+2} is free abelian. Morphisms are triples of group homomorphisms $f = (f_{n+2}, f_{n+1}, f_n)$, $f_i\colon H_i \to H_i'$, such that there exists a group homomorphism $\Omega\colon \pi_{n+1} \to \pi_{n+1}'$ making the diagram

$$\begin{tikzcd}
H_{n+2} \arrow[r] \arrow[d, "f_{n+2}"] & \Gamma_n^1(H_n) \arrow[r] \arrow[d, "\Gamma_n^1(f_n)"] & \pi_{n+1} \arrow[r] \arrow[d, "\Omega"] & H_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & 0 \\
H_{n+2}' \arrow[r] & \Gamma_n^1(H_n') \arrow[r] & \pi_{n+1}' \arrow[r] & H_{n+1}' \arrow[r] & 0
\end{tikzcd}$$

commutative. Objects in Γ-sequences^{n+2} are called Γ-sequences, and morphisms in the category are called Γ-morphisms.

On the one hand, we can assign to an A_n^2-polyhedron X an object in Γ-sequences^{n+2} by considering the associated exact sequence (2.2). We call such an object the Γ-*sequence of* X. On the other hand, to a continuous map $\alpha\colon X \to X'$ of A_n^2-polyhedra, we can assign a morphism between the corresponding Γ-sequences by considering the induced homomorphisms

$$\begin{tikzcd}
H_{n+2}(X) \arrow[r] \arrow[d, "H_{n+2}(\alpha)"] & \Gamma_n^1(H_n(X)) \arrow[r] \arrow[d, "\Gamma_n^1(H_n(\alpha))"] & \pi_{n+1}(X) \arrow[r] \arrow[d, "\pi_{n+1}(\alpha)"] & H_{n+1}(X) \arrow[r] \arrow[d, "H_{n+1}(\alpha)"] & 0 \\
H_{n+2}(X') \arrow[r] & \Gamma_n^1(H_n(X')) \arrow[r] & \pi_{n+1}(X') \arrow[r] & H_{n+1}(X') \arrow[r] & 0.
\end{tikzcd}$$

Therefore, we have a functor A_n^2-polyhedra $\to$ Γ-sequences^{n+2} which clearly restricts to the homotopy category of A_n^2-polyhedra, $\mathcal{H}o A_n^2$-polyhedra. It is obvious that this functor sends homotopy equivalences to isomorphisms between the corresponding Γ-sequences. Thus we can classify homotopy types of A_n^2-polyhedra through isomorphism classes of Γ-sequences.

Theorem 2.3 ([4, Ch. I, § 8]). *The functor* $\mathcal{H}o A_n^2$-polyhedra $\to$ Γ-sequences^{n+2} *previously defined is full. Moreover, for any object in* Γ-sequences^{n+2}, *there exists an* A_n^2*-polyhedron whose* Γ*-sequence is the given object in* Γ-sequences^{n+2}. *In fact, there exists a 1–1 correspondence between homotopy types of* A_n^2*-polyhedra and isomorphism classes of* Γ*-sequences.*

Following the ideas of [6], we introduce the following.

Definition 2.4. Let X be an A_n^2-polyhedron. We denote by $\mathcal{B}^{n+2}(X)$ the group of Γ-isomorphisms of the Γ-sequence of X.

Let $\Psi\colon \mathcal{E}(X) \to \mathcal{B}^{n+2}(X)$ be the map that associates to $\alpha \in \mathcal{E}(X)$ the Γ-isomorphism $\Psi(\alpha) = (H_{n+2}(\alpha), H_{n+1}(\alpha), H_n(\alpha))$. Then Ψ is a group homomorphism: its kernel is the subgroup of self-homotopy equivalences inducing the identity map on the homology groups of X, that is, $\mathcal{E}_*(X)$. Also, Ψ is onto as a consequence of Theorem 2.3. Hence, we immediately obtain the following result.

Proposition 2.5. *Let* X *be an* A_n^2*-polyhedron,* $n \geq 2$. *Then*

$$\mathcal{B}^{n+2}(X) \cong \mathcal{E}(X)/\mathcal{E}_*(X).$$

3 Self-homotopy equivalences of finite type A_n^2-polyhedra

Henceforth, an A_n^2-polyhedron will mean an $(n-1)$-connected, $(n+2)$-dimensional CW-complex of finite type. Recall that, for simply connected and finite type spaces, the homology and homotopy groups $H_n(X)$ and $\pi_n(X)$ are finitely generated and abelian for $n \geq 1$.

The Γ-sequence tool introduced in Section 2 will help us to illustrate, from an algebraic point of view, how different restrictions on an A_n^2-polyhedron X affect the quotient group $\mathcal{E}(X)/\mathcal{E}_*(X)$. We devote this section to that matter. We also obtain several results that are needed in the proof of Theorem 1.1 and Theorem 1.3. The following result is a generalisation of [6, Theorem 4.5].

Proposition 3.1. *Let X be an A_n^2-polyhedron and suppose that the Hurewicz homomorphism $h_{n+2}\colon \pi_{n+2}(X) \to H_{n+2}(X)$ is onto. Then every automorphism of $H_{n+2}(X)$ is realised by a self-homotopy equivalence of X.*

Proof. As part of the exact sequence (2.1) for X, we have

$$\cdots \longrightarrow \pi_{n+2}(X) \xrightarrow{h_{n+2}} H_{n+2}(X) \xrightarrow{b_{n+2}} \Gamma_n^1(H_n(X)) \longrightarrow \pi_{n+1}(X) \longrightarrow \cdots.$$

Then, since h_{n+2} is onto by hypothesis, b_{n+2} is the trivial homomorphism. Thus, for every $f_{n+2} \in \operatorname{Aut}(H_{n+2}(X))$, we have $b_{n+2} f_{n+2} = b_{n+2} = 0$, so if $\Omega = \mathrm{id}$, $(f_{n+2}, \mathrm{id}, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$. Then there exists $f \in \mathcal{E}(X)$ with $H_{n+2}(f) = f_{n+2}$, $H_{n+1}(f) = \mathrm{id}$, $H_n(f) = \mathrm{id}$. □

We can easily prove that automorphism groups can be realised; a result that can also be obtained as a consequence of [14, Theorem 2.1].

Example 3.2. Let G be a group isomorphic to $\operatorname{Aut}(H)$ for some finitely generated abelian group H. Then, for any integer $n \geq 2$, there exists an A_n^2-polyhedron X such that $G \cong \mathcal{B}^{n+2}(X)$: take the Moore space $X = M(H, n+1)$, which in particular is an A_n^2-polyhedron. The Γ-sequence of X is

$$H_{n+2}(X) = 0 \to \Gamma_n^1(H_n(X)) = 0 \to H \xrightarrow{=} H \to 0.$$

Then, for every $f \in \operatorname{Aut}(H)$, taking $\Omega = f$, we see that $(\mathrm{id}, f, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$, and those are the only possible Γ-isomorphisms. Thus $\mathcal{B}^{n+2}(X) \cong \operatorname{Aut}(H) \cong G$.

The use of Moore spaces is not required in the $n = 2$ case.

Example 3.3. Let G be a group isomorphic to $\operatorname{Aut}(H)$ for some finitely generated abelian group H. Consider the following object in Γ-sequences[4]:

$$\mathbb{Z} \overset{b_4}{\twoheadrightarrow} \Gamma(\mathbb{Z}_2) = \mathbb{Z}_4 \longrightarrow H \xrightarrow{=} H \longrightarrow 0. \tag{3.1}$$

By Theorem 2.3, there exists an A_2^2-polyhedron X realising this object. In particular, $H_4(X) = \mathbb{Z}$, $H_3(X) = \pi_3(X) = H$ and $H_2(X) = \mathbb{Z}_2$. It is clear from (3.1) that $(\mathrm{id}, f, \mathrm{id})$ is a Γ-isomorphism for every $f \in \mathrm{Aut}(H)$. Now, $\mathrm{Aut}(\mathbb{Z}_2)$ is the trivial group while $\mathrm{Aut}(\mathbb{Z}) = \{-\mathrm{id}, \mathrm{id}\}$. It is immediate to check that $(-\mathrm{id}, f, \mathrm{id})$ is not a Γ-isomorphism since $\mathrm{id}\, b_4 \neq b_4(-\mathrm{id})$. Then we obtain $\mathcal{B}^4(X) \cong \mathrm{Aut}(H)$.

Observe that not every group G is isomorphic to the automorphism group of an abelian group (for example $\mathbb{Z}_p$ if p is odd). Hence, examples from above only provide a partial positive answer to the realisability problem for $\mathcal{B}^{n+2}(X)$. Indeed, the automorphism group of an abelian group (other than $\mathbb{Z}_2$) has elements of even order. The following results go in that direction.

Lemma 3.4. *Let X be an A_n^2-polyhedron, $n \geq 2$. If $H_n(X)$ is not an elementary abelian 2-group, then $\mathcal{B}^{n+2}(X)$ has an element of order 2.*

Proof. Since $H_n(X)$ is not an elementary abelian 2-group, it admits a non-trivial involution $-\mathrm{id}\colon H_n(X) \to H_n(X)$. But $\Gamma_n^1(-\mathrm{id}) = \mathrm{id}$ for every $n \geq 2$, so we have $(\mathrm{id}, \mathrm{id}, -\mathrm{id}) \in \mathcal{B}^{n+2}(X)$, and the result follows. □

Notice a key difference between the $n = 2$ and the $n \geq 3$ cases: $\Gamma_2^1(A) = \Gamma(A)$ is never an elementary abelian 2-group when A is finitely generated and abelian, as can be deduced from Proposition 2.1. However, for $n \geq 3$, $\Gamma_n^1(A) = A \otimes \mathbb{Z}_2$ is always an elementary abelian 2-group. Taking advantage of this fact we can prove the following result.

Lemma 3.5. *Let X be an A_n^2-polyhedron, $n \geq 3$. If any of the homology groups of X is not an elementary abelian 2-group (in particular, if $H_{n+2}(X) \neq 0$), then $\mathcal{B}^{n+2}(X)$ contains a non-trivial element of order 2.*

Proof. Under our assumptions, $\Gamma_n^1(H_n(X))$ is an elementary abelian 2-group. For $\Omega = -\mathrm{id}$, the triple $(-\mathrm{id}, -\mathrm{id}, -\mathrm{id})$ is a Γ-isomorphism of order 2 unless $H_{n+2}(X)$, $H_{n+1}(X)$ and $H_n(X)$ are all elementary abelian 2-groups. □

We remark that this result does not hold for A_2^2-polyhedra. Indeed, if we consider the construction in Example 3.3 for $H = \mathbb{Z}_2$, then $\mathcal{B}^4(X) \cong \mathrm{Aut}(\mathbb{Z}_2) = \{*\}$ does not contain a non-trivial element of order 2 although $H_4(X) = \mathbb{Z}$ is not an elementary abelian 2-group.

We now prove some results regarding the finiteness of $\mathcal{B}^{n+2}(X)$.

Proposition 3.6. *Let X be an A_n^2-polyhedron, $n \geq 2$, with* $\mathrm{rank}\, H_{n+2}(X) \geq 2$ *and every element of $\Gamma_n^1(H_n(X))$ of finite order. Then $\mathcal{B}^{n+2}(X)$ is an infinite group.*

Proof. Since rank $H_{n+2}(X) \geq 2$, we may write $H_{n+2}(X) = \mathbb{Z}^2 \oplus G$, G a (possibly trivial) free abelian group. Consider the Γ-sequence of X,

$$\mathbb{Z}^2 \oplus G \xrightarrow{b_{n+2}} \Gamma_n^1(H_n(X)) \xrightarrow{i_n} \pi_{n+1}(X) \xrightarrow{h_{n+1}} H_{n+1}(X) \longrightarrow 0.$$

Since $b_{n+2}(\mathbb{Z}^2) \leq \Gamma_n^1(H_n(X))$ is a finitely generated $\mathbb{Z}$-module with finite order generators, it is a finite group. Define $k = \exp(b_{n+2}(\mathbb{Z}^2))$, and consider the automorphism of $\mathbb{Z}^2$ given by the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}),$$

which is of infinite order. If we take $f \oplus \mathrm{id}_G \in \mathrm{Aut}(\mathbb{Z}^2 \oplus G)$, then we have $b_{n+2}(f \oplus \mathrm{id}) = b_{n+2}$, thus $(f \oplus \mathrm{id}_G, \mathrm{id}, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$, which is an element of infinite order. □

As previously mentioned, $\Gamma_n^1(H_n(X))$ is an elementary abelian 2-group for $n \geq 3$. Hence, from Proposition 3.6 we get:

Corollary 3.7. *Let X be an A_n^2-polyhedron, $n \geq 3$, with* rank $H_{n+2}(X) \geq 2$. *Then $\mathcal{B}^{n+2}(X)$ is an infinite group.*

This result does not hold, in general, for $n = 2$. However, if A is a finite group, Proposition 2.1 implies that $\Gamma(A)$ is finite as well, so from Proposition 3.6, we get the following.

Corollary 3.8. *Let X be an A_2^2-polyhedron with* rank $H_4(X) \geq 2$ *and $H_2(X)$ finite. Then $\mathcal{B}^4(X)$ is an infinite group.*

We end this section with one more result on the infiniteness of $\mathcal{B}^{n+2}(X)$.

Proposition 3.9. *Let X be an A_n^2-polyhedron, $n \geq 3$. If $H_n(X) = \mathbb{Z}^2 \oplus G$ for a certain abelian group G, then $\mathcal{B}^{n+2}(X)$ is an infinite group.*

Proof. If $H_n(X) = \mathbb{Z}^2 \oplus G$, then

$$\Gamma_n^1(H_n(X)) = H_n(X) \otimes \mathbb{Z}_2 = \mathbb{Z}_2^2 \oplus (G \otimes \mathbb{Z}_2).$$

Hence $\mathrm{GL}_2(\mathbb{Z}) \leq \mathrm{Aut}(H_n(X))$ and $\mathrm{GL}_2(\mathbb{Z}_2) \leq \mathrm{Aut}(H_n(X) \otimes \mathbb{Z}_2)$. Moreover, for every $f \in \mathrm{GL}_2(\mathbb{Z})$, we have $f \oplus \mathrm{id}_G \in \mathrm{Aut}(H_n(X))$, which yields, through Γ_n^1, an automorphism $(f \oplus \mathrm{id}_G) \otimes \mathbb{Z}_2 = (f \otimes \mathbb{Z}_2) \oplus \mathrm{id}_{G \otimes \mathbb{Z}_2} \in \mathrm{Aut}(H_n(X) \otimes \mathbb{Z}_2)$. This means that the functor Γ_n^1 restricts to $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}_2)$. Moreover, we have that $-\otimes \mathbb{Z}_2\colon \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}_2)$ has an infinite kernel. Hence, there are infinitely many morphisms $f \in \mathrm{Aut}(H_n(X))$ such that $f \otimes \mathbb{Z}_2 = \mathrm{id}$. For any such a morphism f, $(\mathrm{id}, \mathrm{id}, f)$ is an element of $\mathcal{B}^{n+2}(X)$. Therefore, $\mathcal{B}^{n+2}(X)$ is infinite. $\square$

4 Obstructions to the realisability of groups

We have seen in Section 3 that the group $\mathcal{B}^{n+2}(X)$ contains elements of even order unless strong restrictions are imposed on the homology groups of the A_n^2-polyhedron X. Since we are interested in realising an arbitrary group G as $\mathcal{B}^{n+2}(X)$ for X a finite type A_n^2-polyhedron, in this section, we focus our attention on the remaining situations and prove Theorems 1.1 and 1.3. We first give some previous results.

Lemma 4.1. *For G an elementary abelian 2-group, $\Gamma(-)\colon \mathrm{Aut}(G) \to \mathrm{Aut}(\Gamma(G))$ is injective.*

Proof. Let us show that the kernel of $\Gamma(-)$ is trivial. Assume that G is generated by $\{e_j \mid j \in J\}$, J an ordered set. If $f \in \mathrm{Aut}(G)$ is in the kernel of $\Gamma(-)$, then, for each $j \in J$, there exists a finite subset $I_j \subset J$ such that $f(e_j) = \sum_{i \in I_j} e_i$, and

$$\gamma(e_j) = \Gamma(f)\gamma(e_j) = \gamma f(e_j) = \gamma\Big(\sum_{i\in I_j} e_i\Big) = \sum_{i \in I_j} \gamma(e_i) + \sum_{i<k} e_i \otimes e_k,$$

as a consequence of Proposition 2.1 (3), so $I_j = \{j\}$ and $f(e_j) = e_j$ for every $j \in J$. $\square$

Lemma 4.2. *Let $H_2 = \bigoplus_{i=1}^n \mathbb{Z}_2$, and let $\chi \in \Gamma(H_2)$ be an element of order 4. If there exists a non-trivial automorphism of odd order $f \in \mathrm{Aut}(H_2)$ such that $\Gamma(f)(\chi) = \chi$, then there exists $g \in \mathrm{Aut}(H_2)$ of order 2 such that $\Gamma(g)(\chi) = \chi$.*

Proof. Notice that according to [15, p. 66], we can write $h \otimes h = 2\gamma(h)$ for any element $h \in H_2$. Therefore, given a basis $\{h_1, h_2, \ldots, h_n\}$ of H_2, and replacing $3\gamma(h_i)$ by $\gamma(h_i) + h_i \otimes h_i$ if needed, we can write

$$\chi = \sum_{i=1}^n a(i)\gamma(h_i) + \sum_{i,j=1}^n a(i,j) h_i \otimes h_j,$$

where every coefficient $a(i)$, $a(i,j)$ is either 0 or 1. We now inductively construct a basis $\{e_1, e_2, \ldots, e_n\}$ of H_2 as follows. Without loss of generality, assume

$a(1) = 1$, and define $e_1 = \sum_{i=1}^{n} a(i)h_i$. Then $\{e_1, h_2, \dots, h_n\}$ is again a basis of H_2 and

$$\chi = \gamma(e_1) + \alpha_1 e_1 \otimes e_1 + \beta_1 e_1 \otimes \left(\sum_{s=2}^{n} b(1,s)h_s\right) + \sum_{i,j>1}^{n} a_1(i,j)h_i \otimes h_j,$$

where every coefficient in the equation is either 0 or 1. Assume a basis

$$\{e_1, \dots, e_r, h_{r+1}, \dots, h_n\}$$

has been constructed such that

$$\begin{aligned}\chi = \gamma(e_1) + \sum_{j=1}^{r} \alpha_j e_j \otimes e_j + \sum_{j=1}^{r-1} \beta_j e_j \otimes e_{j+1} \\ + \beta_r e_r \otimes \left(\sum_{s=r+1}^{n} b(r,s)h_s\right) + \sum_{i,j>r}^{n} a_r(i,j)h_i \otimes h_j,\end{aligned}$$

where every coefficient is either 0 or 1. We may assume $b(r, r+1) = 1$ and define $e_{r+1} = \sum_{s=r+1}^{n} b(r,s)h_s$. Thus $\{e_1, \dots, e_{r+1}, h_{r+2}, \dots, h_n\}$ is again a basis of H_2 and

$$\begin{aligned}\chi = \gamma(e_1) + \sum_{j=1}^{r+1} \alpha_j e_j \otimes e_j + \sum_{j=1}^{r} \beta_j e_j \otimes e_{j+1} \\ + \beta_{r+1} e_{r+1} \otimes \left(\sum_{s=r+2}^{n} b(r+1,s)h_s\right) + \sum_{i,j>r+1}^{n} a_{r+1}(i,j)h_i \otimes h_j.\end{aligned}$$

Finally, we obtain a basis $\{e_1, e_2, \dots, e_n\}$ of H_2 such that

$$\chi = \gamma(e_1) + \sum_{j=1}^{n} \alpha_j e_j \otimes e_j + \sum_{j=1}^{n-1} \beta_j e_j \otimes e_{j+1} \tag{4.1}$$

for some coefficients

$$\alpha_j \in \{0,1\}, \quad j = 1, 2, \dots, n, \quad \text{and} \quad \beta_j \in \{0,1\}, \quad j = 1, 2, \dots, n-1.$$

Now, for $n = 1$, $H_2 = \mathbb{Z}_2$ has a trivial group of automorphisms, so the result holds. For $n = 2$, assume that there exists $f \in \mathrm{Aut}(H_2)$ such that $\Gamma(f)(\chi) = \chi$. From equation (4.1), $\chi = \Gamma(f)(\gamma(e_1)) + \Gamma(f)(P)$, where

$$P \in \Omega_1(\Gamma(H_2)) = \{h \in \Gamma(H_2) : \mathrm{ord}(h) \mid 2\}.$$

Then $\Gamma(f)(\gamma(e_1))$ has a multiple of $\gamma(e_1)$ as its only summand of order 4, which implies $f(e_1) = e_1$. Then either $f(e_2) = e_2$, so f is trivial, or $f(e_2) = e_1 + e_2$, so f has order 2.

For $n \geq 3$, we define $g \in \mathrm{Aut}(H_2)$ by $g(e_j) = e_j$ for $j = 1, 2, \ldots, n-2$, and $g(e_{n-1})$ and $g(e_n)$, depending on α_{n-j} and β_{n-1-j}, for $j = 0, 1$, in equation (4.1), according to the following table.

α_n	β_{n-1}	α_{n-1}	β_{n-2}	$g(e_{n-1})$	$g(e_n)$
0	0	0 or 1	0 or 1	e_{n-1}	$e_{n-1} + e_n$
0	1	0	0	e_n	e_{n-1}
0	1	0	1	$e_{n-2} + e_n$	$e_{n-2} + e_{n-1}$
0	1	1	0	$e_{n-1} + e_n$	e_n
0	1	1	1	$e_{n-2} + e_{n-1} + e_n$	e_n
1	0	0	0	$e_{n-2} + e_{n-1}$	e_n
1	0	0	1	$e_{n-2} + e_{n-1}$	$e_{n-2} + e_n$
1	0	1	0	e_n	e_{n-1}
1	0	1	1	$e_{n-2} + e_{n-1}$	e_n
1	1	0 or 1	0 or 1	e_{n-1}	$e_{n-1} + e_n$

A simple computation shows that, in all cases, g has order 2 and $\Gamma(g)(\chi) = \chi$, so the result follows. □

Definition 4.3. Let $f\colon H \to K$ be a morphism of abelian groups. We say that a non-trivial subgroup $A \leq K$ is f-split if there exist groups $B \leq H$ and $C \leq K$ such that $H \cong A \oplus B$, $K = A \oplus C$ and f can be written as

$$\mathrm{id}_A \oplus g\colon A \oplus B \to A \oplus C \quad \text{for some } g\colon B \to C.$$

Henceforward, we will make extensive use of this notation applied to

$$h_{n+1}\colon \pi_{n+1}(X) \to H_{n+1}(X),$$

the Hurewicz morphism. We prove the following.

Lemma 4.4. *Let X be an A_n^2-polyhedron, $n \geq 2$. Let $A \leq H_{n+1}(X)$ be an h_{n+1}-split subgroup; thus $H_{n+1}(X) = A \oplus C$ for some abelian group C. Then, for every $f_A \in \mathrm{Aut}(A)$, there exists $f \in \mathcal{E}(X)$ inducing* $(\mathrm{id}, f_A \oplus \mathrm{id}_C, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$.

Proof. By hypothesis, $H_{n+1}(X) = A \oplus C$, $\pi_{n+1}(X) \cong A \oplus B$ for some abelian group B, and h_{n+1} can be written as $\mathrm{id}_A \oplus g$ for some morphism $g\colon B \to C$. Thus, for every $f_A \in \mathrm{Aut}(A)$, we have a commutative diagram

$$\begin{tikzcd}
H_{n+2}(X) \arrow[r, "b_{n+2}"'] \arrow[d, "\mathrm{id}"] & \Gamma_n^1(H_n(X)) \arrow[r] \arrow[d, "\mathrm{id}"] & A \oplus B \arrow[r, "h_{n+1}"'] \arrow[d, "f_A \oplus \mathrm{id}_B"] & A \oplus C \arrow[r] \arrow[d, "f_A \oplus \mathrm{id}_C"] & 0 \\
H_{n+2}(X) \arrow[r, "b_{n+2}"] & \Gamma_n^1(H_n(X)) \arrow[r] & A \oplus B \arrow[r, "h_{n+1}"] & A \oplus C \arrow[r] & 0.
\end{tikzcd}$$

Hence $(\mathrm{id}, f_A \oplus \mathrm{id}_C, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$, and by Theorem 2.3, there exists $f \in \mathcal{E}(X)$ such that $H_{n+1}(f) = f_A \oplus \mathrm{id}_C$, $H_{n+2}(f) = \mathrm{id}$ and $H_n(f) = \mathrm{id}$. □

The following lemma is crucial in the proof of Theorems 1.1 and 1.3.

Lemma 4.5. *Let X be an A_n^2-polyhedron, $n \geq 2$. Suppose that there exist h_{n+1}-split subgroups of $H_{n+1}(X)$.*

(1) *If $n \geq 3$, then $\mathcal{B}^{n+2}(X)$ is either trivial or it has elements of even order.*

(2) *If $\mathcal{B}^4(X)$ is finite and non-trivial, then it has elements of even order.*

Proof. First of all, observe that we just need to consider when $H_n(X)$ is an elementary abelian 2-group. Otherwise, the result is a consequence of Lemma 3.4.

Let A be an arbitrary h_{n+1}-split subgroup of $H_{n+1}(X)$. If $A \neq \mathbb{Z}_2$, there is an involution $\iota \in \mathrm{Aut}(A)$ that induces an element $(\mathrm{id}, \iota \oplus \mathrm{id}, \mathrm{id}) \in \mathcal{B}^{n+2}(X)$ of order 2 by Lemma 4.4, and the result follows. Hence we can assume that every h_{n+1}-split subgroup of $H_{n+1}(X)$ is $\mathbb{Z}_2$.

Both assumptions, namely $H_n(X)$ being an elementary abelian 2-group and every h_{n+1}-split subgroup of $H_{n+1}(X)$ being $\mathbb{Z}_2$, imply that $H_{n+1}(X)$ is a finite 2-group. Indeed, since $H_n(X)$ is finitely generated, $\Gamma_n^1(H_n(X))$ is a finite 2-group and so is $\operatorname{coker} b_{n+2}$. Then, since $H_{n+1}(X)$ is also finitely generated, any direct summand of $H_{n+1}(X)$ which is not a 2-group would be h_{n+1}-split, contradicting our assumption that every h_{n+1}-split subgroup of $H_{n+1}(X)$ is $\mathbb{Z}_2$.

To prove our lemma, we start with the case $A = H_{n+1}(X)$ is h_{n+1}-split. When $H_{n+2}(X) = 0$, the Γ-sequence of X becomes then the short exact sequence

$$0 \to \Gamma_n^1(H_n(X)) \to \Gamma_n^1(H_n(X)) \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0.$$

Notice that any automorphism of order 2 in $H_n(X)$ yields an automorphism of order 2 in $\Gamma_n^1(H_n(X))$ since Γ_n^1 is injective on morphisms: it is immediate for $n \geq 3$, and for $n = 2$, apply Lemma 4.1. As our sequence is split, any $f \in \mathrm{Aut}(H_n(X))$

induces the Γ-isomorphism (id, id, f) of the same order. Hence, for $H_n(X) \neq \mathbb{Z}_2$, it suffices to consider an involution. For $H_n(X) = \mathbb{Z}_2$, since by hypothesis

$$H_{n+1}(X) = \mathbb{Z}_2 \quad \text{and} \quad H_{n+2}(X) = 0,$$

the only Γ-isomorphism is (id, id, id), and therefore $\mathcal{B}^{n+2}(X)$ is trivial as claimed.

When $H_{n+2}(X) \neq 0$, for $n \geq 3$, the result follows directly from Lemma 3.5. For $n = 2$, we also assume that $\mathcal{B}^4(X)$ is finite and non-trivial. Hence, since $H_2(X)$ is an elementary abelian 2-group, Proposition 3.6 implies that $H_4(X) = \mathbb{Z}$. Then, if a Γ-isomorphism of the form ($-$id, f, id) exists, it is of even order. In particular, if $\operatorname{Im} b_4$ is a subgroup of $\Gamma(H_2(X))$ of order 2, ($-$id, id, id) is a Γ-isomorphism of even order.

Assume otherwise that $\operatorname{Im} b_4$ is a group of order 4. If a Γ-isomorphism (id, f, id) of odd order exists, then $\Gamma(f) \circ b_4 = b_4$. In this situation, by Lemma 4.2 for $\chi = b_4(1)$, there exists $g \in \operatorname{Aut}(H_2(X))$, an automorphism of order 2 such that $\Gamma(g)b_4(1) = b_4(1)$. Moreover, as we are in the case $A = H_3(X)$ being h_3-split, $(\mathrm{id}, g, \mathrm{id}) \in \mathcal{B}^4(X)$ is a Γ-isomorphism of order 2.

We deal now with the case $A \subsetneqq H_{n+1}(X)$. Since $A = \mathbb{Z}_2$ is a proper h_{n+1}-split subgroup of $H_{n+1}(X)$, there exist non-trivial groups B and C such that

$$\begin{aligned} \pi_{n+1}(X) = \mathbb{Z}_2 \oplus B &\xrightarrow{h_{n+1}} \mathbb{Z}_2 \oplus C = H_{n+1}(X), \\ (t, b) &\longmapsto (t, g(b)) \end{aligned}$$

for some group morphism $B \xrightarrow{g} C$. Moreover, $H_{n+1}(X)$ is a finite 2-group; thus C is a (non-trivial) finite 2-group, and there exists an epimorphism $C \xrightarrow{\tau} \mathbb{Z}_2$.

Define

$$\begin{aligned} f &\in \operatorname{Aut}(\mathbb{Z}_2 \oplus C) = \operatorname{Aut}(H_{n+1}(X)), \\ \Omega &\in \operatorname{Aut}(\mathbb{Z}_2 \oplus B) = \operatorname{Aut}(\pi_{n+1}(X)) \end{aligned}$$

to be the non-trivial involutions given by

$$f(t, c) = (t + \tau(c), c) \quad \text{and} \quad \Omega(t, b) = (t + \tau(g(b)), b).$$

By construction, $h_{n+1}\Omega = f h_{n+1}$, and if $(t, b) \in \operatorname{coker} b_{n+2} = \ker h_{n+1}$ (thus $g(b) = 0$), then $\Omega(t, b) = (t, b)$. In other words, (id, f, id) $\in \mathcal{B}^{n+2}(X)$, and it has order 2. $\square$

We now prove our main results.

Proof of Theorem 1.1. Assume that $H_n(X)$ and $H_{n+1}(X)$ are elementary abelian 2-groups, and $H_{n+2}(X) = 0$. Otherwise, there would already be elements of order 2 in $\mathcal{B}^{n+2}(X)$ as a consequence of Lemma 3.5.

Write $H_n(X) = \oplus_I \mathbb{Z}_2$, I an ordered set. Since $n \geq 3$, we have $\Gamma_n^1 = - \otimes \mathbb{Z}_2$, so $\Gamma_n^1(H_n(X)) = H_n(X)$. We can also assume that there are no subgroups in

$H_{n+1}(X)$ that are h_{n+1}-split. Otherwise, we would deduce from Lemma 4.5 that there are elements of order 2 in $\mathcal{B}^{n+2}(X)$. Thus $H_{n+1}(X) = \oplus_J \mathbb{Z}_2$ with $J \subset I$, and the Γ-sequence corresponding to X is

$$0 \to \bigoplus_I \mathbb{Z}_2 \xrightarrow{b} \Big(\bigoplus_{I-J} \mathbb{Z}_2\Big) \oplus \Big(\bigoplus_J \mathbb{Z}_4\Big) \xrightarrow{h} \bigoplus_J \mathbb{Z}_2 \to 0.$$

We may rewrite the sequence as

$$0 \to \Big(\bigoplus_{I-J} \mathbb{Z}_2\Big) \oplus \Big(\bigoplus_J \mathbb{Z}_2\Big) \xrightarrow{b} \Big(\bigoplus_{I-J} \mathbb{Z}_2\Big) \oplus \Big(\bigoplus_J \mathbb{Z}_4\Big) \xrightarrow{h} \bigoplus_J \mathbb{Z}_2 \to 0$$

and assume that $b(x, y) = (x, 2y)$ and $h(x, y) = y \bmod 2$. It is clear that any $f \in \mathrm{Aut}(\bigoplus_{I-J} \mathbb{Z}_2)$ induces a Γ-isomorphism $(0, \mathrm{id}, f \oplus \mathrm{id})$ of the same order.

On the one hand, for $|I - J| \geq 2$, $\bigoplus_{I-J} \mathbb{Z}_2$ has an involution, and therefore $\mathcal{B}^{n+2}(X)$ has elements of even order. On the other hand, for $|I - J| < 2$, we consider the remaining possibilities.

Suppose that $|I - J| = 1$. Then $\pi_{n+1}(X) = \mathbb{Z}_2 \oplus (\bigoplus_J \mathbb{Z}_4)$. If J is trivial, then $\mathcal{B}^{n+2}(X)$ is clearly trivial as well. Otherwise, suppose that $I - J = \{i\}$ and choose $j \in J$. Define

$$f \in \mathrm{Aut}\Big(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \Big(\bigoplus_{I-\{i,j\}} \mathbb{Z}_2\Big)\Big) \quad \text{by} \quad f(x, y, z) = (x, x + y, z),$$

$$g \in \mathrm{Aut}\Big(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \Big(\bigoplus_{I-\{i,j\}} \mathbb{Z}_4\Big)\Big) \quad \text{by} \quad g(x, y, z) = (x, 2x + y, z).$$

Then $(\mathrm{id}, \mathrm{id}, f)$ is a Γ-isomorphism of order 2 since we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \begin{matrix}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \oplus (\bigoplus_{I-\{i,j\}} \mathbb{Z}_2)\end{matrix} \arrow[r] \arrow[d, "f"] & \begin{matrix}\mathbb{Z}_2 \oplus \mathbb{Z}_4 \\ \oplus (\bigoplus_{I-\{i,j\}} \mathbb{Z}_4)\end{matrix} \arrow[r] \arrow[d, "g"] & \begin{matrix}\mathbb{Z}_2 \\ \oplus (\bigoplus_{J-\{j\}} \mathbb{Z}_2)\end{matrix} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \begin{matrix}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \oplus (\bigoplus_{I-\{i,j\}} \mathbb{Z}_2)\end{matrix} \arrow[r] & \begin{matrix}\mathbb{Z}_2 \oplus \mathbb{Z}_4 \\ \oplus (\bigoplus_{I-\{i,j\}} \mathbb{Z}_4)\end{matrix} \arrow[r] & \begin{matrix}\mathbb{Z}_2 \\ \oplus (\bigoplus_{J-\{j\}} \mathbb{Z}_2)\end{matrix} \arrow[r] & 0.
\end{tikzcd}$$

Suppose that $I = J$. If $H_n(X) = H_{n+1}(X) = \mathbb{Z}_2$, $\mathcal{B}^{n+2}(X)$ is trivial. If not, choose $i, j \in I$, and define maps

$$f \in \mathrm{Aut}\Big(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \Big(\bigoplus_{I-\{i,j\}} \mathbb{Z}_2\Big)\Big) \quad \text{by} \quad f(x, y, z) = (y, x, z),$$

$$g \in \mathrm{Aut}\Big(\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \Big(\bigoplus_{I-\{i,j\}} \mathbb{Z}_4\Big)\Big) \quad \text{by} \quad g(x, y, z) = (y, x, z).$$

We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \begin{array}{c}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \oplus\, (\bigoplus_{I-\{i,j\}} \mathbb{Z}_2)\end{array} & \longrightarrow & \begin{array}{c}\mathbb{Z}_4 \oplus \mathbb{Z}_4 \\ \oplus\, (\bigoplus_{I-\{i,j\}} \mathbb{Z}_4)\end{array} & \longrightarrow & \begin{array}{c}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \oplus\, (\bigoplus_{I-\{i,j\}} \mathbb{Z}_2)\end{array} & \longrightarrow 0 \\
 & \Big\downarrow f & & \Big\downarrow g & & \Big\downarrow f & \\
0 \longrightarrow & \begin{array}{c}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \oplus\, (\bigoplus_{I-\{i,j\}} \mathbb{Z}_2)\end{array} & \longrightarrow & \begin{array}{c}\mathbb{Z}_4 \oplus \mathbb{Z}_4 \\ \oplus\, (\bigoplus_{I-\{i,j\}} \mathbb{Z}_4)\end{array} & \longrightarrow & \begin{array}{c}\mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \oplus\, (\bigoplus_{I-\{i,j\}} \mathbb{Z}_2)\end{array} & \longrightarrow 0.
\end{array}$$

Then $(0, f, f)$ is a Γ-isomorphism of order 2. □

As a consequence, we obtain a negative answer to the problem of realising groups as self-homotopy equivalences of A_n^2-polyhedra.

Corollary 4.6. *Let G be a non-nilpotent finite group of odd order. Then, for any $n \geq 3$ and for any A_n^2-polyhedron X, we have $G \ncong \mathcal{E}(X)$.*

Proof. Assume that there exists an A_n^2-polyhedron X such that $\mathcal{E}(X) \cong G$. Then, if $\mathcal{E}(X) \neq \mathcal{E}_*(X)$, the quotient $\mathcal{E}(X)/\mathcal{E}_*(X)$ is a finite group of odd order, which contradicts Theorem 1.1. Thus $G \cong \mathcal{E}(X) = \mathcal{E}_*(X)$. However, since X is a 1-connected and finite-dimensional CW-complex, $\mathcal{E}_*(X)$ is a nilpotent group, [8, Theorem D], which contradicts the fact that G is non-nilpotent. □

We end this paper by proving our second main result.

Proof of Theorem 1.3. By hypothesis, $\mathcal{B}^4(X)$ is a finite group of odd order. From Lemma 3.4, we deduce that $H_2(X)$ is an elementary abelian 2-group, and from Proposition 2.1, we deduce that $\Gamma(H_2(X))$ is a 2-group. In particular, every element of $\Gamma(H_2(X))$ is of finite order, and therefore rank $H_4(X) \leq 1$ by Proposition 3.6, so we have Theorem 1.3 (1). Now, any element in $\mathcal{B}^4(X)$ is of the form $(0, f_2, f_3)$ if $H_4(X) = 0$, or (id, f_2, f_3) if $H_4(X) = \mathbb{Z}$. Notice that a Γ-morphism of the form $(-\mathrm{id}, f_2, f_3)$ has even order thus it cannot be a Γ-isomorphism under our hypothesis. Therefore, if $H_4(X) = \mathbb{Z}$, then $b_4(1)$ generates a $\mathbb{Z}_4$ factor in $\Gamma(H_2(X))$, and under our hypothesis, the equation

$$\operatorname{rank} \Gamma(H_2(X)) = \operatorname{rank} H_4(X) + \operatorname{rank}(\operatorname{coker} b_4)$$

holds for rank $H_4(X) \leq 1$.

Observe that any Γ-isomorphism of X induces a chain morphism of the short exact sequence

$$0 \longrightarrow \operatorname{coker} b_4 \longrightarrow \pi_3(X) \xrightarrow{h_3} H_3(X) \longrightarrow 0.$$

We will draw our conclusions from this induced morphism, which can be seen as an automorphism of $\pi_3(X)$ that maps the subgroup $i_2(\operatorname{coker} b_4)$ to itself, thus inducing an isomorphism on the quotient, $H_3(X)$.

As we mentioned above, $\Gamma(H_2(X))$ is a 2-group. Then $\operatorname{coker} b_4$ is a quotient of a 2-group so a 2-group itself. We claim that $H_3(X)$ is also a 2-group; otherwise, $H_3(X)$ has a summand whose order is either infinite or odd, and therefore this summand would be h_3-split, which from Lemma 4.5 implies that $\mathcal{B}^4(X)$ has elements of even order, leading to a contradiction. Since $\operatorname{coker} b_4$ and $H_3(X)$ are 2-groups, so is $\pi_3(X)$, proving thus Theorem 1.3 (2).

Moreover, as a consequence of Lemma 4.5, no subgroup of $H_3(X)$ can be h_3-split, and thus $\operatorname{rank} H_3(X) \leq \operatorname{rank}(\operatorname{coker} b_4) = \operatorname{rank} \Gamma(H_2(X)) - \operatorname{rank} H_4(X)$. We can compute $\operatorname{rank} \Gamma(H_2(X))$ using Proposition 2.1 and immediately obtain Theorem 1.3 (3).

Now, for a 2-group G, define the subgroup $\Omega_1(G) = \{g \in G : \operatorname{ord}(g) \mid 2\}$. One can easily check that $\Omega_1(\pi_3(X)) \leq i_2(\operatorname{coker} b_4)$, and from [10, Ch. 5, Theorem 2.4], we obtain that any automorphism of odd order of $\pi_3(X)$ acting as the identity on $i_2(\operatorname{coker} b_4)$ must be the identity.

Then, if $(\mathrm{id}, f_3, f_2) \in \mathcal{B}^4(X)$ is a Γ-morphism with f_3 non-trivial, f_3 has odd order, so we may assume that $\Omega\colon \pi_3(X) \to \pi_3(X)$ (see Definition 2.2) has odd order too. By the argument above, it must induce a non-trivial homomorphism on $i_2(\operatorname{coker} b_4)$, and therefore f_2 is non-trivial as well. So the natural action of $\mathcal{B}^4(X)$ on $H_2(X)$ must be faithful since any Γ-automorphism $(\mathrm{id}, f_3, f_2) \in \mathcal{B}^4(X)$ induces a non-trivial $f_2 \in \operatorname{Aut}(H_2(X))$. Then Theorem 1.3 (4) follows. □

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