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Blocks of small defect in alternating groups and squares of Brauer character degrees

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Abstract. Let p be a prime. We show that other than a few exceptions, alternating groups will have p-blocks with small defect for p equal to 2 or 3. Using this result, we prove that a finite group G has a normal Sylow p-subgroup P and G/P is nilpotent if and only if $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for every irreducible Brauer character φ of G.

1 Introduction

In this note, all groups are finite. We will write $\mathrm{Irr}(G)$ for the set of ordinary irreducible characters of G and $\mathrm{IBr}(G)$ for the set of irreducible Brauer characters of G for a given prime p. In [8], the third author along with Stephen Gagola, Jr. proved that a group G is nilpotent if and only if $\chi(1)^2$ divides $|G : \ker(\chi)|$ for all $\chi \in \mathrm{Irr}(G)$. We now ask whether a similar result holds for Brauer characters.

First, we must recall the definition of $\ker(\varphi)$ from [18] for a Brauer character φ. Let p be a prime and let F be an algebraically closed field of characteristic p. If $\varphi \in \mathrm{IBr}(G)$, then $\ker(\varphi) = \ker(\mathcal{X})$ where $\mathcal{X}$ is an F-representation affording φ. (It is not difficult to see that $\ker(\varphi)$ is well-defined, see [18, p. 39].) We find a necessary and sufficient condition for a group G to have $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{IBr}(G)$.

Theorem 1. *Let p be a prime and let G be a group. Then $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{IBr}(G)$ if and only if G has a normal Sylow p-subgroup P and G/P is nilpotent.*

We first prove the 'if' part of Theorem 1 in Lemma 3.4. The proof is very similar to the proof for ordinary characters and takes relatively little additional work. So,

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we will focus on the 'only if' part. Now suppose that $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{IBr}(G)$. If G has a normal Sylow p-subgroup P, then it is easy to deduce from the main result in [8] that G/P is nilpotent. Therefore, we will need to show that G has a normal Sylow p-subgroup.

To prove this result, assume that the Sylow p-subgroup P of G is not normal in G. We will consider the p-solvable and non-p-solvable cases separately. For the p-solvable case, we will make use of the B_π-characters defined in [11] and the large orbit result in [14] along with recycling some of the ideas from [8]. For the non-p-solvable case, we will compare the r-parts, for various primes r, of the degrees of the Brauer characters φ and that of $|G : \ker(\varphi)|$ for $\varphi \in \mathrm{IBr}(G)$. We will separate the proof into two cases depending on whether a certain nonabelian composition factor S of the group possesses an ordinary irreducible character of p-defect zero or not. If the former case occurs, we can construct a Brauer character of G whose degree has a large p-part, which leads quickly to a contradiction.

When S does not have an ordinary irreducible character of p-defect zero, we know from [9, Corollary 1] that $p = 2$ or 3 and S is isomorphic to a handful of sporadic simple groups or an alternating group $\mathrm{Alt}(n)$ for some integer $n \geq 7$. The sporadic simple groups cases can be handled using GAP [24].

For the alternating groups, when $p = 2$ or 3, we know that 'most' of the groups $\mathrm{Alt}(n)$ have no p-blocks of defect zero. However, we will show that $\mathrm{Alt}(n)$ possesses a p-block B whose defect is not too large in comparison with the exponent of the Sylow p-subgroup of $\mathrm{Alt}(n)$. We then use the p-parts of the degrees of the Brauer characters in B to obtain a contradiction. Thus the following is key in our proof of Theorem 1 in the non-p-solvable case. This result will certainly be very useful in the study of the p-parts of ordinary and Brauer character degrees in finite groups.

Theorem 2. *Let $n \geq 5$ be an integer, let $p = 2$ or 3 and let $p^a = |\mathrm{Alt}(n)|_p$. Then* $\mathrm{Alt}(n)$ *has a p-block B of defect $d = d(B)$ such that $d \leq (a-1)/2$ for $p = 3$, and $d \leq (a-2)/2$ for $p = 2$ with the following exceptions:*

(1) $n = 7$ *if* $p = 3$.

(2) $n = 7, 9, 11, 13, 22, 24$ *and* 26 *if* $p = 2$.

This is related to a question posed in [6] by Espuelas and Navarro, where they asked whether a finite group G with $\mathbf{O}_p(G) = 1$ and $|G|_p = p^a$, where $p \geq 5$ is a prime, contains a block of defect less than $[a/2]$.

Our notation for Brauer characters is standard and follows [18]. For the organization of this paper, we will prove Theorem 2 in Section 2 and Theorem 1 will be proved in Section 3.

2 Alternating groups

We begin with some background material. (See [15] for more details.) Recall that the ordinary irreducible characters of $\mathrm{Sym}(n)$ are indexed by the partitions of n. Each partition λ has an associated Young diagram, and each box in the Young diagram has an associated hook length: for the box in row i, column j, the hook length is the number of boxes in the Young diagram directly to the right of (i, j), plus the number of boxes in the Young diagram directly below (i, j), plus one (for the node (i, j)). The rim hook corresponding to a box is the projection of the boxes of the hook to the outer edge of the Young diagram. In other words, the rim hook corresponding to the box (i, j) is the set of boxes starting at the end of column j, and going to the end of row i, along the edge of the Young diagram of λ. The number of boxes in the rim hook corresponding to (i, j) is necessarily equal to the hook length of that box.

A p-core is a partition that has no rim hooks of length divisible by p. For each partition λ of n, we can compute the associated p-core $\widehat{\lambda}$ by successively removing all the rim p-hooks (and recomputing the hook lengths of the remaining boxes each time we remove a rim p-hook) from the Young diagram of λ. It can be shown that the end result is independent of the order the hooks are removed. Brauer [3] and Robinson [21] proved that two characters χ_λ and χ_μ are in the same p-block of $\mathrm{Sym}(n)$ if and only if λ and μ have the same p-core.

Note that in removing rim p-hooks to get the core of λ, the number r of removed boxes is necessarily a multiple of p, and we let $w = r/p$ be the weight of λ. Characters of $\mathrm{Sym}(n)$ in the same block necessarily have the same weight. It is known (see [19]) that the defect group of a block B of weight w of $\mathrm{Sym}(n)$ is a Sylow p-subgroup of $\mathrm{Sym}(pw)$.

We set some notation. For a prime p and a positive integer m, we let $\nu_p(m) = a$, where $m = p^a b$ and b is a positive integer not divisible by p. We write $a_{p,n} = \nu_p(|\mathrm{Alt}(n)|)$ and $s_{p,n} = \nu_p(|\mathrm{Sym}(n)|)$. Of course, for odd p we have $a_{p,n} = s_{p,n}$ and $a_{2,n} = s_{2,n} - 1$. Recall (see [20]) that the p-blocks of $\mathrm{Sym}(n)$ correspond to the p-blocks of $\mathrm{Alt}(n)$. For a p-block B, we write $d_{B,n}$ to be the defect of the block of $\mathrm{Alt}(n)$, and $e_{B,n}$ to be the defect of the block of $\mathrm{Sym}(n)$. For odd p, we know $d_{B,n} = e_{B,n}$. However, for $p = 2$ we know that $d_{B,n} = e_{B,n} - 1$ if $e_{B,n} \geq 1$, and if $e_{B,n} = 0$, then there are two blocks of $\mathrm{Alt}(n)$ associated to B, each of which having defect zero. Thus for $p = 3$ our desired inequality is equivalent to $2e_{B,n} + 1 \leq s_{p,n}$. For $p = 2$, our desired inequality is $d_{B,n} \leq (a_{2,n} - 2)/2$, which is equivalent (for blocks of positive defect) to $e_{B,n} - 1 \leq (s_{2,n} - 1 - 2)/2$, which also reduces to $2e_{B,n} + 1 \leq s_{2,n}$.

Consequently, Theorem 2 follows from the following results on the symmetric groups.

Theorem 2.1. *For all values of $n \geq 5$ except $n = 7, 9, 11, 13, 22, 24, 26$, the symmetric group* $\mathrm{Sym}(n)$ *has a* 2*-block B of defect d such that $2d + 1 \leq s_{2,n}$. Moreover, for all values of $n \geq 5$ except $n = 7$,* $\mathrm{Sym}(n)$ *has a* 3*-block B of defect d such that $2d + 1 \leq s_{3,n}$.*

We will need the following easy lemma.

Lemma 2.2. *Fix a prime p. For a symmetric group* $\mathrm{Sym}(m)$*, let*

$$s_{p,m} = \nu_p(|\mathrm{Sym}(m)|).$$

Then there is a function $f_p : \mathbb{N} \to \mathbb{N}$ such that if $n \geq f_p(w)$, then

$$2s_{p,pw} + 1 \leq s_{p,n}.$$

In particular, if B is a block of $\mathrm{Sym}(n)$ *of weight w and defect d, then*

$$2d + 1 \leq s_{p,n}$$

if $n \geq f_p(w)$.

Proof. The first statement is obvious by comparing p-parts of the orders of the symmetric groups, and the second statement follows from the first by recalling that a block of $\mathrm{Sym}(n)$ of weight w has defect group isomorphic to a Sylow p-subgroup of $\mathrm{Sym}(pw)$ (see [19]). □

Of course, we have not uniquely defined f_p above. But by computing $s_{p,pw}$ and $s_{p,n}$ for small values of w (for $p = 2, 3$) it can be easily seen that we may take

$$\begin{aligned} &f_3(1) = 9, \quad f_3(2) = 12, \quad f_3(3) = 21, \quad f_3(4) = 27,\\ &f_3(5) = 27, \quad f_3(6) = 36, \quad f_3(7) = 42, \quad f_3(8) = 45.\end{aligned}$$

Moreover, it is easily seen that

$$\begin{aligned} &f_2(1) = 4, \quad f_2(2) = 8, \quad f_2(3) = 12, \quad f_2(4) = 16, \quad f_2(5) = 20,\\ &f_2(6) = 24, \quad f_2(7) = 26, \quad f_2(8) = 32, \quad f_2(9) = 36, \quad f_2(10) = 40,\\ &f_2(11) = 42, \quad f_2(12) = 48, \quad f_2(13) = 50, \quad f_2(14) = 60, \quad f_2(15) = 62,\\ &f_2(16) = 68, \quad f_2(17) = 70.\end{aligned}$$

2.1 The case for $p = 3$

We first generate some (in fact, all, though we will not need this) 3-cores. We note that the 3-cores and their sizes are well known. Let j and k be positive integers.

Then we set $\alpha_{j,k}$ to be the partition

$$(k+2j, k+2j-2, k+2j-4, \dots, k+4, k+2, k, k, k-1, k-1, \dots, 2, 2, 1, 1).$$

In addition, we let $\beta_{j,k}$ be the partition

$$(k+2j+1, k+2j-1, k+2j-3, \dots, k+3, k+1, k, k, k-1, k-1, \dots, 2, 2, 1, 1).$$

It is easily seen that both $\alpha_{j,k}$ and $\beta_{j,k}$ are 3-cores. Moreover, $\alpha_{j,k}$ is a partition of $jk + j(j+1) + k(k+1)$, and thus $\alpha_{0,k}$ is a partition of $k^2 + k$. We also see that $\alpha_{1,k}$ is a partition of $k^2 + 2k + 2$, and $\beta_{0,k}$ is a partition of $k^2 + 2k + 1$.

Notice that $|\alpha_{0,k}| \equiv 0 \bmod 3$ if and only if k is congruent to 0 or 2 mod 3. Similarly, $|\alpha_{1,k}| \equiv 2 \bmod 3$ if and only if k is congruent to 0 or 1 mod 3. And finally, $|\beta_{0,k}| \equiv 1 \bmod 3$ if and only if k is congruent to 0 or 1 mod 3.

Lemma 2.3. *Let $n \geq 90$. Then there is a* 3*-block B of* $\mathrm{Sym}(n)$ *of defect d such that* $2d + 1 \leq s_{3,n}$.

Proof. We consider three cases.

Case 1. Assume $n \equiv 0 \bmod 3$. Let m be the largest integer such that $m^2+m \leq n$. If $m^2 + m$ is divisible by 3, set $k = m$. If $m^2 + m$ is not divisible by 3, then $(m-1)^2 + (m-1)$ is necessarily divisible by 3, and in this case we take $k = m-1$. By assumption, then, we have that $n - (k^2 + k)$ is divisible by 3, and thus $\mathrm{Sym}(n)$ has a 3-block B with core partition $\alpha_{0,k}$. Note we certainly have

$$k^2 + k \leq n < (k+2)^2 + (k+2),$$

and thus

$$n - (k^2 + k) < (k+2)^2 + (k+2) - (k^2 + k) = 4k + 6.$$

Thus any partition of n with core $\alpha_{0,k}$ has 3-weight $w \leq (4k+5)/3$. Therefore the defect group P of such a block is contained in a Sylow 3-subgroup of $\mathrm{Sym}(4k+5)$. Thus

$$\mathrm{Sym}(4k+5) \times \mathrm{Sym}(4k+5) \times \mathrm{Sym}(3)$$

contains a copy of

$$P \times P \times \mathbb{Z}_3.$$

If $k \geq 9$, then $k \geq (7 + \sqrt{101})/2$, thus $k^2 - 7k - 13 \geq 0$, and so

$$2(4k+5) + 3 = 8k + 13 \leq k^2 + k \leq n.$$

Therefore

$$\mathrm{Sym}(4k+5) \times \mathrm{Sym}(4k+5) \times \mathrm{Sym}(3)$$

is contained in $\mathrm{Sym}(n)$ if $k \geq 9$, i.e. $n \geq (9^2 + 9) = 90$, and hence $3|P|^2 \leq 3^{s_{3,n}}$. Thus $2d + 1 \leq s_{3,n}$, where d is the defect of B, and hence $d \leq (s_{3,n} - 1)/2$, as desired, for $n \geq 90$ in this case.

The other two cases follow similarly, though we will include the details here.

Case 2. Assume $n \equiv 1 \bmod 3$. We will use $\beta_{0,k}$ here, thus let m be the largest integer such that $m^2 + 2m + 1 \leq n$. If $m^2 + 2m + 1 \equiv 1 \bmod 3$, set $k = m$. If m^2+2m+1 is not equivalent to 1 mod 3, then $(m-1)^2+2(m-1)+1 \equiv 1 \bmod 3$, and we take $k = m - 1$ in this case. By assumption, $n - (k^2 + 2k + 1)$ is divisible by 3, and thus $\mathrm{Sym}(n)$ has a 3-block with core partition $\beta_{0,k}$. Note that we certainly have $k^2 + 2k + 1 \leq n < (k + 2)^2 + 2(k + 2) + 1$, and thus

$$n - (k^2 + 2k + 1) < (k + 2)^2 + 2(k + 2) + 1 - (k^2 + 2k + 1) = 4k + 8.$$

We see that any partition of n with core $\beta_{0,k}$ has 3-weight $w \leq (4k + 7)/3$. Therefore the defect group P of such a block is contained in a Sylow 3-subgroup of $\mathrm{Sym}(4k + 7)$. We now have that

$$\mathrm{Sym}(4k + 7) \times \mathrm{Sym}(4k + 7) \times \mathrm{Sym}(3)$$

contains a copy of

$$P \times P \times \mathbb{Z}_3.$$

If $k \geq 8$, then $k^2 - 6k - 16 \geq 0$, and hence

$$2(4k + 7) + 3 = 8k + 17 \leq k^2 + 2k + 1 \leq n.$$

We have shown that

$$\mathrm{Sym}(4k + 7) \times \mathrm{Sym}(4k + 7) \times \mathrm{Sym}(3)$$

is contained in $\mathrm{Sym}(n)$ if $k \geq 8$, i.e. $n \geq 82$, and therefore $3|P|^2 \leq 3^{s_{n,3}}$. Thus $2d + 1 \leq s_{n,3}$, where d is the defect of B, and thus $d \leq (s_{n,3} - 1)/2$, as desired, for $n \geq 82$ in this case.

Case 3. Assume $n \equiv 2 \bmod 3$. We will use $\alpha_{1,k}$ here, thus let m be the largest integer such that $m^2 + 2m + 2 \leq n$. If $m^2 + 2m + 2 \equiv 2 \bmod 3$, set $k = m$. If m^2+2m+2 is not congruent to 2 mod 3, then $(m-1)^2+2(m-1)+2 \equiv 2 \bmod 3$, and in this case we set $k = m - 1$. By assumption, $n - (k^2 + 2k + 2)$ is divisible by 3, and thus $\mathrm{Sym}(n)$ has a 3-block with core partition $\alpha_{1,k}$. Note that we certainly have

$$k^2 + 2k + 2 \leq n < (k + 2)^2 + 2(k + 2) + 2$$

and so

$$n - (k^2 + 2k + 2) < (k + 2)^2 + 2(k + 2) + 2 - (k^2 + 2k + 2) = 4k + 8.$$

Hence, any partition of n with core $\alpha_{1,k}$ has 3-weight $w \leq (4k+7)/3$. Therefore the defect group P of such a block is contained in a Sylow 3-subgroup of $\mathrm{Sym}(4k+7)$. We deduce that

$$\mathrm{Sym}(4k+7) \times \mathrm{Sym}(4k+7) \times \mathrm{Sym}(3)$$

contains a copy of

$$P \times P \times \mathbb{Z}_3.$$

If $k \geq 8$, then $k^2 - 6k - 15 \geq 0$, and so

$$2(4k+7) + 3 = 8k + 17 \leq k^2 + 2k + 2 \leq n.$$

We have shown that

$$\mathrm{Sym}(4k+7) \times \mathrm{Sym}(4k+7) \times \mathrm{Sym}(3)$$

is contained in $\mathrm{Sym}(n)$ if $n \geq 8$, and therefore $3|P|^2 \leq 3^{s_{3,n}}$. We conclude that $2d+1 \leq s_{n,3}$, where d is the defect of B, and thus $d \leq (s_{n,3}-1)/2$, as desired, for $n \geq 83$ in this case. □

Proposition 2.4. *Theorem* 2.1 *holds for* $p = 3$.

Proof. Assume that $p = 3$. By Lemma 2.3, it remains to check that the result holds for $5 \leq n \leq 90$. Of course, the result is trivially true for all values of $n \geq 5$ for which $\mathrm{Sym}(n)$ has a block of defect zero. Similarly, if $\mathrm{Sym}(n)$ has a block of defect 1, then the result holds for all values of $n \geq f_3(1) = 9$. Continuing, we see the result holds for all values of $n \geq f_3(2) = 12$ for which $\mathrm{Sym}(n)$ has a block of defect two. By using the characterization of 3-cores, we see that continuing in this manner covers all values of n with $5 \leq n \leq 90$ except $n = 7$. Thus we have proven our result for $p = 3$. □

2.2 The case where $p = 2$

The case $p = 2$ follows similarly as the $p = 3$ case. Only here, there is only one 'type' of 2-core, the partition

$$\delta_k = (k, k-1, k-2, \ldots, 2, 1),$$

a partition of $k(k+1)/2$.

Notice that $k(k+1)/2$ is even if and only if $k \equiv 0$ or 3 mod 4.

Proposition 2.5. *Theorem* 2.1 *holds for* $p = 2$.

Proof. We let k be the largest integer such that $k(k+1)/2 \leq n$ and $n-k(k+1)/2$ is even, and we note that

$$n - \frac{k(k+1)}{2} < \frac{(k+3)(k+4)}{2} - \frac{k(k+1)}{2} = 3k + 6.$$

Thus we have $n - k(k+1)/2 \leq 3k + 5$, and therefore $\mathrm{Sym}(n)$ has a 2-block of weight at most $(3k+5)/2$, with defect group isomorphic to a Sylow 2-subgroup of $\mathrm{Sym}(3k+5)$. Note that if $k \geq 13$, then $k^2 - 11k - 26 \geq 0$, and thus

$$6k + 13 \leq \frac{k(k+1)}{2} \leq n.$$

Therefore there is a copy of

$$\mathrm{Sym}(3k+5) \times \mathrm{Sym}(3k+5) \times \mathrm{Sym}(3)$$

inside of $\mathrm{Sym}(n)$, and thus we are done for $k \geq 13$, i.e. $n \geq 91$.

As before, we handle the smaller values of n by finding the blocks of $\mathrm{Sym}(n)$ with the smallest weight w and comparing the weight w to the values of $f_2(w)$ computed above. Notice that the smallest weights for $n = 7, 9, 11, 13, 22, 24$, and 26, respectively, are $w = 2, 3, 4, 5, 6, 7$, and 8. However, these values of n are not large enough to use the function f_2 defined above to cover these cases. Thus, for these values of n we do not have 2-defect groups small enough. In all other cases we are done, however. □

Finally, the proof of Theorem 2.1 follows from Propositions 2.4 and 2.5.

3 Squares of Brauer character degrees

We now work to prove Theorem 1. We first prove that if the squares of the degrees of the Brauer characters all divide the index of their kernels, then the group must be p-solvable. We then consider the p-solvable case where we will use the B_π-characters due to I. M. Isaacs.

3.1 Non-p-solvable groups

We will need the following result.

Lemma 3.1. *Let S be any nonabelian simple group and let p be a prime with p dividing $|S|$. Then $|\mathrm{Out}(S)|_p < |S|_p$.*

Proof. If the pair (S, p) lies in the list

$$\mathcal{L} = \{(\mathrm{Alt}(7), 3), (\mathrm{Alt}(7), 2), (\mathrm{Alt}(11), 2), (\mathrm{Alt}(13), 2), (\mathrm{M}_{22}, 2)\},$$

then the lemma follows easily by checking [4].

Now suppose that $(S, p) \notin \mathcal{L}$. By [7, Lemma 1.2], there exists $\psi \in \mathrm{Irr}(S)$ such that $|\mathrm{Aut}(S)|_p < \psi(1)_p^2$. Since $\psi(1)$ divides $|S|$, we deduce that $\psi(1)_p \leq |S|_p$ so $|\mathrm{Aut}(S)|_p < |S|_p^2$ which implies that $|\mathrm{Out}(S)|_p < |S|_p$ as wanted. □

Theorem 3.2. *Let p be a prime and let G be a group. If $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{IBr}(G)$, then G is p-solvable.*

Proof. We may assume that G is not p-solvable. We first observe that the hypothesis of the theorem inherits to quotient groups. Suppose that G has two distinct minimal normal subgroups, say N_1 and N_2. Then G/N_i, $i = 1, 2$, are p-solvable by induction. Thus G is p-solvable, a contradiction. Therefore, G has a unique minimal normal subgroup N which is not abelian and p divides $|N|$. So we have $N \cong \prod_{i=1}^{k} S_i$, where $S_i \cong S$ and S is a nonabelian simple group with p dividing $|S|$. Observe that $\mathbf{C}_G(N) = 1$ so G embeds into $\mathrm{Aut}(N) \cong \mathrm{Aut}(S) \wr \mathrm{Sym}(k)$. Let $B = \mathrm{Aut}(S)^k \cap G$. Then $G = BH$ where $H \cong G/B$ is a subgroup of $\mathrm{Sym}(k)$.

Let r be a prime not necessarily equal to p. Then we write $|S|_r = r^{a(r)}$ and $|\mathrm{Out}(S)|_r = r^{c(r)}$. Let $\theta \in \mathrm{IBr}(S)$ be non-principal with r dividing $\theta(1)$ and let $\beta = \theta^k \in \mathrm{IBr}(N)$. Write $\theta(1)_r = r^{b(r)}$ for some integer $b(r) \geq 1$ depending on r. Let $\varphi \in \mathrm{IBr}(G)$ lie over β. Since G/N is p-solvable, $e := \varphi(1)/\beta(1)$ divides $|G/N|$ by the Swan Theorem [18, Theorem 8.22]. Since N is a unique minimal normal subgroup of G and $\theta \in \mathrm{IBr}(S)$ is faithful, $\beta = \theta^k$ is faithful and so is φ. By hypothesis, we have that $\varphi(1)^2 m = |G|$ for some integer $m \geq 1$. It follows that

$$|H| \cdot |B/N| \cdot |N| = m\varphi(1)^2 = me^2\beta(1)^2 = me^2\theta(1)^{2k}.$$

Now taking the r-parts of the equation above, we obtain that

$$|H|_r |B/N|_r r^{a(r)k} = m_r e_r^2 r^{2b(r)k}.$$

As $B/N \leq \mathrm{Out}(S)^k$, we have

$$r^{2b(r)k} \leq |\mathrm{Out}(S)|_r^k \cdot |H|_r \cdot r^{a(r)k}.$$

Furthermore, since $H \leq \mathrm{Sym}(k)$, we see that $|H|_r < r^{k/(r-1)}$ (see [8, proof of Case 1] for instance) so

$$r^{2b(r)k} < r^{a(r)k+k/(r-1)} r^{c(r)k} \leq r^{k(a(r)+c(r)+1)},$$

which implies that

$$2b(r) < a(r) + c(r) + \frac{1}{r-1} \leq a(r) + c(r) + 1$$

and so

$$2b(r) \leq a(r) + c(r). \qquad (*)$$

In what follows, we consider the cases when S has a p-block of defect zero or when S does not have a p-block of defect zero, separately. The first case is straight forward. For the latter, we can use GAP [24] to handle the sporadic simple groups and alternating groups of small degrees. For alternating groups of large degrees, we will need the results on block theory proved earlier.

(a) Suppose first that S has a p-block of defect zero. Then S has an irreducible ordinary character, say μ, of p-defect zero and thus $\theta = \mu^\circ$ is an irreducible p-Brauer character of S with $\theta(1)_p = |S|_p$. It follows that $a(p) = b(p)$. Moreover, by Lemma 3.1, we know that $c(p) < a(p)$ so $c(p) + 1 \leq a(p)$. Thus

$$2b(p) = 2a(p) \geq a(p) + c(p) + 1$$

which contradicts $(*)$ with $r = p$.

(b) Suppose that S has no characters of p-defect zero. Using the classification of p-blocks of defect zero (see, for example, [9, Corollary 2]), one of the following holds:

- $p = 3$ and $S \cong \mathrm{Suz}, \mathrm{Co}_3$ or $\mathrm{Alt}(n)$ for some integer $n \geq 7$.
- $p = 2$ and $S \cong \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{24}, \mathrm{J}_2, \mathrm{HS}, \mathrm{Suz}, \mathrm{Ru}, \mathrm{Co}_1, \mathrm{Co}_3, \mathrm{B}$ or $\mathrm{Alt}(n)$ for some integer $n \geq 7$.

If B is a p-block of S with defect $d = d(B)$, then $b = b(p) \geq a - d$ with $|S|_p = p^a$. Moreover, since $|\mathrm{Out}(S)| \leq 2$, we have $c = c(p) = \log_p(|\mathrm{Out}(S)|_p)$. Hence $(*)$ implies that $a + c \geq 2b \geq 2a - 2d(B)$ or equivalently $a \leq 2d(B) + c$. So, to get a contradiction, it suffices to find a block B such that

$$a \geq 2d(B) + c + 1, \quad \text{or equivalently} \quad d(B) \leq \frac{1}{2}(a - c - 1). \qquad (**)$$

(i) Using the p-parts. Assume that $p = 3$ and $S \cong \mathrm{Suz}$. Then S has a 3-Brauer character θ of degree $189540 = 2^2 \cdot 3^6 \cdot 5 \cdot 13$. Also $|S|_3 = 3^7$ and $|\mathrm{Out}(S)|_3 = 1$. Thus $a(3) + c(3) + 1 = 7 + 0 + 1 = 8 < 2b(3) = 12$ violating $(*)$.

Similarly, if $p = 3$ and $S \cong \mathrm{Co}_3$, then $|S|_3 = 3^7$, $|\mathrm{Out}(S)| = 1$ and S has an irreducible 3-Brauer character θ of degree $93312 = 2^7 \cdot 3^6$. Thus

$$a(3) + c(3) + 1 = 8 < 2b(3) = 12,$$

which is a contradiction.

Assume that $(S, p) = (\mathrm{M}_{12}, 2)$. Then we can choose

$$\theta \in \mathrm{IBr}_2(S)$$

with $\theta(1) = 2^4$. We have that $|S|_2 = 2^6$ and $|\mathrm{Out}(S)|_2 = 2$, so

$$a(2) + c(2) + 1 = 6 + 1 + 1 = 8 = 2b(2)$$

violating $(*)$.

Assume that $(S, p) = (\mathrm{M}_{24}, 2)$. Then we can choose $\theta \in \mathrm{IBr}_2(S)$ with

$$\theta(1) = 1792 = 2^8 \cdot 7.$$

We have that $|S|_2 = 2^{10}$ and $|\mathrm{Out}(S)| = 1$, so

$$a(2) + c(2) + 1 = 10 + 0 + 1 = 11 < 16 = 2b(2)$$

violating $(*)$.

Assume that $(S, p) = (\mathrm{J}_2, 2)$. Then we can choose $\theta \in \mathrm{IBr}_2(S)$ with

$$\theta(1) = 2^6.$$

We have that $|S|_2 = 2^7$ and $|\mathrm{Out}(S)|_2 = 2$, so

$$a(2) + c(2) + 1 = 7 + 1 + 1 = 9 < 12 = 2b(2)$$

violating $(*)$.

Assume that $(S, p) = (\mathrm{HS}, 2)$. Then we can choose $\theta \in \mathrm{IBr}_2(S)$ with

$$\theta(1) = 1408 = 2^7 \cdot 11.$$

We have that $|S|_2 = 2^9$ and $|\mathrm{Out}(S)|_2 = 2$, so

$$a(2) + c(2) + 1 = 9 + 1 + 1 = 11 < 14 = 2b(2)$$

violating $(*)$.

Assume that $(S, p) = (\mathrm{Suz}, 2)$. Then we can choose $\theta \in \mathrm{IBr}_2(S)$ with

$$\theta(1) = 102400 = 2^{12} \cdot 5^2.$$

We have that $|S|_2 = 2^{13}$ and $|\mathrm{Out}(S)|_2 = 2$, so

$$a(2) + c(2) + 1 = 13 + 1 + 1 = 15 < 24 = 2b(2)$$

violating $(*)$.

Assume that $(S, p) = (\mathrm{Ru}, 2)$. Then we can choose $\theta \in \mathrm{IBr}_2(S)$ with

$$\theta(1) = 8192 = 2^{13}.$$

We have that $|S|_2 = 2^{14}$ and $|\mathrm{Out}(S)|_2 = 1$, so

$$a(2) + c(2) + 1 = 14 + 0 + 1 = 15 < 26 = 2b(2)$$

violating $(*)$.

Assume that $(S, p) = (\mathrm{Co}_3, 2)$. Then we can choose $\theta \in \mathrm{IBr}_2(S)$ with

$$\theta(1) = 131584 = 2^9 \cdot 257.$$

We have that $|S|_2 = 2^{10}$ and $|\mathrm{Out}(S)|_2 = 1$, so

$$a(2) + c(2) + 1 = 10 + 0 + 1 = 11 < 18 = 2b(2)$$

violating $(*)$.

(ii) Using the r-parts with $r \neq p$. Suppose that $(S, p) = (\mathrm{M}_{22}, 2)$. Then S has an irreducible character $\theta \in \mathrm{IBr}_2(S)$ with $\theta(1) = 2 \cdot 17$. We see that

$$|S|_{17} = 1 = |\mathrm{Out}(S)|_{17}.$$

So $a(17) + c(17) + 1 = 1 < 2 = 2b(17)$ violating $(*)$.

(iii) Assume that $(S, p) = (\mathrm{Co}_1, 2)$ or $(\mathrm{B}, 2)$. In these two cases, the 2-Brauer character tables of S are not available. Recall that $a = a(p)$, so $|S|_p = p^a$. Moreover, $c = c(p) = 0$ as $|\mathrm{Out}(S)| = 1$.

If $(S, p) = (\mathrm{Co}_1, 2)$, then $a = 21$ and S has a 2-block B with defect $d(B) = 3$. Similarly, if $(S, p) = (\mathrm{B}, 2)$, then $a = 41$ and S has a 2-block B with defect $d(B) = 3$. In both cases, we have $a + c > 2d(B) + 1$ and the result follows.

(iv) Assume that $S = \mathrm{Alt}(7)$ and $p = 3$. Then we can choose $\theta \in \mathrm{IBr}_3(S)$ with $\theta(1) = 13$. Then $|S|_{13} = 1$ and $|\mathrm{Out}(S)|_{13} = 1$ so

$$2b(13) = 2 > 1 = a(13) + c(13) + 1$$

violating $(*)$.

(v) Assume that $S \cong \mathrm{Alt}(n)$ with $n > 7$ and $p = 3$. In this case, $c = 0$ since $|\mathrm{Out}(S)| = 2$. It follows from Theorem 2 that $\mathrm{Alt}(n)$ has a 3-block B with defect $d = d(B)$ satisfying the inequality

$$d \leq \frac{1}{2}(a - 1).$$

Therefore, $(**)$ occurs and thus this case cannot happen.

(vi) Assume that $S = \mathrm{Alt}(n)$ with $n \leq 13$ and $p = 2$. If $S = \mathrm{Alt}(7)$, then we can choose $\theta \in \mathrm{IBr}_2(S)$ with $\theta(1) = 2 \cdot 7$. Then $|S|_7 = 7$ and $|\mathrm{Out}(S)|_7 = 1$ so

$$2b(7) = 2 \geq a(7) + c(7) + 1$$

violating $(*)$.

If $S = \mathrm{Alt}(9)$, then there exists $\theta \in \mathrm{IBr}_2(S)$ with $\theta(1) = 2^5 \cdot 5$. Moreover, we have $|S|_2 = 2^6$ and $|\mathrm{Out}(S)|_2 = 2$ so

$$a(2) + c(2) + 1 = 8 < 10 = 2b(2),$$

a contradiction.

If $S = \mathrm{Alt}(11)$, then there exists $\theta \in \mathrm{IBr}_2(S)$ with $\theta(1) = 2^5 \cdot 13$. Moreover, $|S|_2 = 2^7$ and $|\mathrm{Out}(S)|_2 = 2$ so

$$a(2) + c(2) + 1 = 9 < 10 = 2b(2),$$

a contradiction.

If $S = \mathrm{Alt}(13)$, then there exists $\theta \in \mathrm{IBr}_2(S)$ with $\theta(1) = 4224 = 2^7 \cdot 3 \cdot 11$. Moreover, $|S|_2 = 2^9$ and $|\mathrm{Out}(S)|_2 = 2$ so

$$a(2) + c(2) + 1 = 11 < 14 = 2b(2),$$

a contradiction.

If n is 5 or 6, then $\mathrm{Alt}(n)$ is a simple group of Lie type and we are done by the above comments. If $n \in \{8, 10, 12\}$, then we are done by Theorem 2.

(vii) Assume that $S \cong \mathrm{Alt}(n)$ with $n > 13$ and $p = 2$. In this case, we see that $c = 1$ and the result follows by applying Theorem 2 unless $n = 22, 24$ or 26. For these cases, we will use some 2-Brauer character degrees of $\mathrm{Alt}(n)$ which can be found in [2, Proposition 5.2], for example.

Let $\lambda = (n-2, 2)$ and $\mu = (n-3, 3)$. Since $n \geq 22$, we know that both λ and μ are 2-regular partitions of n. It follows from [1] that the simple modules in characteristic p, D^λ and D^μ, labelled by λ and μ respectively, remain irreducible upon restriction to $\mathrm{Alt}(n)$ and their dimensions are given in [2, Proposition 5.2]. We have

$$\dim D^\lambda = \begin{cases} \frac{1}{2}(n-1)(n-4) & \text{if } n \equiv 0 \bmod 4, \\ \frac{1}{2}n(n-3) - 1 & \text{if } n \equiv 1 \bmod 4, \\ \frac{1}{2}(n-1)(n-4) - 1 & \text{if } n \equiv 2 \bmod 4, \\ \frac{1}{2}n(n-3) & \text{if } n \equiv 3 \bmod 4, \end{cases}$$

and

$$\dim D^\mu = \begin{cases} \frac{1}{6}n(n-2)(n-7) & \text{if } n \equiv 0 \bmod 4, \\ \frac{1}{6}n(n-1)(n-5) & \text{if } n \equiv 1 \bmod 4, \\ \frac{1}{6}(n-1)(n-2)(n-6) & \text{if } n \equiv 2 \bmod 4, \\ \frac{1}{6}(n+1)(n-1)(n-6) & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

If $\varphi \in \mathrm{IBr}_2(\mathrm{Alt}(n))$ is such that $\varphi(1)$ is divisible by a prime $r > n/2 > 2$, then $a(r) = 1$ or 0 depending on whether $r \leq n$ or $r > n$ and $c(r) = 0$ while $b(r) = 1$, therefore $2b(r) = 2 > a(r) + c(r)$ violating $(*)$.

If $n = 22$, then $n \equiv 2 \bmod 4$ so

$$\dim D^\lambda = \frac{1}{2}(22-1)(22-4) - 1 = 2^2 \cdot 47.$$

Similarly, if $n = 26$, then $n \equiv 2 \bmod 4$ so

$$\dim D^\lambda = \frac{1}{2}(26-1)(26-4) - 1 = 2 \cdot 137.$$

If $n = 24$, then $n \equiv 0 \bmod 4$ so

$$\dim D^{\mu} = \frac{1}{6} 24(24-2)(24-7) = 2^3 \cdot 11 \cdot 17.$$

Thus in each case, $\mathrm{Alt}(n)$ has a 2-Brauer character of degree divisible by a prime $r > n/2$ and so the result follows. The proof is now complete. □

3.2 *p*-solvable groups

In order to prove Theorem 1 under the assumption that G is p-solvable, we use the B_{π}-characters of Isaacs that were initially defined in [11, Definition 5.1]. Since the definition of B_{π}-characters is somewhat complicated, we do not repeat it here, but refer the interested reader to [11] or to the expository accounts in [12] and [13]. For our purposes, it is enough to know that if π is a set of primes and G is a π-separable group, then $\mathrm{B}_{\pi}(G)$ is a subset of $\mathrm{Irr}(G)$. In particular, if $\pi = p'$ and G is p-solvable, then restriction to the p-regular elements of G yields a bijection between $\mathrm{B}_{p'}(G)$ and $\mathrm{IBr}(G)$. We write G^o for the set of p-regular elements of G and if Θ is a character of G, then we write Θ^o for the restriction of Θ to G^o. Hence, the map $\chi \mapsto \chi^o$ is the bijection from $\mathrm{B}_{p'}(G)$ to $\mathrm{IBr}(G)$. For our work, we need to establish that this map preserves kernels.

Lemma 3.3. *Let p be a prime and let G be a p-solvable group. If $\varphi \in \mathrm{IBr}(G)$ and $\chi \in \mathrm{B}_{p'}(G)$ are such that $\chi^o = \varphi$, then $\ker(\varphi) = \ker(\chi)$.*

Proof. Let $N = \ker(\varphi)$ and $M = \ker(\chi)$. It follows that $\varphi = \chi^o$ is a Brauer character of G/M. Hence, φ has an F-representation $\mathcal{X}$ of G/M that affords φ. Observe that $N = \ker(\mathcal{X}) = \ker(\varphi)$ must contain M, so $M \leq N$. Conversely, suppose that $\mathcal{Y}$ is an F-representation that affords φ. Note that $N = \ker(\mathcal{Y})$. It follows that we can view $\varphi \in \mathrm{IBr}(G/N)$. We know that there exists $\psi \in \mathrm{B}_{p'}(G/N)$ so that $\psi^o = \varphi$. By [16], we showed that

$$\mathrm{B}_{\pi}(G) \cap \mathrm{Irr}(G/N) = \mathrm{B}_{\pi}(G/N)$$

for any set of primes π. This shows that $\mathrm{B}_{p'}(G/N) \subseteq \mathrm{B}_{p'}(G)$. Since there is a unique character in $\mathrm{B}_{p'}(G)$ that lifts φ, we conclude that $\chi = \psi$, and thus, $N \leq M$. We now have $M = N$ as desired. □

The next lemma will prove the 'if' part of Theorem 1.

Lemma 3.4. *Let p be a prime. If G has a normal Sylow p-subgroup P and G/P is nilpotent, then $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{IBr}(G)$.*

Proof. Suppose that G has a normal Sylow p-subgroup P and G/P is nilpotent. Let $\varphi \in \mathrm{IBr}(G)$. There is a character $\chi \in \mathrm{B}_{p'}(G)$ so that $\chi^o = \varphi$. As $P = \mathbf{O}_p(G)$, we have via [11, Corollary 5.3] that $P \leq \ker(\chi)$. Now, G/P is a p'-group and hence $\chi \in \mathrm{Irr}(G/P)$. Applying [8, Theorem A], we have $\chi(1)^2 \mid |G : \ker(\chi)|$. Since $\varphi(1) = \chi(1)$ and $\ker(\varphi) = \ker(\chi)$, we obtain the desired conclusion that $\varphi(1)^2$ divides $|G : \ker(\varphi)|$. □

We now prove the 'only if' part of Theorem 1 when G is p-solvable.

Theorem 3.5. *Let p be a prime, and let G be a p-solvable group. If $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{IBr}(G)$, then G has a normal Sylow p-subgroup P and G/P is nilpotent.*

Proof. Suppose that $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{IBr}(G)$. We will work by induction on $|G|$. Assume first that $\mathbf{O}_p(G) > 1$. It is easily seen that the hypothesis holds in $G/\mathbf{O}_p(G)$, so $G/\mathbf{O}_p(G)$ has a normal Sylow p-subgroup, meaning G has a normal Sylow p-subgroup. Moreover, induction (applied to $G/\mathbf{O}_p(G)$) shows that $G/\mathbf{O}_p(G)$ is nilpotent.

Thus we may assume $\mathbf{O}_p(G) = 1$. If we can prove that G is a p'-group, then we will have that $\mathrm{Irr}(G) = \mathrm{IBr}(G)$ and we will be done by [8, Theorem A]. We work to prove that G is a p'-group.

Let M be a minimal normal subgroup of G. Since $\mathbf{O}_p(G) = 1$, we know that M is a p'-subgroup of G. It is not difficult to see that G/M will satisfy the hypothesis. By induction, we see that G/M will have a normal Sylow p-subgroup N/M and G/N will be nilpotent. If K is a Sylow p-subgroup of G, then $N = MK$. The Frattini argument implies that $G = N\mathbb{N}_G(K) = MK\mathbb{N}_G(K) = M\mathbb{N}_G(K)$.

Note that if M_1 is a minimal normal subgroup of G other than M, then M_1 would also be a p'-subgroup of G, so $M_1 \cap N = 1$, so M_1 centralizes K. Applying the argument of the previous paragraph with M_1 in place of M, we obtain $G = M_1\mathbb{N}_G(K)$, and since M_1 centralizes K, we have $G = \mathbb{N}_G(K)$, and K is normal in G. Since $\mathbf{O}_p(G) = 1$, we have $K = 1$, and G is a p'-group as desired.

Therefore, we may assume that M is the unique minimal normal subgroup of G. Assuming M is not abelian, we now mimic the argument in [8, Case 1 of the proof of Theorem A] to obtain a contradiction. We know that M is the direct product of k copies of some nonabelian simple group S. Since M is unique, we know that $\mathbb{C}_G(M) = 1$. This implies that G is isomorphic to a subgroup of $\mathrm{Aut}(M)$, and $\mathrm{Aut}(M)$ is an extension of the direct product of k copies of $\mathrm{Aut}(S)$ by the symmetric group $\mathrm{Sym}(k)$.

Using [8, Proposition 2.1], there is a prime q and a character $\sigma \in \mathrm{Irr}(S)$ so that $\sigma(1)_q = |S|_q$ and $|\mathrm{Out}(S)|_q < |S|_q$. Since M is a p'-group and G is p-solvable, there is a character $\chi \in \mathrm{B}_{p'}(G) \cap \mathrm{Irr}(G \mid \sigma \times \cdots \times \sigma)$, and observe that $\sigma(1)^k$

divides $\chi(1)$. Since σ has q-defect 0, we see that $|M|_q$ divides $\chi(1)_q$. Using the same computation as in [8, Case 1 of the proof of Theorem A], we see that

$$|G|_q < {|M|_q}^2 \leq \chi(1)_q^2,$$

and so $\chi(1)^2$ does not divide $|G|$. If we let $\varphi = \chi^o \in \mathrm{I}_{p'}(G)$ $(= \mathrm{IBr}(G))$, then $\varphi(1)^2$ does not divide $|G : \ker(\varphi)|$, a contradiction.

Thus, we have that M is an abelian minimal normal subgroup of G. This implies that M is a q-group for some prime q. Observe that $\mathbb{N}_G(K)$ will normalize $\mathbb{C}_K(M)$, and obviously, M normalizes $\mathbb{C}_K(M)$. Hence, $\mathbb{C}_K(M)$ is normal in $G = M\mathbb{N}_G(K)$. Since $M \cap K = 1$ and M is the unique minimal normal subgroup of G, we conclude that $\mathbb{C}_K(M) = 1$. Hence, we can view $\mathrm{Irr}(M)$ as a faithful module for K with characteristic q.

Applying Isaacs' large orbit result [14, Theorem A], we see that there exists a character $\lambda \in \mathrm{Irr}(M)$ so that $|\mathbb{C}_K(\lambda)| < \sqrt{|K|}$, and observe that this implies that $|K : \mathbb{C}_K(\lambda)| > \sqrt{|K|}$. We see that $M\mathbb{C}_K(\lambda)$ is the stabilizer of λ in N. We know that λ extends to $\theta \in \mathrm{B}_{p'}(M\mathbb{C}_K(\lambda))$ and then $\gamma = \theta^N \in \mathrm{B}_{p'}(N)$. We have $\gamma(1) = |N : M\mathbb{C}_K(\lambda)| = |K : \mathbb{C}_K(\lambda)|$ and so, $\gamma(1)^2$ does not divide $|K|$. Let χ be an irreducible constituent of γ^G, and we know that $\chi \in \mathrm{B}_{p'}(G)$. Observe that $\chi(1)_p = \gamma(1)$ and $|K| = |G|_p$. It follows that $\chi(1)^2$ does not divide $|G|$ which is a contradiction. □

Now the 'only if' part of Theorem 1 follows from Theorems 3.2 and 3.5.

In [11], Isaacs used the B_π-characters to extend the ideas of Brauer characters to the set of primes π in place of a single prime p for π-separable groups. Thus, we consider our question in this setting. To do this, we review some of the key points of π-theory.

We fix a set of primes π and a π-separable group G. We define the *π-partial characters* of G to be the restrictions of characters of G to G^*, where G^* denotes the set of elements of G with order only divisible by primes in π. We define $\mathrm{I}_\pi(G)$ to be those π-partial characters that cannot be written as the sum of other partial characters. In [11, Theorem A], Isaacs proves that $\mathrm{I}_\pi(G)$ forms a basis for the set of complex-valued π-class functions on G. (Note that in [11] the symbol $I^\pi(G)$ is used to denote the π'-partial characters of G.) In [11, Corollary 10.2] it is shown that restriction is a bijection from $\mathrm{B}_\pi(G)$ to $\mathrm{I}_\pi(G)$. Recall that $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ and note that $\mathrm{I}_\pi(G)$ plays the role for π-partial characters that $\mathrm{IBr}(G)$ plays for Brauer characters.

Following Lemma 3.3, if $\varphi \in \mathrm{I}_\pi(G)$, then we define $\ker(\varphi) = \ker(\chi)$, where $\chi \in \mathrm{B}_\pi(G)$ such that $\chi^o = \varphi$. It now makes sense to ask whether the condition for p-solvable groups translates to a similar condition for π-separable groups. It is easy to obtain the following generalization of Lemma 3.4: Let π be a set of primes.

If G has a normal Hall π-complement N and G/N is nilpotent, then $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{I}_\pi(G)$.

For the converse of this statement, we can prove the following.

Theorem 3.6. *Let π be a set of primes, and let G be a finite π-separable group. If $\varphi(1)^2$ divides $|G : \ker(\varphi)|$ for all $\varphi \in \mathrm{I}_\pi(G)$, then one of the following occurs:*

(1) *G has a normal Hall π-complement N and G/N is nilpotent.*

(2) *There is a prime $p \in \pi$ and a π'-group K with a faithful module V of characteristic p so that $|\mathbb{C}_K(v)| \geq \sqrt{|K|}$ for all $v \in V$ and G has a section isomorphic to VK.*

In particular, the key for proving Theorem 3.5 is Isaacs' Large Orbit Theorem stating that if K is a q-group that acts coprimely on a group V, then there exists an element $v \in V$ so that $|\mathbb{C}_K(v)| < \sqrt{|K|}$. Recently, Halasi and Podoski have proven in [10, Corollary 1.4] that if p is a prime and K is a p'-group with a faithful module V of characteristic p, there exist $v, w \in V$ so that $\mathbb{C}_K(v) \cap \mathbb{C}_K(w) = 1$. We note that this had been proved earlier for supersolvable groups by Wolf in [23], for groups of odd order by Moretó and Wolf in [17], and for solvable groups by Dolfi [5] and Vdovin in [22]. If there exists $v \in V$ so that $|\mathbb{C}_K(v)| < \sqrt{|K|}$, then (2) does not hold. Thus, in a group as in conclusion (2), we have $|\mathbb{C}_K(v)| = |\mathbb{C}_K(w)| = \sqrt{|K|}$ and $G = \mathbb{C}_K(v)\mathbb{C}_K(w)$, so there exist at least two elements of v whose centralizers in K have size $\sqrt{|K|}$. However, for our purposes, we need the existence of an element $v \in V$ with $|\mathbb{C}_K(v)| < \sqrt{|K|}$. At this time, we do not see how to prove the existence of such an element, but we also do not know of any examples where this happens.

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Bibliography

[1] D. Benson, Spin modules for symmetric groups, *J. Lond. Math. Soc. (2)* **38** (1988), no. 2, 250–262.

[2] C. Bessenrodt and H. Weber, On p-blocks of symmetric and alternating groups with all irreducible Brauer characters of prime power degree, *J. Algebra* **320** (2008), no. 6, 2405–2421.

[3] R. Brauer, On a conjecture by Nakayama, *Trans. Roy. Soc. Canada. Sect. III. (3)* **41** (1947), 11–19.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford University Press, Oxford, 1985.

[5] S. Dolfi, Large orbits in coprime actions of solvable groups, *Trans. Amer. Math. Soc.* **360** (2008), no. 1, 135–152.

[6] A. Espuelas and G. Navarro, Blocks of small defect, *Proc. Amer. Math. Soc.* **114** (1992), no. 4, 881–885.

[7] S. M. Gagola, Jr., A character theoretic condition for $F(G) > 1$, *Comm. Algebra* **33** (2005), no. 5, 1369–1382.

[8] S. M. Gagola, Jr. and M. L. Lewis, A character-theoretic condition characterizing nilpotent groups, *Comm. Algebra* **27** (1999), no. 3, 1053–1056.

[9] A. Granville and K. Ono, Defect zero p-blocks for finite simple groups, *Trans. Amer. Math. Soc.* **348** (1996), no. 1, 331–347.

[10] Z. Halasi and K. Podoski, Every coprime linear group admits a base of size two, *Trans. Amer. Math. Soc.* **368** (2016), no. 8, 5857–5887.

[11] I. M. Isaacs, Characters of π-separable groups, *J. Algebra* **86** (1984), no. 1, 98–128.

[12] I. M. Isaacs, Characters of solvable groups, in: *The Arcata Conference on Representations of Finite Groups* (Arcata 1986), Proc. Sympos. Pure Math. 47, American Mathematical Society, Providence (1987), 103–109.

[13] I. M. Isaacs, The π-character theory of solvable groups, *J. Aust. Math. Soc. Ser. A* **57** (1994), no. 1, 81–102.

[14] I. M. Isaacs, Large orbits in actions of nilpotent groups, *Proc. Amer. Math. Soc.* **127** (1999), no. 1, 45–50.

[15] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia Math. Appl. 16, Addison-Wesley, Reading, 1981.

[16] M. L. Lewis, B_π-characters and quotients, preprint (2016), https://arxiv.org/abs/1609.02029.

[17] A. Moretó and T. R. Wolf, Orbit sizes, character degrees and Sylow subgroups, *Adv. Math.* **184** (2004), no. 1, 18–36.

[18] G. Navarro, *Characters and Blocks of Finite Groups*, London Math. Soc. Lecture Note Ser. 250, Cambridge University Press, Cambridge, 1998.

[19] J. B. Olsson, Lower defect groups in symmetric groups, *J. Algebra* **104** (1986), no. 1, 37–56.

[20] J. B. Olsson, On the p-blocks of symmetric and alternating groups and their covering groups, *J. Algebra* **128** (1990), no. 1, 188–213.

[21] G. D. B. Robinson, On a conjecture by Nakayama, *Trans. Roy. Soc. Canada. Sect. III. (3)* **41** (1947), 20–25.

[22] E. P. Vdovin, Regular orbits of solvable linear p'-groups, *Sib. Èlektron. Mat. Izv.* **4** (2007), 345–360.

[23] T. R. Wolf, Large orbits of supersolvable linear groups, *J. Algebra* **215** (1999), no. 1, 235–247.

[24] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.5, 2016, http://www.gap-system.org.

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