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Group varieties not closed under cellular covers and localizations

Daniel Herden, Montakarn Petapirak and José L. Rodríguez

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Dedicated to the memory of Rüdiger Göbel

Abstract. A group homomorphism $e : H \to G$ is a *cellular cover* of G if for every homomorphism $\varphi : H \to G$ there is a unique homomorphism $\bar{\varphi} : H \to H$ such that $\bar{\varphi}e = \varphi$. Group localizations are defined dually. The main purpose of this paper is to establish $2^{\aleph_0}$ varieties of groups which are not closed under taking cellular covers. This will use the existence of a special Burnside group of exponent p for a sufficiently large prime p as a key witness. This answers a question raised by Göbel in [12]. Moreover, by using a similar witness argument, we can prove the existence of $2^{\aleph_0}$ varieties not closed under localizations. Finally, the existence of $2^{\aleph_0}$ varieties of groups neither closed under cellular covers nor under localizations is presented as well.

1 Introduction

In recent years, there has been an increasing interest in cellular covers of groups. These were first treated categorically by the third named author and Scherer in [28] to compute the fundamental group of cellular approximations of spaces in the sense of Bousfield and Dror Farjoun. Many initially significant results were later published in [8] by Dror Farjoun, Göbel and Segev. They mainly studied cellular covers of nilpotent groups and finite groups, and also some properties of groups inherited by their cellular covers. It was observed that cellular covers of nilpotent groups of class n are nilpotent of class n as well, in particular, cellular covers of abelian groups are abelian. This shows that the variety $\mathfrak{N}_n$ of nilpotent groups of class n is closed under taking cellular covers as well as the variety $\mathfrak{A}_n$ of abelian groups of exponent n. Hence, we obtain examples of countably many distinct varieties which are closed under taking cellular covers. Some more studies on cellular covers of particular groups and of groups with specific additional properties have

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been conducted and can be found in a considerable amount of literature; see e.g. [6, 7, 9–12, 26, 29]. At the moment, this area of research is still active with many interesting open questions.

In 2010, Göbel [12] examined cellular covers of R-modules and of varieties of groups, and established an example of countably many distinct varieties which are not closed under taking cellular covers by considering the Burnside variety $\mathfrak{B}_p$ of exponent p for primes $p > 10^{75}$:

Theorem 1.1. *It is clear that the only abelian subvarieties of $\mathfrak{B}_p$ are the trivial variety $\{1\}$ and the variety $\mathfrak{A}_p$ of abelian groups of exponent p. But the variety*

$$\operatorname{cell}\mathfrak{B}_p := \langle H \mid H \text{ is a cellular cover of } G \in \mathfrak{B}_p\rangle$$

contains (for any prime $p > 10^{75}$) all abelian groups. Thus $\mathfrak{B}_p$ is not cellular closed.

However, the question of whether or not there exist uncountably many such varieties was left open, and it is natural to ask:

Question 1. Are there $2^{\aleph_0}$ varieties of groups which are not closed under cellular covers?

This constitutes the main motivation of this paper, where we will establish $2^{\aleph_0}$ pairwise distinct varieties of groups which are not cellular closed in Theorem 3.4. This result combines Ol'shanskii's 1970 proof (see [23]) of the existence of a continuum of distinct varieties with his construction of a special Burnside group $\mathscr{B}_p$ of exponent p for any sufficiently large prime p. In addition, in Section 2 we provide the construction of countably many distinct varieties neither cellular closed nor finitely based with the help of an explicit system of relations. The existence of $2^{\aleph_0}$ pairwise distinct varieties of groups not closed under localizations, which acts as a dual result to Theorem 3.4, is presented in Section 4. In Section 5, the existence of $2^{\aleph_0}$ varieties of groups which are neither closed under cellular covers nor under localizations is established.

Our notation is standard, as found in [15, 17, 21, 25]. We write homomorphisms on the right-hand side.

2 Countably many varieties neither cellular closed nor finitely based

As in [12], we deal with varieties of groups, i.e., classes of groups which are closed under taking subgroups, quotients, and direct products, and consider the additional operation of taking cellular covers.

Definition 2.1. A homomorphism $e : H \to G$ is a *cellular cover* of G if every homomorphism $\varphi : H \to G$ factors uniquely through e, i.e., there is a unique homomorphism $\bar{\varphi} : H \to H$ such that $\bar{\varphi} e = \varphi$:

$$\begin{tikzcd} H \arrow[d, dotted, "\exists!\bar{\varphi}"'] \arrow[dr, "\forall\varphi"] & \\ H \arrow[r, "e"'] & G. \end{tikzcd}$$

In a slight abuse of notation, we will often identify $e : H \to G$ with the group H and therefore call H a *cellular cover* of G. With this in mind, a variety $\mathfrak{V}$ of groups is *closed under cellular covers* if it contains all cellular covers of groups in $\mathfrak{V}$. By cell $\mathfrak{V}$ we will denote the smallest variety containing $\mathfrak{V}$ which is closed under taking cellular covers.

In general, if $e : H \to G$ is a cellular cover, then $e : H \to \operatorname{Im} e$ is a cellular cover too, and $1 \to \operatorname{Ker} e \to H \to \operatorname{Im} e \to 1$ is a central extension ($\operatorname{Ker} e \subseteq \mathfrak{z}H$). Moreover, if $e : H \to G$ is a surjective cellular cover, then $H/\mathfrak{z}H \cong G/\mathfrak{z}G$.

For any variety of groups $\mathfrak{V}$, let $\mathfrak{V}'$ be the variety generated by the quotients $G/\mathfrak{z}G$ of groups $G \in \mathfrak{V}$, and let $\mathfrak{V}^c$ denote the variety of central extensions of groups in $\mathfrak{V}'$. Then cell $\mathfrak{V} \subseteq \mathfrak{V}^c$ holds (see [12, Theorem 6.3]). As the variety $\mathfrak{N}_n$ satisfies $\mathfrak{N}_n = (\mathfrak{N}_n)^c$, for any integer n, it follows that these varieties are closed under cellular covers; for more details see [12].

The *Burnside variety* $\mathfrak{B}_n$ *of exponent* n is the variety defined by the law $x^n = 1$. That is, for any group $G \in \mathfrak{B}_n$, $g^n = 1$ for all $g \in G$. For convenience, we call any group in a Burnside variety a *Burnside group*.

In 1902, Burnside [3] asked if it is the case that every finitely generated group in which every element is of finite order must necessarily be finite. In 1968 Novikov and Adjan constructed a counterexample in [22], which established a finitely generated but infinite group for odd exponents $n \geq 4381$. Later, Ol'shanskii shortened their lengthy construction for odd $n > 10^{10}$ in [24]. By applying geometric ideas to defining relations in groups, Ol'shanskii gave in [25] for primes $p > 10^{75}$ an inductive construction for a particular group $\mathcal{B}_p$ which is infinite such that every nontrivial proper subgroup is cyclic of order p. This particular group is a so-called *Tarski monster group* which turns out to be a negative answer to Burnside's question as well. We recall some essential properties of $\mathcal{B}_p$.

(i) $\mathcal{B}_p$ is infinite.

(ii) Every nontrivial proper subgroup of $\mathcal{B}_p$ is cyclic of order p.

(iii) $\mathcal{B}_p$ is generated by two elements.

Any such group must be trivially simple.

Lemma 2.2. *If the group B satisfies conditions* (i) *and* (ii)*, then B is simple.*

Proof. To show that B is simple, suppose that $N \neq B$ is a nontrivial normal subgroup of B. It follows that N is a proper subgroup of order p. Let g be any element in $B \setminus N$. Because N is normal and g is of finite order, $\langle g, N \rangle = \langle g \rangle N$ is a proper finite subgroup of order $> p$, a contradiction. □

We recall one more useful result, cf. [25, Theorem 31.1].

Theorem 2.3. *If $\mathcal{B}_p = F/N$ is a free presentation of $\mathcal{B}_p$, then $N/[F,N]$ is a free abelian group of countable rank.*

Remark 2.4. Let $X = \{x_1, x_2\}$ be a set of two generators and $F = F(X)$ the associated free group. We have an aspherical representation $\mathcal{B}_p = \langle X \mid \mathcal{R} \rangle$ for a suitable set $\mathcal{R}$ of relators. In particular, we have $\mathcal{B}_p = F/N$ with $N = \mathcal{R}^F$ the normal closure of the set of relators $\mathcal{R}$. It is clear that $N/[F,N]$ is abelian, however, the asphericity of the presentation implies that $\{r = 1 \mid r \in \mathcal{R}\}$ is an independent set of relations and, consequently, that $\{r[F,N] \mid r \in \mathcal{R}\}$ is a basis of $N/[F,N]$.

Let F/N be a free presentation of the Burnside group $\mathcal{B}_p$. Since the Schur multiplier $H_2(\mathcal{B}_p, \mathbb{Z}) = (F' \cap N)/[F,N]$ is a subgroup of $N/[F,N]$, $H_2(\mathcal{B}_p, \mathbb{Z})$ is free abelian as well. More precisely, it is well known that the Schur multiplier $H_2(\mathcal{B}_p, \mathbb{Z})$ is free abelian of infinite countable rank, cf. [25, Corollary 31.2].

For the next step, we need some more tools: A group G is said to be *perfect* if it is equal to its own commutator subgroup, i.e., $G = G' = [G,G]$; see e.g. [27]. Following [2], an epimorphism $f : G \to Q$ is called a *perfect cover* of Q if G is perfect and $\operatorname{Ker} f \subseteq \mathfrak{z}G$. We note that [17, Theorem 2.10.3] transfers immediately from the case of finite perfect groups to the infinite perfect group $\mathcal{B}_p$.

Lemma 2.5. *The group $\mathcal{B}_p = F/N$ has a central extension*

$$1 \to K \to A \xrightarrow{e} \mathcal{B}_p \to 1,$$

where $K := (F' \cap N)/[F,N]$, $A := F'/[F,N]$, and

$$e : F'/[F,N] \to F/N \quad (f[F,N] \mapsto fN)$$

is a perfect cover.

Remark 2.6. This central extension is also known as the *universal perfect cover* of $\mathcal{B}_p$ and has a number of important properties; see for example [2] and [17]. In a similar fashion, a universal perfect cover is possible for any perfect group. Observe here that $\mathcal{B}_p$ is clearly perfect as it is a non-abelian simple group.

The perfect cover e is also a cellular cover of $\mathcal{B}_p$. The following argument is due to Göbel [12].

Lemma 2.7. *The group $\mathcal{B}_p$ has a cellular cover $e : A \to \mathcal{B}_p$, where $\mathcal{B}_p = F/N$, $A = F'/[F,N]$, and*

$$e : F'/[F,N] \to F/N \quad (f[F,N] \mapsto fN).$$

The group A contains a copy of $\mathbb{Z}$ as subgroup.

Proof. We have to show that $e : A \to \mathcal{B}_p$ induces a bijection

$$e_* : \mathrm{Hom}(A,A) \to \mathrm{Hom}(A,\mathcal{B}_p) \quad (\bar{\varphi} \mapsto \bar{\varphi}e).$$

First, we claim that the map e_* is surjective: Let $\varphi \in \mathrm{Hom}(A,\mathcal{B}_p)$. We must find $\bar{\varphi} \in \mathrm{Hom}(A,A)$ such that $\bar{\varphi}e = \varphi$. If $\varphi = 0$, then we choose $\bar{\varphi} = 0$. So $\bar{\varphi}e = \varphi$ clearly holds. Next, we assume that $\varphi \neq 0$, then $1 \neq A\varphi \subseteq \mathcal{B}_p$. If $A\varphi \neq \mathcal{B}_p$, then, by the choice of $\mathcal{B}_p$, $A\varphi$ is cyclic hence abelian. Since A is perfect, we have that

$$A\varphi = [A,A]\varphi = [A\varphi, A\varphi],$$

i.e., $A\varphi$ is perfect as well. Consequently, we obtain that $A\varphi = [A\varphi, A\varphi] = 1$ which contradicts the assumption that $\varphi \neq 0$. Thus, we now have that $A\varphi = \mathcal{B}_p$, i.e., φ is surjective. As $\mathcal{B}_p$ is simple, we know that $\mathfrak{z}\mathcal{B}_p = 1$. Moreover, we can see that $(\mathfrak{z}A)\varphi \subseteq \mathfrak{z}\mathcal{B}_p = 1$ because φ is surjective. From the above central extension, we obtain that $K \subseteq \mathfrak{z}A \subseteq \mathrm{Ker}\,\varphi$ and $\varphi = e\beta$ follows for some $\beta \in \mathrm{Hom}(\mathcal{B}_p,\mathcal{B}_p)$. Next, we claim that $\beta \in \mathrm{Aut}(\mathcal{B}_p)$. Since $\varphi \neq 0$ and $\mathcal{B}_p$ is simple, we have that $\mathrm{Ker}\,\beta = 1$, which shows that β is injective. We can see that β is also surjective by using the fact that φ is surjective. Thus, we finally have that $\beta \in \mathrm{Aut}(\mathcal{B}_p)$, hence $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,e = K$.

We next consider the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N/[F,N] & \longrightarrow & F/[F,N] & \xrightarrow{\lambda} & \mathcal{B}_p & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \\
1 & \longrightarrow & K & \longrightarrow & A & \xrightarrow{e} & \mathcal{B}_p & \longrightarrow & 1,
\end{array}$$

where the map λ is an epimorphism defined by sending $f[F,N] \in F/[F,N]$ to $fN \in F/N$. By the help of [17, Lemma 2.4.1], we obtain a homomorphism $\alpha : F/[F,N] \to A$ such that the above diagram commutes, i.e., $\alpha e = \lambda\beta$. Recall that $A = F'/[F,N] \subseteq F/[F,N]$. Hence, $\alpha : F/[F,N] \to A$ can be restricted to $\bar{\varphi} := \alpha \restriction A$. Then $\bar{\varphi} \in \mathrm{Hom}(A,A)$. But $e = \lambda \restriction A$, and hence $\varphi = e\beta = \bar{\varphi}e$.

Next, we will show that e_* is injective: Let $\varphi_1, \varphi_2 \in \mathrm{Hom}(A, A)$ such that $\varphi_1 e = \varphi_2 e$. Define

$$\psi : A \to A \quad (a \mapsto (a\varphi_1)(a^{-1}\varphi_2)).$$

It is easy to see that $A\psi \subseteq \mathrm{Ker}\, e = K$. For $a, b \in A$, we obtain that

$$(ab)\psi = (ab)\varphi_1(ab)^{-1}\varphi_2 = a\varphi_1(b\varphi_1 b^{-1}\varphi_2)a^{-1}\varphi_2 = a\varphi_1 b\psi a^{-1}\varphi_2.$$

Using the fact that $K = \mathfrak{z}A$, since $b\psi \in K$ and $a^{-1}\varphi_2 \in A$, we then obtain

$$(ab)\psi = a\varphi_1 a^{-1}\varphi_2 b\psi = a\psi b\psi,$$

i.e., $\psi \in \mathrm{Hom}(A, K)$. Moreover, as A is perfect and $A\psi \subseteq K = \mathfrak{z}A$ is abelian, we obtain $A\psi = [A, A]\psi = [A\psi, A\psi] = 1$. As a result, $\psi = 0$ and then

$$1 = a\varphi_1 a^{-1}\varphi_2 = a\varphi_1(a\varphi_2)^{-1}$$

for all $a \in A$. This shows that $\varphi_1 = \varphi_2$. Thus e_* is injective, as desired.

Therefore, we have that $e : A \to \mathcal{B}_p$ is a cellular cover of $\mathcal{B}_p$. Note that A contains $K = H_2(\mathcal{B}_p, \mathbb{Z})$ which is free abelian of infinite countable rank. □

Remark 2.8. In fact, Lemma 2.5 provides the *universal central extension* of the perfect group $\mathcal{B}_p$. In this context, Lemma 2.7 gives a more direct proof of the general result [8, Lemma 3.10] that every universal central extension corresponds to a surjective cellular cover.

For our countably many varieties neither cellular closed nor finitely based, let $\mathfrak{V}_p$ be the variety defined by the following infinite system of group laws:

$$\begin{aligned} [x_1, x_2]^p &= 1, \\ ([x_1, x_2][x_3, x_4])^p &= 1, \\ &\vdots \\ ([x_1, x_2] \dots [x_{2k-1}, x_{2k}])^p &= 1, \\ &\vdots \end{aligned} \tag{2.1}$$

Ol'shanskii proved in [25] for all primes $p > 10^{10}$ that $\mathfrak{V}_p$ is not finitely based. In fact, [25, Lemma 31.4] provides a system of groups witnessing the non-finiteness of (2.1) within $\mathfrak{B}_{p^2}$. Thus the infinite system of group laws

$$\begin{aligned} x_1^{p^2} &= 1, \\ [x_1, x_2]^p &= 1, \\ &\vdots \\ ([x_1, x_2] \dots [x_{2k-1}, x_{2k}])^p &= 1, \\ &\vdots \end{aligned} \tag{2.2}$$

defines a variety $\mathfrak{W}_p \subseteq \mathfrak{B}_{p^2}$ which is not finitely based as well. In addition, for primes $p > 10^{75}$, this variety $\mathfrak{W}_p$ will contain the group $\mathcal{B}_p$ but clearly not the group $\mathbb{Z}$ of integers. By Lemma 2.7, the cellular cover A of $\mathcal{B}_p$ cannot belong to $\mathfrak{W}_p$. This shows that this variety is not closed under taking cellular covers, which establishes the following result.

Theorem 2.9. *There exist countably many pairwise distinct varieties of groups which are neither cellular closed nor finitely based.*

3 Uncountably many varieties not closed under cellular covers

As every variety can be described by a suitable system of group laws and the set of possible laws has cardinality $\aleph_0$, this immediately implies that there exist not more than $2^{\aleph_0}$ varieties, and the question whether there indeed exist $2^{\aleph_0}$ pairwise distinct varieties was raised by B. H. Neumann [20] in 1937. One might expect that a positive answer to this question can be given as follows: Start with a countable system of laws which is not equivalent to any finite system of laws. It should then be possible to construct $2^{\aleph_0}$ pairwise distinct varieties by choosing suitable subsets of this system of laws. This idea was realized in the constructions of $2^{\aleph_0}$ varieties of groups by Adjan [1] and Vaughan-Lee [30]. Unfortunately, not every infinite system of laws qualifies for this approach and, in particular, the system of laws (2.1) is unsuitable as every law implies all its predecessors (by replacing some variables by the identity). Here we will abstain from this approach and utilize instead the ideas of a construction presented by Ol'shanskii [23] in 1970.

We denote the variety of abelian groups of exponent n by $\mathfrak{A}_n$. Applying some properties of locally finite varieties and monolithic groups, Ol'shanskii's paper [23] proved the following result.

Lemma 3.1. *Let e be a positive integer, and assume that there exists an infinite series of finite groups T_i, $i \in \omega$, such that for each i,*

(i) *T_i is from a fixed locally finite variety $\mathfrak{V} \subseteq \mathfrak{B}_e$,*

(ii) *T_i is monolithic, and*

(iii) *T_i is not isomorphic to any factor of T_j for $i \neq j$.*

If p is a prime not dividing e, then the product variety $\mathfrak{A}_p\mathfrak{V}$ has $2^{\aleph_0}$ pairwise distinct subvarieties.

Remark 3.2. Note that $\mathfrak{A}_p\mathfrak{V}$ as product of locally finite varieties is again locally finite, cf. [27, Theorem 14.3.1].

The main implication from [23] is the following corollary.

Corollary 3.3. *The following statements hold.*

(a) *For any $e = 8p_1$, where p_1 is an odd prime, the conditions of Lemma* 3.1 *are satisfied by a suitable series T_i, $i \in \omega$, of finite solvable groups of length* 4.

(b) *For any distinct odd primes p_1 and p_2 there exist $2^{\aleph_0}$ pairwise distinct subvarieties of the locally finite variety of length 5 solvable groups of exponent $8p_1p_2$.*

We apply these results to obtain the existence of $2^{\aleph_0}$ pairwise distinct varieties of groups which are not closed under cellular covers. Moreover, this answers a question raised in [12].

Theorem 3.4. *There exist $2^{\aleph_0}$ pairwise distinct varieties of groups which are not closed under cellular covers.*

Proof. Let $p_3 > 10^{75}$ be a prime, and let $\mathfrak{B}_{p_3}$ be the associated Burnside variety and $\mathcal{B}_{p_3} \in \mathfrak{B}_{p_3}$ the Burnside group mentioned in Section 2. Next, we apply the idea of enlarging two given varieties $\mathfrak{U}$ and $\mathfrak{V}$ by considering the product variety $\mathfrak{U}\mathfrak{V}$, which consists of all extensions of a group in $\mathfrak{U}$ by a group in $\mathfrak{V}$.

Since $\mathfrak{B}_{p_3}$ is not the variety of all groups, by applying the right cancellation law for product varieties, we have that $\mathfrak{V}\mathfrak{B}_{p_3} \neq \mathfrak{V}'\mathfrak{B}_{p_3}$ for any varieties $\mathfrak{V} \neq \mathfrak{V}'$. See [21] for more details on varieties of groups.

Thus, there remain $2^{\aleph_0}$ pairwise distinct product varieties $\mathfrak{V}\mathfrak{B}_{p_3}$, where $\mathfrak{V}$ is a variety obtained from Corollary 3.3 (b). It is clear that $\mathfrak{V}\mathfrak{B}_{p_3}$ contains the group $\mathcal{B}_{p_3}$, as $\mathcal{B}_{p_3} \in \mathfrak{B}_{p_3} \subseteq \mathfrak{V}\mathfrak{B}_{p_3}$. Since $\mathfrak{V}$ is of exponent $8p_1p_2$ and $\mathfrak{B}_{p_3}$ is of exponent p_3, the product variety $\mathfrak{V}\mathfrak{B}_{p_3}$ is of exponent $8p_1p_2p_3$. But, as shown in Lemma 2.7, there exists a cellular cover $e : A \to \mathcal{B}_{p_3}$ such that $\mathbb{Z} \subseteq A$. Therefore, A cannot be of finite exponent and obviously is not in the product variety $\mathfrak{V}\mathfrak{B}_{p_3}$. This shows the existence of $2^{\aleph_0}$ pairwise distinct varieties which are not closed under cellular covers. □

The next result actually strengthens Theorem 2.9.

Corollary 3.5. *There exist $2^{\aleph_0}$ pairwise distinct varieties of groups which are neither cellular closed nor finitely based.*

Proof. From Theorem 3.4 we obtain $2^{\aleph_0}$ varieties which are not cellular closed. As there are only countably many finitely based varieties, there actually must exist $2^{\aleph_0}$ pairwise distinct varieties which are neither cellular closed nor finitely based. □

Remark 3.6. Observe that the existence of $2^{\aleph_0}$ pairwise distinct not finitely based varieties in Corollary 3.5 is derived only implicitly by set-theoretic considerations, while in Theorem 2.9 countably many such varieties are explicitly given with the help of a suitable system of group laws (2.2).

4 Uncountably many varieties not closed under localizations

In this section, we will present a result dual to Theorem 3.4 for localizations. Cellular covers are also known as co-localizations. We can see the resemblance between cellular covers and localizations in the following definition which can also be found in [13] and [14] (cf. [4, Lemma 2.1]).

Definition 4.1. A group homomorphism $e : G \to H$ is a *localization* of G if every homomorphism $\varphi : G \to H$ factors uniquely through e, i.e., there is a unique homomorphism $\bar{\varphi} : H \to H$ such that $e\bar{\varphi} = \varphi$. It is equivalent to saying that $e : G \to H$ is a localization of G if the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "e"] \arrow[dr, "\forall\varphi"'] & H \arrow[d, dotted, "\exists!\bar{\varphi}"] \\ & H. \end{tikzcd}$$

In a slight abuse of notation, we will often identify $e : G \to H$ with the group H and therefore call H a *localization* of G.

As with cellular covers, there has been research into which properties of groups transfer to their localizations; see [4] or [5] for motivating examples. Libman showed in [19] that localizations fail to preserve finiteness of groups, i.e., a localization of a finite group need not be finite. In particular, he proved for all even integers $n \geq 10$ that the canonical irreducible representation $\rho : A_n \to \mathrm{SO}(n-1, \mathbb{R})$ is a localization of A_n, where A_n is the alternating group on n letters and $\mathrm{SO}(n, \mathbb{R})$ is the special orthogonal group of orthogonal $n \times n$ matrices with determinant 1 over the field of real numbers. Thus, there exists a localization of infinite order of the finite group A_n. So it is reasonable to ask about the existence of an upper bound for the cardinality of localizations of A_n. The answer to this question is given by the following result, keeping in mind that A_n is non-abelian and simple for all $n \geq 5$. A proof of this result can be found in [13] under the set-theoretic assumption ZFC + GCH and, more generally, without any set-theoretic restrictions in [14].

For more details, we quote the main theorem from [14] in Theorem 4.3 which needs the following definition.

Definition 4.2. We say that a group G is *suitable* if the following conditions hold:

(i) $G \neq 1$ is finite with trivial center.

(ii) $\mathrm{Aut}(G)$ is complete.

(iii) If $\bar{G} \subseteq \mathrm{Aut}(G)$ and $\bar{G} \cong G$, then $\bar{G} = \mathrm{Inn}(G)$.

Recall that a group H is said to be *complete* if its center $\mathfrak{z}H$ and outer automorphism group $\mathrm{Out}(H) := \mathrm{Aut}(H)/\mathrm{Inn}(H)$ are both trivial; cf. [27, p. 412].

Theorem 4.3. *Let $\mathcal{A}$ be a family of suitable groups and μ an infinite cardinal such that $\mu^{\aleph_0} = \mu$. Then we can find a group H of cardinality $\lambda = \mu^+$ such that the following holds:*

(i) *The group H is simple. Moreover, if $1 \neq h \in H$, then any element of H is a product of at most four conjugates of h.*

(ii) *Any $G \in \mathcal{A}$ is a subgroup of H and any two different groups in $\mathcal{A}$ have trivial intersection* 1 *when considered as subgroups of H. If $\mathcal{A}$ is not empty, then $H[\mathcal{A}] = H$, where the $\mathcal{A}$*-socle *$H[\mathcal{A}]$ is the subgroup of H generated by all copies of groups $G \in \mathcal{A}$ in H.*

(iii) *Any monomorphism $\varphi : G \to H$, $G \in \mathcal{A}$, is induced by some $h \in H$, i.e., there is an element $h \in H$ such that $\varphi = h^* \restriction G$.*

(iv) *If $\bar{G} \subseteq H$ is an isomorphic copy of some $G \in \mathcal{A}$, then the centralizer $C_H(\bar{G})$ is trivial.*

(v) *Any monomorphism $\varphi : H \to H$ is an inner automorphism, i.e., $\varphi = h^*$ for some $h \in H$.*

(vi) *The group H contains a free subgroup of rank μ.*

Note that condition (vi) is an extra property of the group H derived from [14, Construction 3.4 (i)]. In particular, H is complete with conditions (i) and (v).

Next, consider a non-abelian finite simple group G. We see that condition (i) of Definition 4.2 holds trivially, and condition (ii) is an immediate consequence of [27, Theorem 13.5.10]. For condition (iii), let now $\bar{G} \subseteq \mathrm{Aut}(G)$ with $\bar{G} \cong G$. If $\bar{G} \subseteq \mathrm{Inn}(G)$, then $\bar{G} = \mathrm{Inn}(G)$ by cardinality. Hence, assume $\bar{G} \not\subseteq \mathrm{Inn}(G)$. In this case, the simple group $\bar{G}$ embeds canonically as a subgroup of $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. According to Schreier's conjecture [27, p. 403], $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is solvable. Thus, G is both non-abelian finite simple and solvable, a contradiction. Therefore, one obtains the following (cf. [13, Proposition 2 (a)]):

Lemma 4.4. *Any non-abelian finite simple group is suitable.*

As an immediate consequence of Lemma 4.4 and Theorem 4.3 we have the following

Proposition 4.5. *The following statements hold.*

(a) *For any even $n \geq 10$ there exists a localization of A_n with a free subgroup of countable rank.*

(b) *If G is a non-abelian finite simple group and κ an infinite cardinal, then there exists a localization of G with a free subgroup of rank κ.*

Proof. (a) This is an immediate consequence of ((b)) as A_n is simple for all $n \geq 5$. Alternatively, we may use Libman's localization $\rho : A_n \to \mathrm{SO}(n-1, \mathbb{R})$: We know that $\mathrm{SO}(3, \mathbb{R})$, and hence any $\mathrm{SO}(n, \mathbb{R})$ for $n \geq 3$, contains a free subgroup of rank 2; cf. [16, pp. 469–472]. But it is well known that a free group of rank 2 has a free subgroup of countable rank; cf. [18, Vol. II].

(b) All non-abelian finite simple groups are suitable, by Lemma 4.4. Thus, applying Theorem 4.3 for the choice $\mathcal{A} = \{G\}$ and $\mu = \kappa^{\aleph_0}$, there exists a group H containing G with the following properties:

(i) The group H is simple.

(iii) Any monomorphism from G to H is a restriction of an inner automorphism of H.

(iv) The centralizer of G in H is trivial.

(v) Any monomorphism from H to H is an inner automorphism of H.

(vi) The group H contains a free subgroup of rank $\kappa^{\aleph_0} \geq \kappa$.

It remains to show that the inclusion $e : G \to H$ is a localization of G, i.e., we will show that for any $\varphi : G \to H$, there exists a unique $\bar{\varphi} : H \to H$ such that $e\bar{\varphi} = \varphi$.

The proof taken from [13, Section 1] is the following. If $\varphi = 0$, then choose $\bar{\varphi} = 0$. We obtain $Ge\bar{\varphi} = G\bar{\varphi} = 1 = G\varphi$. Since H is simple and $1 \neq G \subseteq \operatorname{Ker} \bar{\varphi}$, the zero map is the only possibility for $\bar{\varphi}$.

If $\varphi \neq 0$, then we obtain that $\operatorname{Ker} \varphi = 1$ because $\operatorname{Ker} \varphi \trianglelefteq G$ and G is simple. Thus, φ is a monomorphism. By (iii), φ is a restriction of an inner automorphism of H, say $\varphi = \bar{h}^* \restriction G$, where $\bar{h} \in H$. Choose $\bar{\varphi} = \bar{h}^*$. Then $e\bar{\varphi} = \varphi$ follows.

Next, assume that $\varphi' : H \to H$ is another homomorphism such that $e\varphi' = \varphi$. Consequently, $\varphi' \neq 0$, and hence φ' is a monomorphism of H, by (i). Thus, by (v), there exists $h \in H$ such that $\varphi' = h^*$. We then obtain that

$$gh^* = g\varphi' = ge\varphi' = g\varphi = ge\bar{\varphi} = g\bar{\varphi} = g\bar{h}^*$$

for all $g \in G$. Hence, $h^* = \bar{h}^*$ on G, and $h^{-1}gh = \bar{h}^{-1}g\bar{h}$ for all $g \in G$. This implies that $\bar{h}h^{-1} \in C_H(G)$. Therefore, (iv) implies $\bar{h} = h$, hence $\bar{\varphi} = \varphi'$. This shows the uniqueness of $\bar{\varphi}$ and completes the proof. □

Proposition 4.5 (b) directly implies:

Corollary 4.6. *Any non-abelian finite simple group has a localization which contains a free subgroup of arbitrarily large cardinality.*

This observation can be utilized to give an explicit example of $2^{\aleph_0}$ varieties neither closed under localizations nor finitely based.

Theorem 4.7. *There exist $2^{\aleph_0}$ pairwise distinct varieties of groups which are neither closed under localizations nor finitely based.*

Proof. Consider the result by Adjan [1] which shows that, for all odd $n \geq 4381$, the set of relations

$$(x^{rn}y^{rn}x^{-rn}y^{-rn})^n = 1, \tag{4.1}$$

where r is a prime, is irreducible. In particular, any infinite set of such relations automatically generates a variety which is not finitely based. Let $\mathfrak{V}$ be any such variety. It is easy to see that every group of exponent n satisfies all the laws (4.1), more precisely, $\mathfrak{B}_n \subseteq \mathfrak{V}$.

Let now $n \geq 4381$ be odd and divisible by 15 (e.g., $n = 4395$). Note that A_5 is a non-abelian finite simple group with exponent 30. For any element $x \in A_5$, it follows that $(x^{rn})^2 = x^{2rn} = 1$ holds because $30 \mid 2n$. Thus,

$$x^{rn} = x^{-rn}$$

which implies that the relation

$$(x^{rn}y^{rn}x^{-rn}y^{-rn})^n = (x^{rn}y^{rn}x^{rn}y^{rn})^n = (x^{rn}y^{rn})^{2n} = 1$$

is satisfied in A_5. This shows that A_5 is a member of $\mathfrak{V}$. But, by Corollary 4.6, this gives us that $\mathfrak{V}$ is not closed under localizations as otherwise $\mathfrak{V}$ would contain free groups of arbitrary rank, hence all groups, which is a contradiction. This establishes $2^{\aleph_0}$ pairwise distinct varieties of groups neither closed under localizations nor finitely based. □

Remark 4.8. It is not apparent whether or not Adjan's varieties $\mathfrak{V}$ are closed under cellular covers. In particular, our argumentation from Theorem 2.9 fails as $\mathfrak{A} \subseteq \mathfrak{V}$ for the variety $\mathfrak{A}$ of all abelian groups, a problem which seems difficult to bypass.

5 Uncountably many varieties neither closed under cellular covers nor under localizations

In this section, we will discuss some further consequences of Corollary 3.3 (b).

Theorem 5.1. *There exist $2^{\aleph_0}$ pairwise distinct locally finite varieties of groups which are not closed under localizations.*

Proof. Let S be any non-abelian finite simple group and set $\mathfrak{W} := \mathrm{var}\{S\}$. Then $\mathfrak{W}$ is locally finite, cf. [21, Theorem 15.71]. Applying the same argument as in Theorem 3.4, we have $\mathfrak{V}\mathfrak{W} \neq \mathfrak{V}'\mathfrak{W}$ for any varieties $\mathfrak{V} \neq \mathfrak{V}'$. Thus, there exist $2^{\aleph_0}$ distinct product varieties $\mathfrak{V}\mathfrak{W}$, where $\mathfrak{V}$ is a locally finite variety obtained from Corollary 3.3 (b). It is clear that each product variety $\mathfrak{V}\mathfrak{W}$ contains the group S, as $S \in \mathfrak{W} \subseteq \mathfrak{V}\mathfrak{W}$. Furthermore, $\mathfrak{V}\mathfrak{W}$ is locally finite as it is a product variety of locally finite varieties. The same argument as in Theorem 4.7 applies to show that none of these $2^{\aleph_0}$ varieties can be closed under localizations. □

Remark 5.2. We make the following remarks.

(i) Note that Adjan's varieties from Theorem 4.7 do contain the variety $\mathfrak{A}$ of all abelian groups and thus are not locally finite.

(ii) Instead of the product variety $\mathfrak{V}\mathfrak{W}$, we might use $\mathfrak{V} \cup \mathfrak{W}$, the minimal variety containing both $\mathfrak{V}$ and $\mathfrak{W}$. If p is a prime with $\gcd(p, |S|) = 1$ and $\mathfrak{V}$ a variety of exponent p^k for some k, then $\mathfrak{V} \cup \mathfrak{W}$ will be the collection of all direct products of a group from $\mathfrak{V}$ and a group from $\mathfrak{W}$. As $\mathfrak{V}$ is uniquely determined by $\mathfrak{V} \cup \mathfrak{W}$ via $\mathfrak{V} = (\mathfrak{V} \cup \mathfrak{W}) \cap \mathfrak{B}_{p^k}$, we have $\mathfrak{V} \cup \mathfrak{W} \neq \mathfrak{V}' \cup \mathfrak{W}$ for any varieties $\mathfrak{V} \neq \mathfrak{V}'$. Thus, the statement would follow from the existence of $2^{\aleph_0}$ pairwise distinct locally finite varieties of exponent p^k.

Once again, by set-theoretic considerations, we obtain the following result which is dual to Corollary 3.5.

Corollary 5.3. *There exist $2^{\aleph_0}$ pairwise distinct locally finite varieties of groups which are neither closed under localizations nor finitely based.*

It is not surprising that, combining the ideas from Theorem 3.4 and Theorem 5.1, we obtain the following result as an extra gift.

Theorem 5.4. *There exist $2^{\aleph_0}$ pairwise distinct varieties of groups which are neither closed under cellular covers nor under localizations.*

Proof. Recall from Theorem 3.4 that we obtain $2^{\aleph_0}$ pairwise distinct product varieties $\mathfrak{V}\mathfrak{B}_{p_3}$ of exponent $8p_1p_2p_3$, where $\mathfrak{V}$ is a variety obtained from Corollary 3.3 (b). Let S be any non-abelian finite simple group and set $\mathfrak{W} := \mathrm{var}\{S\}$. Since S is finite, $\mathfrak{W}$ is a locally finite variety of exponent at most the order of the group S. By multiplying the variety $\mathfrak{W}$ on the right-hand side of each product variety $\mathfrak{V}\mathfrak{B}_{p_3}$, we obtain product varieties of the form $\mathfrak{V}\mathfrak{B}_{p_3}\mathfrak{W}$ for each variety $\mathfrak{V}$ obtained from Corollary 3.3(b). Because $\mathfrak{W}$ is not the variety of all groups and the $2^{\aleph_0}$ varieties $\mathfrak{V}\mathfrak{B}_{p_3}$ are all pairwise distinct, the right cancellation law for product varieties can be applied to assure that there still remain $2^{\aleph_0}$ pairwise distinct product varieties of the form $\mathfrak{V}\mathfrak{B}_{p_3}\mathfrak{W}$.

However, we see that the two groups $\mathcal{B}_{p_3}$ and S are members of each variety $\mathfrak{V}\mathfrak{B}_{p_3}\mathfrak{W}$ as $\mathcal{B}_{p_3} \in \mathfrak{B}_{p_3} \subseteq \mathfrak{V}\mathfrak{B}_{p_3} \subseteq \mathfrak{V}\mathfrak{B}_{p_3}\mathfrak{W}$ and $S \in \mathfrak{W} \subseteq \mathfrak{V}\mathfrak{B}_{p_3}\mathfrak{W}$. From Section 2 we know that $\mathcal{B}_{p_3}$ has cellular covers of infinite exponent and from Section 4 we know that S has localizations of infinite exponent while each variety $\mathfrak{V}\mathfrak{B}_{p_3}\mathfrak{W}$ is of finite exponent (at most $8p_1p_2p_3q$ where q is the exponent of $\mathfrak{W}$). It follows that each product variety $\mathfrak{V}\mathfrak{B}_{p_3}\mathfrak{W}$ is neither closed under taking cellular covers nor closed under taking localizations. □

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Author information

Daniel Herden, Department of Mathematics, Baylor University,
One Bear Place #97328, Waco, TX 76798-7328, USA.
E-mail: daniel_herden@baylor.edu

Montakarn Petapirak, Department of Mathematics and Statistics,
Prince of Songkla University, Hat Yai, Songkhla, 90110, Thailand.
E-mail: montakarn.p@psu.ac.th

José L. Rodríguez, Departamento de Matemáticas, Universidad de Almería,
E-04120 Almería, Spain.
E-mail: jlrodri@ual.es