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On products of groups with abelian subgroups of small index

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Abstract. It is proved that every group of the form $G = AB$ with two subgroups A and B each of which is either abelian or has a quasicyclic subgroup of index 2 is soluble of derived length at most 3. In particular, if A is abelian and B is a locally quaternion group, this gives a positive answer to Question 18.95 of the "Kourovka notebook" posed by A. I. Sozutov.

1 Introduction

Let the group $G = AB$ be the product of two subgroups A and B, i.e. G is of the form $G = \{ab \mid a \in A,\ b \in B\}$. It was proved by N. Itô that the group G is metabelian if the subgroups A and B are abelian (see [1, Theorem 2.1.1]).

In connection with Itô's theorem a natural question is whether every group $G = AB$ with abelian-by-finite subgroups A and B is metabelian-by-finite (see [1, Question 3]) or at least soluble-by-finite. However, this seemingly simple question is very difficult to attack and only partial results in this direction are known. A positive answer was given for linear groups G by the second author in [8] (see also [9]) and for residually finite groups G by J. Wilson [1, Theorem 2.3.4]. Furthermore, N. S. Chernikov proved that every group $G = AB$ with central-by-finite subgroups A and B is soluble-by-finite (see [1, Theorem 2.2.5]).

It is natural to consider first groups $G = AB$ where the two factors A and B have abelian subgroups with small index. There are a few known results in the case when both factors A and B have an abelian subgroup of index at most 2. It was shown in [3] that G is soluble and metacyclic-by-finite if A and B have cyclic subgroups of index at most 2, and it is proved in [2] that G is soluble if A and B

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are periodic locally dihedral subgroups. A more general result that $G = AB$ is soluble if each of the factors A and B is either abelian or generalized dihedral was obtained in [4] by another approach. Here a group is called generalized dihedral if it contains an abelian subgroup of index 2 and an involution which inverts the elements of this subgroup. Clearly dihedral groups and locally dihedral groups, i.e. groups with a local system of dihedral subgroups, are generalized dihedral.

We recall that a group is called quasicyclic (or a Prüfer group) if it is an infinite locally cyclic p-group for some prime p. It is well known that quasicyclic subgroups of abelian groups are their direct factors. Furthermore, it seems to be known and will be shown below that every non-abelian group having a quasicyclic subgroup of index 2 is either an infinite locally dihedral or a locally quaternion group. It should be noted that for each prime p, up to isomorphism, there exists a unique locally dihedral group whose quasicyclic subgroup is a p-group, and there is only one locally quaternion group. These and other details about such groups can be found in [6, pp. 45–50].

Theorem 1.1. *Let the group $G = AB$ be the product of two subgroups A and B each of which is either abelian or has a quasicyclic subgroup of index* 2*. Then G is soluble with derived length at most* 3*. Moreover, if the subgroup B is non-abelian and X is its quasicyclic subgroup, then $AX = XA$ is a metabelian subgroup of index* 2 *in G.*

As a direct consequence of this theorem, we have an affirmative answer to Question 18.95 of the "Kourovka notebook" [7] posed by A. I. Sozutov.

Corollary 1.2. *If a group $G = AB$ is the product of an abelian subgroup A and a locally quaternion subgroup B, then G is soluble.*

It is also easy to see that if each of the factors A and B in Theorem 1.1 has a quasicyclic subgroup of index 2, then their quasicyclic subgroups are permutable. As a result of this the following holds.

Corollary 1.3. *Let the group $G = A_1 A_2 \cdots A_n$ be the product of pairwise permutable subgroups $A_1, \dots, A_n$ each of which contains a quasicyclic subgroup of index* 2*. Then the derived subgroup G' is a direct product of the quasicyclic subgroups and the factor group G/G' is elementary abelian of order 2^m for some positive integer $m \leq n$.*

The notation is standard. If H is a subgroup of a group G and $g \in G$, then the normal closure of H in G is the normal subgroup of G generated by all conjugates of H in G, and g^G is the conjugacy class of G containing g, respectively.

2 Preliminary lemmas

Our first lemma lists some simple facts concerning groups with quasicyclic subgroups of index 2 which will be used without further explanation.

Lemma 2.1. *Let G be a non-abelian group containing a quasicyclic p-subgroup X of index 2 and $y \in G \setminus X$. Then $y^2 \in X$ and the following statements hold:*

(1) *Every subgroup of X is characteristic in G.*

(2) *The group G is either locally dihedral or locally quaternion.*

(3) *The derived subgroup G' coincides with X.*

(4) *Every proper normal subgroup of G is contained in X.*

(5) *If G is locally quaternion, then $p = 2$, $y^4 = 1$, $x^y = x^{-1}$ for all $x \in X$, the center $Z(G)$ coincides with $\langle y^2 \rangle$ and is contained in every non-trivial subgroup of G, the coset yX coincides with the conjugacy class $y^G = y^X$.*

(6) *If G is locally dihedral, then $y^2 = 1$, $x^y = x^{-1}$ for all $x \in X$, $Z(G) = 1$ and the coset yX coincides with the conjugacy class $y^G = y^X$ for $p > 2$ and $Z(G)$ is the subgroup of order 2 in X for $p = 2$.*

(7) *The factor group $G/Z(G)$ is locally dihedral.*

Proof. In fact, only statement (2) needs an explanation. Clearly $G = X\langle y \rangle$ for some $y \in G$ with $y^2 \in X$ and each cyclic subgroup $\langle x \rangle$ of X is normal in G. Therefore for $p > 2$ we have $y^2 = 1$ and either $x^y = x$ or $x^y = x^{-1}$. Since X contains a unique cyclic subgroup of order p^n for each $n \geq 1$, the equality $x^y = x$ for some $x \neq 1$ holds for all $x \in X$, contrary to the hypothesis that G is non-abelian. Therefore $x^y = x^{-1}$ for all $x \in X$ and hence the group G is locally dihedral. In the case $p = 2$ each subgroup $\langle x \rangle$ of X properly containing the subgroup $\langle y^2 \rangle$ has index 2 in the subgroup $\langle x, y \rangle$. If x is of order 2^n for some $n > 3$, then the element y can be chosen such that either $y^4 = 1$ and $\langle x, y \rangle$ is a generalized quaternion group with $x^y = x^{-1}$ or $y^2 = 1$ and $\langle x, y \rangle$ is one of the following groups: dihedral with $x^y = x^{-1}$, semidihedral with $x^y = x^{-1+2^{n-2}}$ or a group with $x^y = x^{1+2^{n-2}}$ (see [5, Theorem 5.4.3]). It is easy to see that from this list only generalized quaternion and dihedral subgroups can form an infinite ascending series of subgroups, so that the 2-group G can be either locally quaternion or locally dihedral, as claimed. □

Lemma 2.2. *Let G be a group and M an abelian minimal normal p-subgroup of G for some prime p. Then the factor group $G/C_G(M)$ has no non-trivial finite normal p-subgroup.*

Proof. Indeed, if $N/C_G(M)$ is a finite normal p-subgroup of $G/C_G(M)$ and x is an element of order p in M, then the p-subgroup $K = \langle x^N \rangle$ is finite and N acts on K as a finite p-group of automorphisms. Therefore the centralizer $C_K(N)$ of N in K is non-trivial and hence $C_M(N)$ is a non-trivial normal subgroup of G properly contained in M, contradicting the minimality of M. □

We will say that a subset S of G is normal in G if $S^g = S$ for each $g \in G$ which means that $s^g \in S$ for every $s \in S$.

Lemma 2.3. *Let G be a group, and let A and B be subgroups of G. If a normal subset S of G is contained in the set AB and $S^{-1} = S$, then the normal subgroup of G generated by S is also contained in AB. In particular, if i is an involution with $i^G \subseteq AB$ and N is the normal closure of the subgroup $\langle i \rangle$ in G, then $AN \cap BN = A_1B_1$ with $A_1 = A \cap BN$ and $B_1 = AN \cap B$.*

Proof. If $s, t \in S$, then $t = ab$ and $(s^{-1})^a = cd$ for some elements $a, c \in A$ and $b, d \in B$. Therefore $s^{-1}t = s^{-1}ab = a(s^{-1})^a b = (ac)(db) \in AB$ and hence the subgroup $\langle s \mid s \in S \rangle$ is contained in AB and normal in G. Moreover, if N is a normal subgroup of G and $N \subseteq AB$, it is easy to see that $AN \cap BN = (AN \cap B)N = (AN \cap B)N = (A \cap BN)(AN \cap B)$ (for details see [1, Lemma 1.1.4]). □

The following slight generalization of Itô's theorem was proved in [8] (see also [9, Lemma 9]).

Lemma 2.4. *Let G be a group and let A, B be abelian subgroups of G. If H is a subgroup of G contained in the set AB, then H is metabelian.*

3 The product of an abelian group and a group containing a quasicyclic subgroup of index 2

In this section we consider groups of the form $G = AB$ with an abelian subgroup A and a subgroup $B = X\langle y \rangle$ in which X is a quasicyclic p-subgroup of index 2 and $y \in B \setminus X$.

Lemma 3.1. *Let the group $G = AB$ be the product of an abelian subgroup A and a non-abelian subgroup B with a quasicyclic p-subgroup X of index 2. If G has non-trivial abelian normal subgroups, then one of these is contained in the set AX.*

Proof. Suppose the contrary and let $\mathcal{N}$ be the set of all non-trivial normal subgroups of G contained in the derived subgroup G'. Then $A_G = 1$ and $ANX \neq AX$ for each $N \in \mathcal{N}$. Since $G = AB = AX \cup AXy$ and $AX \cap AXy = \emptyset$, for every $N \in \mathcal{N}$ the intersection $NX \cap AXy$ is non-empty and so $G = ANX$. Moreover, as $X = B' \leq G'$ by Lemma 2.1, it follows that $G' = DNX$ with $D = A \cap G'$. It is also clear that $G = \langle A, X \rangle$, because otherwise $\langle A, X \rangle = AX$ is a normal subgroup of index 2 in G. In particular, $A \cap X = 1$.

For each $N \in \mathcal{N}$ we put $A_N = A \cap BN$ and $B_N = AN \cap B$. Then $A_N N = B_N N = A_N B_N$ by [1, Lemma 1.1.4], and the subgroup B_N is not contained in X, because otherwise N is contained in the set AX, contrary to the assumption. Let $X_N = B_N \cap X$ and $C_N = A_N \cap NX_N$. Then X_N is a subgroup of index 2 in B_N and $C_N N = NX_N$ is a normal subgroup of G, because $(NX_N)^X = NX_N$ and $(NC_N)^A = NC_N$. Put $M = \bigcap_{N \in \mathcal{N}} N$.

Since $G = ANX$ for each $N \in \mathcal{N}$, the factor group G/N is metabelian by Lemma 2.4. Therefore also the factor group G/M is metabelian and so its derived subgroup G'/M is abelian. Clearly if $M = 1$, then $D = A \cap G' \leq A_G = 1$ and hence $G' = \bigcap_{N \in \mathcal{N}} NX = X$, contrary to the assumption. Thus M is the unique abelian minimal normal subgroup of G. We show first that the centralizer $C_G(M)$ of M in G does not contain the subgroup X.

Indeed, otherwise the group $G = A(MX)$ is metabelian by Itô's theorem and so the derived subgroup $G' = DMX$ is abelian. Since $G = AG'$, it follows that $D = A \cap G' \leq A_G = 1$ and so $G' = MX$. If M contains elements of order p, then it is an elementary abelian p-subgroup and hence X is the finite residual of G'. In the other case M has no element of order p, so that X is the maximal p-subgroup of G'. Therefore in both cases X is a characteristic subgroup of G' and so normal in G, contrary to the assumption. Thus $X \nleq C_G(M)$ which implies in particular that the subgroup M is infinite and the centralizer $C_X(M)$ is finite.

Now, if M is a p-subgroup, then the factor group $\bar{G} = G/C_G(M)$ has no non-trivial finite normal p-subgroup by Lemma 2.2. On the other hand, $G = AMX$ and $G' = DMX$ with $D = A \cap G'$, so that $G = AG'$. Using bars for images under the homomorphism $G \to \bar{G}$, we derive that the group $\bar{G} = \bar{A}\bar{X} = \bar{A}\bar{G}'$ is metabelian and the derived subgroup $\bar{G}' = \bar{D}\bar{X}$ is abelian. Therefore the intersection $\bar{A} \cap \bar{G}'$ is a central subgroup of $\bar{G}$ and hence it has no subgroup of order p. Since $\bar{D} \leq \bar{A} \cap \bar{G}'$, it follows that $\bar{X}$ is the maximal p-subgroup of $\bar{G}'$ and so normal in $\bar{G}$. As $\bar{X}$ is the union of its finite p-subgroups each of which is also normal in $\bar{G}$, this implies $\bar{X} = 1$ and thus $X \leq C_G(M)$, contrary to the above.

Suppose next that M is not a p-subgroup and so M has no element of order p. As was shown above, the subgroup $MX_M = C_M M$ is normal in G. If $X_M = X$, then $G = A(MX) = A(C_M M) = AM$ and so $G' = M$. But then $X \leq M$ which is not the case. Therefore the subgroup X_M is finite of order p^k for some $k \geq 0$. If the subgroup MX_M is non-abelian, then its center is trivial, because the subgroup M is minimal normal in G. In particular, $C_M \cap M = 1$ and hence the subgroups A_M and B_M are finite of order $2p^k$. As $A_M M = B_M M = A_M B_M$, the subgroup M is also finite which contradicts what has been proved above.

Thus the subgroup MX_M is abelian and hence X_M is its maximal p-subgroup. Therefore X_M is normal in G and so $X_M = 1$ by assumption. Then $AM \cap X = 1$ and the subgroup $B_M = AM \cap B$ is of order 2, because $B_M \cap X = 1$ and $M \nleq A$ by assumption. Therefore $AM = AB_M$ and $B_M = \langle y \rangle$ with $y^2 = 1$. Since the subgroup B is non-abelian by the hypothesis of the lemma, it follows from Lemma 2.1 that $B = X \rtimes \langle y \rangle$ is locally dihedral and so $x^y = x^{-1}$ for each $x \in X$. Furthermore, the index of A in AM is equal to 2 and so $A \cap M$ is a subgroup of index 2 in M. As M is abelian and minimal normal in G, it follows that M is an elementary abelian 2-subgroup. It is also clear that the subgroup AM is nilpotent and the intersection $A \cap M$ is centralized by y.

It was noted above that $G = AMX$ and $G' = DMX$ with $D = A \cap G'$. Passing to the factor group $\bar{G} = G/M$ and using bars for images under the homomorphism $G \to \bar{G}$, we obtain that the group $\bar{G} = \bar{A}\bar{X}$ is metabelian and so its derived subgroup $\bar{G}' = \bar{D}\bar{X}$ is abelian. Since A is abelian, the subgroup $\bar{D}$ is central in $\bar{G}$ and thus the subgroup DM is normal in G. Furthermore, $(DM)' \neq M$, because DM as a subgroup of AM is nilpotent. As M is the unique minimal normal subgroup of G, the subgroup DM must be abelian. But then $D^2 = 1$, because $D^2 = (DM)^2$ is a normal subgroup of G. Thus $\bar{X}$ is the maximal p-subgroup of $\bar{G}'$. Since $p \neq 2$ and X is quasicyclic, this means that each subgroup of $\bar{X}$ is characteristic in $\bar{G}'$ and so normal in $\bar{G}$. Therefore for each $x \in X$ the subgroup $M\langle x \rangle$ and MX itself are normal in G. In particular, for each $g \in G$ there exists $m \in M$ such that $\langle x \rangle^g = \langle x \rangle^m$ from which it follows that $gm \in N_G(\langle x \rangle)$ and thus $G = MN_G(\langle x \rangle)$. It is easily seen that $M \cap N_G(\langle x \rangle) = 1$ and hence $M = C_G(M)$, because otherwise the intersection $C_G(M) \cap N_G(\langle x \rangle)$ is a non-trivial normal subgroup of G which does not contain M. Moreover, as $G = AB$ and B is contained in $N_G(\langle x \rangle)$, it follows that $N_G(\langle x \rangle) = N_A(\langle x \rangle)B$ and so $N_A(\langle x \rangle)^G = N_A(\langle x \rangle)^B$ is a normal subgroup of G contained in $N_G(\langle x \rangle)$. Therefore we have $N_A(\langle x \rangle) = 1$ and we conclude that $G = AB = M \rtimes B$, the subgroup $B = X \rtimes \langle y \rangle$ is locally dihedral and $A \cap B = 1$.

Finally, taking an element x of order p in X and considering M as an irreducible B-module, we derive from Clifford's theorem (see [5, Theorem 4.1]) that M is decomposed in an infinite direct product $M = M_1 \times \cdots \times M_i \times \cdots$ of finite $\langle x \rangle$-invariant subgroups M_i. Furthermore, it was proved above that $M = C_G(M)$ and $A \cap M$ has index 2 in M and is centralized by y. This gives $[M, y] = \langle a \rangle$ for some involution $a \in A \cap M$. Let N be one of the subgroups M_i which does not contain a. Then the subgroup N^y is also $\langle x \rangle$-invariant and $(A \cap N)^y = A \cap N \neq 1$. Therefore $N^y = N$ and so $[N, y] \leq \langle a \rangle \cap N = 1$. But then we have $1 = [N, y]^x = [N, y^x] = [N, yx^2] = [N, x^2]$ and so the centralizer $C_M(x)$ contains N. Since $M\langle x \rangle$ is a normal subgroup of G, so is $C_M(x)$ and thus $C_M(x) = M$. This final contradiction completes the proof. □

It should be noted that if in Lemma 3.1 the subgroup B is locally dihedral, then the group $G = AB$ is soluble by [4, Theorem 1.1]. Therefore the following assertion is an easy consequence of this lemma.

Corollary 3.2. *If the group $G = AB$ is the product of an abelian subgroup A and a locally dihedral subgroup B containing a quasicyclic subgroup X of index 2, then $AX = XA$ is a metabelian subgroup of index 2 in G.*

Proof. Indeed, let H be a maximal normal subgroup of G with respect to the condition $H \subseteq AX$. If $X \leq H$, then $AH = AX$ is a metabelian subgroup of index 2 in G by Itô's theorem. In the other case the intersection $H \cap X$ is finite and hence HX/H is the quasicyclic subgroup of index 2 in BH/H. Since $G/H = (AH/H)(BH/H)$ is the product of the abelian subgroup AH/H and the locally dihedral subgroup BH/H, the set $(AH/H)(HX/H)$ contains a non-trivial normal subgroup F/H of G/H by Lemma 3.1. But then F is a normal subgroup of G which is contained in the set AX and properly contains H. This contradiction completes the proof. □

In the following lemma $G = AB$ is a group with an abelian subgroup A and a locally quaternion subgroup $B = X\langle y \rangle$ in which X is the quasicyclic 2-subgroup of index 2 and y is an element of order 4, so that $x^y = x^{-1}$ for each $x \in X$ and $z = y^2$ is the unique involution of B. It turns out that in this case the conjugacy class z^G of z in G is contained in the set AX.

Lemma 3.3. *If $G = AB$ and $A \cap B = 1$, then the intersection $z^A \cap AXy$ is empty.*

Proof. Suppose the contrary and let $z^a = bxy$ for some elements $a, b \in A$ and $x \in X$. Then $b^{-1}z = (xy)^{a^{-1}}$ and from the equality $(xy)^2 = z$ it follows that $(b^{-1}z)^4 = 1$ and $b^{-1}zb^{-1}z = z^{a^{-1}}$. Therefore we have $b^{-1}z^a b^{-1} = zz^a$ and hence $bz^a b = z^a z$. As $z^a = bxy$, we have $b(bxy)b = (bxy)z$ and so $bxyb = xyz$. Thus $(xy)^{-1}b(xy) = zb^{-1}$. Furthermore, we have $bxyb^{-1} = (zb^{-1})^a$, so that $bzb^{-1} = ((zb^{-1})^2)^a = (xy)^{-a}b^2(xy)^a$, i.e. the elements z and b^2 are conjugate in G by the element $g = b^{-1}(xy)^{-a}$. Since $g = cd$ for some $c \in B$ and $d \in A$, we have $b^2 = z^g = z^d$ and so $z = (b^2)^{d^{-1}} = b^2$, contrary to the hypothesis of the lemma. Thus $z^A \cap AXy = \emptyset$, as desired. □

Theorem 3.4. *Let the group $G = AB$ be the product of an abelian subgroup A and a locally quaternion subgroup B. If X is the quasicyclic subgroup of B, then $AX = XA$ is a metabelian subgroup of index 2 in G. In particular, G is soluble of derived length at most 3.*

Proof. Let Z be the center of B, N the normal closure of Z in G and $X = B'$, so that X is the quasicyclic subgroup of index 2 in B. If $A \cap B \neq 1$, then Z is contained in $A \cap B$ by statement (4) of Lemma 2.1 and so $N = Z$. Otherwise it follows from Lemma 2.3 that $N = Z^G = Z^A$ is contained in the set AX. Then N is a metabelian normal subgroup of G by Lemma 2.4 and the factor group BN/N is locally dihedral by statement (7) of Lemma 2.1. Since the factor group $G/N = (AN/N)(BN/N)$ is the product of an abelian subgroup AN/N and the locally dihedral subgroup BN/N, it is soluble by [4, Theorem 1.1], and so the group G is soluble.

Now if $X \leq N$, then $AN = AX$ is a metabelian subgroup of index 2 in G and so the derived length of G does not exceed 3. In the other case the intersection $N \cap X$ is finite and hence NX/N is the quasicyclic subgroup of index 2 in BN/N. Therefore $AX = XA$ by Corollary 3.2 and this completes the proof. □

4 The product of groups each of which is locally quaternion or generalized dihedral

Since the groups of the form $G = AB$ with two generalized dihedral subgroups A and B are soluble by [4, Theorem 1.1], in this section we consider the remaining cases in which the subgroup A is locally quaternion and B is either generalized dihedral or locally quaternion. The main part is devoted to the proof that every group G of this form has a non-trivial abelian normal subgroup.

In what follows up to Theorem 4.5 $G = AB$ is a group in which $A = Q\langle c\rangle$ with a quasicyclic 2-subgroup Q of index 2 and an element c of order 4 such that $a^c = a^{-1}$ for each $a \in Q$ and $B = X \rtimes \langle y\rangle$ with an abelian subgroup X and an involution y such that $x^y = x^{-1}$ for each $x \in X$.

Let $d = c^2$ denote the involution of A. The following assertion is concerned with the structure of the centralizer $C_G(d)$ of d in G. It follows from statement (4) of Lemma 2.1 that the normalizer of every non-trivial normal subgroup of A is contained in $C_G(d)$.

Lemma 4.1. *The centralizer $C_G(d)$ is soluble.*

Proof. If $Z = \langle d\rangle$, then the factor group $C_G(d)/Z = (A/Z)(C_B(d)Z/Z)$ is a product of the generalized dihedral subgroup A/Z and the subgroup $C_B(d)Z/Z$ which is either abelian or generalized dihedral. Therefore $C_G(d)/Z$ and thus $C_G(d)$ is a soluble group by [4, Theorem 1.1], as claimed. □

The following lemma shows that if G has no non-trivial abelian normal subgroup, then the index of A in $C_G(d)$ does not exceed 2.

Lemma 4.2. *If $C_B(d) \neq 1$, then either $C_X(d) = 1$ or G contains a non-trivial abelian normal subgroup.*

Proof. If $X_1 = C_X(d)$, then X_1 is a normal subgroup of B and $C_G(d) = AC_B(d)$. Therefore the normal closure $N = X_1^G$ is contained in $C_G(d)$, because $X_1^G = X_1^{BA} = X_1^A$. Since $C_G(d)$ and so N is a soluble subgroup by Lemma 4.1, this completes the proof. □

Consider now the normalizers in A of non-trivial normal subgroups of B.

Lemma 4.3. *Let G have no non-trivial abelian normal subgroup. If U is a non-trivial normal subgroup of B, then $N_A(U) = 1$. In particular, $A \cap B = 1$.*

Proof. If $N_A(U) \neq 1$, then $d \in N_A(U)$ and so the normal closure $\langle d \rangle^G = \langle d \rangle^B$ is contained in the normalizer $N_G(U) = N_A(U)B$. Since $N_A(U) \neq A$, the subgroup $N_A(U)$ is either finite or quasicyclic, so that $N_G(U)$ and thus $\langle d \rangle^G$ is soluble. This contradiction completes the proof. □

Lemma 4.4. *If $C_X(d) = 1$, then G contains a non-trivial abelian normal subgroup.*

Proof. Since $G = AB$, for each $x \in B$ there exist elements $a \in A$ and $b \in B$ such that $d^x = ab$. If $b \notin X$, then $b = a^{-1}d^x$ is an element of order 2 and so $d^x a d^x = a^{-1}$. As $a^{2^k} = d$ for some $k \geq 0$, it follows that $d^x d d^x = d$ and hence $ab = d^x = (d^x)^d = (ab)^d = ab^d$. Therefore $b^d = b$ and so $b \in C_B(d)$. In particular, if $C_B(d) = 1$, then $b \in X$, so that in this case the conjugacy class $d^G = d^B$ is contained in the set AX.

Assume that $C_B(d) \neq 1$ and the group G has no non-trivial normal subgroup. Then $C_X(d) = 1$ by Lemma 4.2 and without loss of generality $C_B(d) = \langle y \rangle$. Then $G = (A\langle y \rangle)X$ and so the quasicyclic subgroup Q of A is normalized by y. In particular, $d^y = d$ and the subgroup $Q\langle y \rangle$ can be either abelian or locally dihedral. We consider first the case when y centralizes Q and show that in this case the conjugacy class d^G is also contained in the set AX.

Indeed, otherwise there exist elements $a \in A$ and $b, x \in B$ such that $d^x = ab$ and $b \notin X$. Then $b \in C_B(d) = \langle y \rangle$ by what was proved above, so that $b = y$ and $d^x = ay$. As $d^B = d^{\langle y \rangle X} = d^X$, we may suppose that $x \in X$. But then $d^{x^{-1}} = (d^x)^y = ay = d^x$ and hence $d^{x^2} = d$. Therefore we have $x^2 \in \langle y \rangle$ and so $x^2 = 1$. In particular, if X has no involution, then $d^G = d^X \subseteq AX$. We show next that the case with an involution $x \in X$ cannot appear.

Clearly in this case x is a central involution in B and so the subgroup $D = \langle d, x \rangle$ generated by the involutions d and x is dihedral. It is easy to see that d and x

cannot be conjugate in G and the center of D is trivial, because otherwise the centralizer $C_G(x)$ properly contains B, contradicting Lemma 4.3. Thus dx is an element of infinite order and so $D = \langle dx \rangle \rtimes \langle x \rangle$ has no automorphism of finite order more than 2. On the other hand, if $u \in A$, $v \in B$ and uv normalizes D, then $\langle d, x^u \rangle = D^u = D^{v^{-1}} = \langle d^{v^{-1}}, x \rangle$ and so $D \le D^u$. Since u is an element of finite order, it follows that $D = D^u = D^{v^{-1}}$ and thus $N_G(D) = N_A(D)N_B(D)$. Therefore $N_A(D) = \langle d \rangle$ and hence $z = (dx)^2$ is an element of infinite order in $N_B(D)$. But then $z \in X$ and so $\langle z \rangle$ is a normal subgroup of B normalized by d, again contradicting Lemma 4.3. Thus X has no involution, as claimed.

Finally, if N is the normal closure of the subgroup $\langle d \rangle$ in G, then $AN = NX = A_1X_1$ with $A_1 = A \cap NX$ and $X_1 = AN \cap X$ by Lemma 2.3. Therefore the subgroup A_1X_1 is soluble by Theorem 3.4, so that N and hence G has a non-trivial abelian normal subgroup, contrary to our assumption.

Thus the subgroup $Q\langle y \rangle$ is locally dihedral and so y inverts the elements of Q. Since $A = Q\langle c \rangle$ with $a^c = a^{-1}$ for all $a \in A$, the element cy centralizes Q and hence the subgroup $Q\langle cy \rangle$ is abelian. But then the group $G = (Q\langle cy \rangle)B$ as the product of an abelian and a generalized dihedral subgroup is soluble by [4, Theorem 1.1]. This final contradiction completes the proof. □

Theorem 4.5. *Let the group $G = AB$ be the product of a locally quaternion subgroup A and a generalized dihedral subgroup B. Then G is soluble. Moreover, if B has a quasicyclic subgroup of index 2, then G is metabelian.*

Proof. If $A \cap X \neq 1$, then the centralizer $C_G(d)$ is of index at most 2 in G and so G is soluble by Lemma 4.1. Let N be a normal subgroup of G maximal with respect to the condition $A \cap NX = 1$. Then $BN = (A \cap BN)B$ and the subgroup $A \cap BN$ is of order at most 2. Therefore the subgroup N is soluble and the factor group $G/N = (AN/N)(BN/N)$ is the product of the locally quaternion subgroup AN/N and the subgroup BN/N which is either abelian or generalized dihedral. Hence it follows from Theorem 3.4 and Lemmas 4.2 and 4.4 that G/N has a non-trivial abelian normal subgroup M/N. Put $L = MQ \cap MX$, $Q_1 = Q \cap MX$ and $X_1 = MQ \cap X$. We have $L = MQ_1 = MX_1$ and $Q_1 \neq 1$, because $A \cap MX \neq 1$ by the choice of M. It is also clear that L is a soluble normal subgroup of G, because $(MQ_1)^A = MQ_1$ and $(MX_1)^B = MX_1$. Therefore the factor group G/L and so the group G is soluble if AL/L is of order 2. In the other case AL/L is locally dihedral and BL/L is abelian or generalized dihedral. Since $G/L = (AL/L)(BL/L)$, it follows that G/L and so G is soluble by [4, Theorem 1.1]. Moreover, if the subgroup X is quasicyclic, then the subgroups Q and X centralize each other by [1, Corollary 3.2.10], so that QX is an abelian normal subgroup of index 2 or 4 in G and thus G is metabelian. □

Our last theorem describes the structure of groups which are products of two locally quaternion subgroups.

Theorem 4.6. *Let the group $G = AB$ be the product of two locally quaternion subgroups A and B. If X and Y are quasicyclic subgroup of A and B, respectively, then $XY = YX$ is an abelian subgroup of index 2 or 4 in G. In particular, G is metabelian.*

Proof. Let x and y be the unique involution of A and B, respectively. If G is soluble, then $XY = YX$ by [1, Corollary 3.2.10]. We show now that the group G satisfies this condition.

Indeed, if $A \cap B \neq 1$, then $x = y$ is a central involution of G and the factor group $G/\langle x\rangle = (A/\langle x\rangle)(B/\langle x\rangle)$ is the product of two locally dihedral subgroups $A/\langle x\rangle$ and $B/\langle x\rangle$. Therefore G is soluble by [4, Theorem 1.1].

Let $A \cap B = 1$ and $D = \langle x, y\rangle$. Then D is a dihedral subgroup of G and the normalizer $N_G(D)$ can be written in the form $N_G(D) = N_A(D)N_B(D)$ by [1, Lemma 1.2.2 (i)]. It is easy to see that $N_A(D) \cap D = \langle x\rangle$ and $N_B(D) \cap D = \langle y\rangle$, so that each of the factor groups $N_A(D)D/D$ and $N_B(D)D/D$ is either abelian or locally dihedral. Since $N_G(D)/D = (N_A(D)D/D)(N_B(D)D/D)$, the factor group $N_G(D)/D$ and so also $N_G(D)$ is soluble by [4, Theorem 1.1]. Then $N_G(D)$ is a 2-group by [1, Corollary 3.2.7], and hence D is a dihedral 2-subgroup of G. Therefore D contains a central involution z which is different from x and y. As $z = ab$ for some $a \in A$ and $b \in B$, it follows that $b \neq 1$ and $x = x^{ab} = x^b$. But then $x = x^y$, because $y \in \langle b\rangle$, so that $D = \langle x\rangle \times \langle y\rangle$ and $C_G(D) = C_A(y)C_B(x)$ is a soluble 2-subgroup.

It is clear that if $C_G(D)$ is of finite index in G, then G is soluble. In the other case one of the centralizers $C_A(y)$ and $C_B(x)$, for example the second one, must be finite and thus the centralizer $C_G(x) = AC_B(x)$ is soluble. But then the normal closure $N = \langle y\rangle^G = \langle y\rangle^A \leq AC_B(x)$ of $\langle y\rangle$ in G is also soluble. Furthermore, in the factor group $G/N = (AN/N)(BN/N$ the subgroup AN/N is either locally quaternion or locally dihedral and BN/N is locally dihedral. Therefore G/N and so also G is soluble by Theorem 4.5 or by [4, Theorem 1.1], as claimed. □

The proof of Theorem 1.1 is completed by a direct application of Corollary 3.2 and Theorems 3.4, 4.5 and 4.6.

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