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A generalization of weak commutativity between two isomorphic groups

Bruno C. R. Lima and Said N. Sidki

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Abstract. The operator χ of weak commutativity between isomorphic groups H and H^{ψ}, defined by

$$\chi(H) = \langle H, H^{\psi} \mid [h, h^{\psi}] = 1 \text{ for all } h \in H \rangle,$$

is known to preserve group properties such as finiteness, solvability and polycyclicity. We introduce here the group construction

$$\mathcal{E}(H) = \langle H, H^{\psi} \mid [[h_1, h_2^{\psi}], h_3^{-1} h_3^{\psi}] = 1 \text{ for all } h_1, h_2, h_3 \in H \rangle.$$

The group $\mathcal{E}(H)$ maps onto $\chi(H)$ and onto $H \otimes H$, the non-abelian tensor square. The operator $\mathcal{E}$ preserves solvability and preserves polycyclicity provided the abelianization $\frac{H}{H'}$ is finite. Moreover, if H is perfect, then $\mathcal{E}(H)$ is perfect and $\mathcal{E}(H) \cong \chi(H)$. Furthermore, $\mathcal{E}(H)$ is a finite group if and only if H is a finite perfect group.

1 Introduction

The concept of weak permutability between two groups was introduced in 1980 [9] and a finiteness criterion result was established for a group generated by two such groups. In Section 4 of the same work, the notion of weak commutativity motivated the definition of the group

$$\chi(H) = \langle H, H^{\psi} \mid [h, h^{\psi}] = 1 \text{ for all } h \in H \rangle.$$

It was shown then that χ, when interpreted as an operator acting on groups, preserves the group properties: finite P-group for any finite set P of primes, solvable, finite nilpotent, perfect. Later, it was proven that if H is finitely generated nilpotent of nilpotency class c, then $\chi(H)$ is nilpotent and bounds for the nilpotency degree of $\chi(H)$ were found in terms of c (see [3]). Recently, the properties polycyclic and polycyclic-by-finite were added to this list [5]. Moreover, a study of homological and homotopical properties of $\chi(H)$ have lead to proving that χ preserves the property of solvable of type FP_{∞} (see [4]).

The definition of $\chi(H)$ implies that the relations

$$[h_1, h_2^{\psi}]^{h_3^{\psi}} = [h_1, h_2^{\psi}]^{h_3}$$

(equivalently, $[[h_1,h_2^{\psi}],h_3^{-1}h_3^{\psi}]=1$) hold for all $h_1,h_2,h_3\in H$. These same relations form an essential part of the definition of the *non-abelian tensor square*

$$H\otimes H=\big\langle h_1\otimes h_2 \mid h_1h_2\otimes h_3=\big((h_1)^{h_2}\otimes(h_3)^{h_2}\big)(h_2\otimes h_3),\\ h_1\otimes h_2h_3=(h_1\otimes h_3)\big((h_1)^{h_2}\otimes(h_3)^{h_2}\big)\\ \text{for all } h_1,h_2,h_3\in H\big\rangle,$$

by Brown–Loday in 1987 [2] and of the definition of

$$\nu(H)=\big\langle H,H^{\psi}\mid [h_1,h_2^{\psi}]^{h_3^{\psi}}=[h_1,h_2^{\psi}]^{h_3}=[h_1^{h_3},(h_2^{h_3})^{\psi}]\\ \text{for all } h_1,h_2,h_3\in H\big\rangle$$

by Rocco in 1991 [7] who proved the subgroup $[H,H^{\psi}]$ of $\nu(H)$ to be isomorphic to $H\otimes H$.

We focus here on the similarity between these constructions by defining the group

$$\mathcal{E}(H)=\langle H,H^{\psi}\mid [[h_1,h_2^{\psi}],h_3^{-1}h_3^{\psi}]=1 \text{ for all } h_1,h_2,h_3\in H)\rangle.$$

The group $\mathcal{E}(H)$ admits an automorphism $h\leftrightarrow h^{\psi}$ of order 2, which will be denoted by the same symbol ψ.

It is easy to see that if H is cyclic of order 2, then $\mathcal{E}(H)$ is the infinite dihedral group. Though $\mathcal{E}$ fails to preserve finiteness and nilpotency, still it preserves many features of χ. Some of our main results are gathered together in

Theorem 1. *The following statements hold.*

(1) *Let H be a solvable group with derived series of length k. Then $\mathcal{E}(H)$ is a solvable group of length at most $k+2$.*

(2) *Suppose H is a polycyclic group. Then $\mathcal{E}(H)$ is polycyclic if and only if the abelianization H/H' is a finite group.*

(3) *Suppose H is a perfect group. Then $\mathcal{E}(H)$ is perfect and $\mathcal{E}(H)\cong\chi(H)$.*

(4) *The group $\mathcal{E}(H)$ is finite if and only if H is a finite perfect group.*

2 Preliminaries

We use standard notation as can be found in [8].

2.1 Some epimorphisms

We start by defining four epimorphisms of $\mathcal{E}(H)$ and stating some of their properties:

$$\rho:\mathcal{E}(H)\to H,\quad h\mapsto h,\ h^{\psi}\mapsto h\quad \text{for all } h\in H,$$

with

$$\ker(\rho) = \mathcal{L}(H) = [H, \psi] = \langle h^{-1}h^{\psi} \mid h \in H \rangle,$$
$$\mathcal{E}(H) = \mathcal{L}(H) \cdot H, \quad \mathcal{L}(H) \cap H = 1;$$

and

$$\pi : \mathcal{E}(H) \to H \times H, \quad h \mapsto (h, 1),\ h^{\psi} \mapsto (1, h) \quad \text{for all } h \in H,$$

with

$$\ker(\pi) = \mathcal{D}(H) = [H, H^{\psi}] = \langle [h_1, h_2^{\psi}] \mid h_1, h_2 \in H \rangle,$$
$$\mathcal{E}(H) = (\mathcal{D}(H) \cdot H).H^{\psi}, \quad \mathcal{D}(H) \cap (H \cdot H^{\psi}) = 1;$$

and

$$\delta : \mathcal{E}(H) \to \chi(H), \quad h \mapsto h,\ h^{\psi} \mapsto h^{\psi} \quad \text{for all } h \in H,$$

with

$$\ker(\delta) = \Delta(H) = \langle [h, h^{\psi}] \mid h \in H \rangle^{\mathcal{E}(H)};$$

and

$$\xi : \mathcal{E}(H) \to \nu(H), \quad h \mapsto h,\ h^{\psi} \mapsto h^{\psi} \quad \text{for all } h \in H.$$

The subgroups $\mathcal{L}(H)$, $\mathcal{D}(H)$, $\Delta(H)$ are ψ-invariant. In addition to these, we have the normal ψ-invariant subgroups of $\mathcal{E}(H)$

$$\mathcal{W}(H) = \mathcal{D}(H) \cap \mathcal{L}(H),$$
$$\mathcal{R}(H) = [[H, \mathcal{L}(H)], H^{\psi}].$$

We mention two more normal subgroups

$$\mathcal{L}_1(H) = [\mathcal{L}(H), H], \quad \mathcal{L}_2(H) = [\mathcal{L}(H), H^{\psi}];$$

these are interchanged by ψ.

Maintaining the notation used for subgroups of $\chi(H)$, we write

$$\mathcal{L}(H)^{\delta} = L(H), \quad \mathcal{D}(H)^{\delta} = D(H), \quad \mathcal{W}(H)^{\delta} = W(H),$$
$$\mathcal{R}(H)^{\delta} = R(H), \quad \mathcal{L}_1(H)^{\delta} = L_1(H), \quad \mathcal{L}_2(H)^{\delta} = L_2(H).$$

The subgroup $\mathcal{R}(H)$ is key to understanding the structure of $\mathcal{E}(H)$. We will prove later

Theorem 2. *Let* $\xi : \mathcal{E}(H) \to \nu(H)$ *be as defined above. Then* $\ker(\xi) = \mathcal{R}(H)$.

Remark 1. (i) It follows directly from the definition of $\mathcal{E}(H)$ that

$$[\mathcal{D}(H), \mathcal{L}(H)] = 1$$

and therefore $\mathcal{W}(H)$ is central in the product $\mathcal{D}(H)\mathcal{L}(H)$. We conclude from

$$[h^{\psi}, h] = [h(h^{-1}h^{\psi}), h] = [h^{-1}h^{\psi}, h] \in \mathcal{D}(H) \cap \mathcal{L}(H) = \mathcal{W}(H)$$

that

$$\Delta(H) \leq \mathcal{W}(H).$$

(ii) Using [9, Proposition 4.1.4], the quotient group $\frac{\mathcal{E}(H)}{\mathcal{W}(H)}(\cong \frac{\chi(H)}{\mathcal{W}(H)})$ embeds into the triple product $H \times H \times H$ by

$$\tau : \mathcal{W}(H)h \to (h, h, 1), \quad \mathcal{W}(H)h^{\psi} \to (1, h, h)$$

for all $h \in H$. Thus,

$$\begin{aligned}
\tau : \mathcal{W}(H)(h^{-1}h^{\psi}) &\to (h^{-1}, 1, h),\\
\mathcal{W}(H)[h_1, h_2^{\psi}] &\to (1, [h_1, h_2], 1),\\
\mathcal{W}(H)([h_1^{-1}h_1^{\psi}, h_2]) &\to ([h_1^{-1}, h_2], 1, 1),\\
\mathcal{W}(H)([h_1^{-1}h_1^{\psi}, h_2^{\psi}]) &\to (1, 1, [h_1, h_2])
\end{aligned}$$

for all $h, h_1, h_2 \in H$.

2.2 Commutator relations

It is convenient to denote $h^{-1}h^{\psi}$ by $[h, \psi]$.

Lemma 1. *The following statements hold.*

(1) *Let h_1, h_2, $h_3 \in H$. Then:*

   (i) $[h_1, \psi]$ *commutes with* $[h_2, h_3{}^{\psi}]$,

   (ii) $[h_1, h_2{}^{\psi}] = [\psi, h_2, h_1][h_1, h_2]$ *(thus $\mathcal{D}(H) \leq [\mathcal{L}(H), H]H'$),*

   (iii) *in $\mathcal{R}(H)$,*

$$\left[h_1, h_3(h_3^{\psi})^{-1}, h_2^{\psi}\right]^{h_3^{\psi}} = \left[h_1, h_2^{\psi}\right]^{-h_3^{\psi}}\left[h_1^{h_3}, (h_2^{h_3})^{\psi}\right].$$

(2) *Let $w = w(z_1^{\epsilon_1}, z_2^{\epsilon_2}, \ldots, z_n^{\epsilon_n})$ be a word in $z_1^{\epsilon_1}, z_2^{\epsilon_2}, \ldots, z_n^{\epsilon_n}$ of length n, where $z_i \in H$, $\epsilon_k \in \{1, \psi,\}$. Also, let $w' = w(z_1, z_2, \ldots, z_n)$. Then*

$$[h_1, h_2^{\psi}]^{w} = [h_1, h_2^{\psi}]^{w'} \quad \textit{for all } h_1, h_2 \in H.$$

(3) *Elements of* $\mathcal{W}(H)$ *have the form*

$$[h_1, h_2^{\psi}] \cdots [h_s, h_{s+1}^{\psi}],$$

where $h_1, h_2, \dots, h_{s+1} \in H$ *and* s *is an odd number such that*

$$[h_1, h_2] \cdots [h_s, h_{s+1}] = 1.$$

Proof. (1) (i) The assertion follows from the definition of $\mathcal{E}(H)$.

(ii) Note that

$$[h_1, h_2^{\psi}] = [h_1, h_2[h_2, \psi]] = [h_1, [h_2, \psi]][h_1, h_2]^{[h_2,\psi]}$$

and since by item (i) $[h_1, h_2^{\psi}]$ commutes with $[\psi, h_2]$, we obtain

$$[h_1, h_2^{\psi}] = [h_1, [h_2, \psi]]^{[\psi,h_2]}[h_1, h_2] = [\psi, h_2, h_1][h_1, h_2].$$

(iii) We compute

$$\begin{aligned}\big[h_1^{h_3}, (h_2^{h_3})^{\psi}\big] &= \big[h_1^{h_3}, (h_2^{\psi})^{h_3^{\psi}}\big] = \big[h_1^{h_3(h_3^{\psi})^{-1}}, h_2^{\psi}\big]^{h_3^{\psi}} \\ &= \big[h_1\big[h_1, h_3(h_3^{\psi})^{-1}\big], h_2^{\psi}\big]^{h_3^{\psi}} \\ &= \Big(\big[h_1, h_2^{\psi}\big]^{[h_1,h_3(h_3^{\psi})^{-1}]}\big[\big[h_1, h_3(h_3^{\psi})^{-1}\big], h_2^{\psi}\big]\Big)^{h_3^{\psi}}.\end{aligned}$$

Since $[h_1, h_2^{\psi}] \in \mathcal{D}(H)$ and $[h_1, h_3(h_3^{\psi})^{-1}] \in \mathcal{L}(H)$, it follows that

$$\begin{aligned}\big[h_1^{h_3}, (h_2^{h_3})^{\psi}\big] &= \Big(\big[h_1, h_2^{\psi}\big]\big[\big[h_1, h_3(h_3^{\psi})^{-1}\big], h_2^{\psi}\big]\Big)^{h_3^{\psi}} \\ &= \big[h_1, h_2^{\psi}\big]^{h_3^{\psi}}\big[\big[h_1, h_3(h_3^{\psi})^{-1}\big], h_2^{\psi}\big]^{h_3^{\psi}}\end{aligned}$$

and therefore,

$$\big[\big[h_1, h_3(h_3^{\psi})^{-1}\big], h_2^{\psi}\big]^{h_3^{\psi}} = \big[h_1, h_2^{\psi}\big]^{-h_3^{\psi}}\big[h_1^{h_3}, (h_2^{h_3})^{\psi}\big].$$

(2) By item (i),

$$[h_1, h_2^{\psi}]^{z_1 z_2^{\psi}} = [h_1, h_2^{\psi}]^{z_1 z_2}.$$

The more general statement follows by induction on n.

(3) Let $\omega = [h_1, h_2^{\psi}] \cdots [h_s, h_{s+1}^{\psi}] \in \mathcal{D}(H)$. Then by item (ii),

$$\omega = [\psi, h_2, h_1][h_1, h_2] \cdots [\psi, h_{s+1}, h_s][h_s, h_{s+1}] = c[h_1, h_2] \cdots [h_s, h_{s+1}],$$

where $c \in \mathcal{L}(H)$. Since $H \cap \mathcal{L}(H) = 1$, we obtain

$$\omega \in \mathcal{W}(H) \iff [h_1, h_2] \cdots [h_s, h_{s+1}] = 1. \qquad \square$$

2.3 The subgroups $\mathcal{L}_1(H), \mathcal{L}_2(H)$

Recall

$$\mathcal{L}_1(H) = [\mathcal{L}(H), H], \quad \mathcal{L}_2(H) = [\mathcal{L}(H), H^{\psi}].$$

By Remark 1 (ii),

$$\begin{aligned} \tau : \mathcal{W}(H)[h_1, h_2^{\psi}] &\to (1, [h_1, h_2], 1), \\ \mathcal{W}(H)([h_1^{-1}h_1^{\psi}, h_2]) &\to ([h_1^{-1}, h_2], 1, 1), \\ \mathcal{W}(H)([h_1^{-1}h_1^{\psi}, h_2^{\psi}]) &\to (1, 1, [h_1, h_2]) \end{aligned}$$

for all $h_1, h_2 \in H$. Therefore

$$\begin{aligned} \tau : \frac{\mathcal{D}(H)}{\mathcal{W}(H)} &\to 1 \times H' \times 1, \\ \frac{\mathcal{W}(H)\mathcal{L}_1(H)}{\mathcal{W}(H)} &\to H' \times 1 \times 1, \\ \frac{\mathcal{W}(H)\mathcal{L}_2(H)}{\mathcal{W}(H)} &\to 1 \times 1 \times H' \end{aligned}$$

are epimorphisms.

Lemma 2. *We have*

$$\begin{aligned} \mathcal{L}_1(H) &= \mathcal{L}(H) \cap (\mathcal{D}(H) \cdot H'), \\ \mathcal{L}_2(H) &= \mathcal{L}(H) \cap (\mathcal{D}(H) \cdot (H')^{\psi}), \\ \mathcal{L}_1(H) \cap \mathcal{L}_2(H) &= \mathcal{W}(H). \end{aligned}$$

Proof. Since

$$[h_1^{\psi}h_1^{-1}, h_2] = [h_1^{\psi}, h_2]^{h_1^{-1}}[h_1^{-1}, h_2]$$

holds for all $h_1, h_2 \in H$, we conclude

$$\mathcal{L}_1(H) = [\mathcal{L}(H), H] \leq \mathcal{D}(H)H' \leq \mathcal{L}(H) \cap (\mathcal{D}(H)H').$$

By item (ii) of the previous lemma,

$$\mathcal{D}(H) \leq [\mathcal{L}(H), H]H' = \mathcal{L}_1(H)H'.$$

Therefore,

$$\begin{aligned} \mathcal{W}(H) &\leq \mathcal{L}(H) \cap (\mathcal{D}(H)H') \leq \mathcal{L}(H) \cap (\mathcal{L}_1(H)H') \\ &\leq \mathcal{L}_1(H)(\mathcal{L}(H) \cap H') = \mathcal{L}_1(H). \end{aligned}$$

Hence,

$$\mathcal{L}_1(H) = \mathcal{L}(H) \cap (\mathcal{D}(H)H').$$

Apply ψ to $\mathcal{L}_1(H)$, to obtain

$$\mathcal{L}_2(H) = \mathcal{L}(H) \cap (\mathcal{D}(H)(H')^{\psi}).$$

Clearly,

$$\mathcal{W}(H) = \mathcal{L}(H) \cap \mathcal{D}(H) \subseteq \mathcal{L}_1(H) \cap \mathcal{L}_2(H).$$

From [9, Lemma 4.1.10 (iii)], we know that the intersection $\frac{L_1(H)}{W(H)} \cap \frac{L_2(H)}{W(H)}$ is trivial in $\frac{\chi(H)}{W(H)} (\cong \frac{\mathcal{E}(H)}{\mathcal{W}(H)})$; therefore,

$$\mathcal{L}_1(H) \cap \mathcal{L}_2(H) = \mathcal{W}(H). \qquad \square$$

3 The groups $\mathcal{R}(H)$ and $\nu(H)$

Proposition 1. *The subgroup $\mathcal{R}(H)$ satisfies the following properties:*

(i) $\mathcal{R}(H) = [\mathcal{L}_1(H), H^{\psi}] = [\mathcal{L}_2(H), H]$,

(ii) $[\mathcal{W}(H), \mathcal{E}(H)] \subseteq \mathcal{R}(H) \subseteq \mathcal{W}(H)$.

Proof. (i) By definition,

$$\mathcal{R}(H) = [H, \mathcal{L}(H), H^{\psi}] = [\mathcal{L}_1(H), H^{\psi}].$$

Since $[\mathcal{D}(H), \mathcal{L}(H)] = [H, H^{\psi}, \mathcal{L}(H)] = 1$, we obtain, by the Hall–Witt identity, that

$$[H^{\psi}, \mathcal{L}, H] = [H, \mathcal{L}, H^{\psi}]$$

and consequently that $\mathcal{R}(H) = [\mathcal{L}_2(H), H]$.

(ii) We note

$$\mathcal{R}(H) \subseteq \mathcal{L}_1(H) \cap \mathcal{L}_2(H) = \mathcal{W}(H);$$

therefore,

$$[\mathcal{W}(H), H], [\mathcal{W}(H), H^{\psi}] \leq \mathcal{R}(H),$$
$$[\mathcal{W}(H), \mathcal{E}(H)] \subseteq \mathcal{R}(H). \qquad \square$$

Proof of Theorem 2. Recall

$$\nu(H) = \langle H, H^{\psi} \mid [h_1, h_2^{\psi}]^{h_3^{\psi}} = [h_1, h_2^{\psi}]^{h_3} = [h_1^{h_3}, (h_2^{h_3})^{\psi}],\ h_1, h_2, h_3 \in H \rangle,$$

and

$$\xi : \mathcal{E}(H) \to \nu(H), \quad h \mapsto h,\ h^{\psi} \mapsto h^{\psi} \quad \text{for all } h \in H.$$

By Lemma 1 (iii), the commutator relations

$$\big[[h_1, h_3(h_3^{\psi})^{-1}], h_2^{\psi}\big] = \left(\left([h_1, h_2^{\psi}]^{h_3^{\psi}}\right)^{-1}[h_1^{h_3}, (h_2^{h_3})^{\psi}]\right)^{(h_3^{\psi})^{-1}}$$

hold in $R(H)$. Therefore, $R(H) \leq \ker(\xi)$ and ξ induces an epimorphism

$$\overline{\xi} : \frac{\mathcal{E}(H)}{\mathcal{R}(H)} \to \nu(H).$$

On the other hand, since both equalities

$$[h_1, h_2^{\psi}]^{h_3\psi} = [h_1, h_2^{\psi}]^{h_3},\ [h_1, h_2^{\psi}]^{h_3^{\psi}} = [h_1^{h_3}, (h_2^{h_3})^{\psi}]$$

hold in $\mathcal{E}(H)$ modulo $\mathcal{R}(H)$, we obtain the epimorphism

$$\xi' : \nu(H) \to \frac{\mathcal{E}(H)}{\mathcal{R}(H)}$$

defined by $h \to \mathcal{R}(H)h$, $h^{\psi} \to \mathcal{R}(H)h^{\psi}$. The composition $\overline{\xi}\xi'$ is the identity map and thus, $\frac{\mathcal{E}(H)}{\mathcal{R}(H)} \cong \nu(H)$. With this the proof of the theorem is finished. □

4 Restrictions and inductions of $\mathcal{E}$

Proposition 2. *Let $K, H_1 \leqslant H \leq \mathcal{E}(H)$. Then:*

(i) $[K, H_1^{\psi}]$ *and* $[K^{\psi}, H_1] \lhd \langle H_1, H_1^{\psi}\rangle$,

(ii) $\langle K, K^{\psi}\rangle \cap \mathcal{D}(H) = [K, K^{\psi}]$,

(iii) $\langle K, K^{\psi}\rangle \cap \mathcal{L}(H) = [K, \psi]$.

Proof. (i) Since H_1^{ψ} normalizes $[K, H_1^{\psi}]$ and

$$[k, h_1^{\psi}]^{y} = [k, h_1^{\psi}]^{y^{\psi}} = [k, h_1^{\psi}]^{-1}[k, h_1^{\psi} y^{\psi}]$$

hold for all $k \in K$ and $h_1, y \in H_1$, we conclude that H_1 normalizes $[K, H_1^{\psi}]$. Analogously, $[H_1, K^{\psi}] \trianglelefteq \langle H_1, H_1^{\psi}\rangle$.

(ii) Denote $\langle K, K^{\psi}\rangle$ by $\hat{K}$. It is clear that $[K, K^{\psi}] \subseteq \hat{K} \cap \mathcal{D}(H)$. On the other hand, consider the epimorphism $\pi : \mathcal{E}(H) \to H \times H$. Since $\ker(\pi) = \mathcal{D}(H)$, we have that π induces an epimorphism

$$\frac{\hat{K}}{\hat{K} \cap \mathcal{D}(H)} \to K \times K.$$

Consider the natural epimorphism

$$K \times K \to \frac{\hat{K}}{[K, K^{\psi}]}.$$

The composition of the last two epimorphisms leads to

$$\frac{\hat{K}}{\hat{K} \cap \mathcal{D}(H)} \to \frac{\hat{K}}{[K, K^{\psi}]}$$

defined by $(\hat{K} \cap \mathcal{D}(H))x \mapsto [K, K^{\psi}]x$ for all $x \in \hat{K}$. Therefore,

$$\hat{K} \cap \mathcal{D}(H) \subseteq [K, K^{\psi}].$$

(iii) The proof here is similar to that of the last item, by considering the epimorphism $\rho : \mathcal{E}(H) \to H$ and by taking the composition of the maps

$$\frac{\hat{K}}{\hat{K} \cap \mathcal{L}} \to K \to \frac{\hat{K}}{[K, \psi]}. \qquad \square$$

Proposition 3. *Let $\phi : H \to K$ be an epimorphism, $N = \ker(\phi)$. Then ϕ extends naturally to an epimorphism $\hat{\phi} : \mathcal{E}(H) \to \mathcal{E}(K)$ such that*

(i) $\hat{\phi}(\mathcal{D}(H)) = \mathcal{D}(K)$, $\hat{\phi}(\mathcal{L}(H)) = \mathcal{L}(K)$,

(ii) $\ker(\hat{\phi}) = \langle N, N^{\psi}\rangle [N, H^{\psi}][N^{\psi}, H]$,

(iii) $\ker(\hat{\phi} \mid_{\mathcal{D}(H)}) = [N, H^{\psi}][N^{\psi}, H]$.

Proof. (i) The extension $\hat{\phi}$ of ϕ is determined by $h^{\hat{\phi}} = h^{\phi}$ and $(h^{\psi})^{\hat{\phi}} = (h^{\phi})^{\psi}$ for all $h \in H$. Then

$$\hat{\phi}(\mathcal{D}(H)) = \hat{\phi}([H, H^{\psi}]) = [\hat{\phi}(H), \hat{\phi}(H^{\psi})] = [K, K^{\psi}] = \mathcal{D}(K)$$

and

$$\hat{\phi}(\mathcal{L}(H)) = \hat{\phi}([H, \psi]) = [\hat{\phi}(H), \psi]) = [K, \psi].$$

(ii) Let $M = \langle N, N^{\psi}\rangle [N, H^{\psi}][H, N^{\psi}]$. We have that $M \leq \ker(\hat{\phi})$, because $N, N^{\psi} \subseteq \ker(\hat{\phi})$. By the previous proposition, M is normal in $\mathcal{E}(H)$. The map

$$\mu : K \cup K^{\psi} \to \frac{\mathcal{E}(H)}{M}, \quad k \to M\phi^{-1}(k), \quad k^{\psi} \to M(\phi^{-1}(k))^{\psi}$$

is well-defined, since $N, N^{\psi} \subseteq M$. The restrictions of μ to K and K^{ψ} are homomorphisms and they extend uniquely to a homomorphism

$$\mu^* : K * K^{\psi} \to \mathcal{E}(H)/M.$$

Since the relations

$$[k_1, k_2^{\psi}]^{k_3} = [k_1, {k_2}^{\psi}]^{{k_3}^{\psi}} \quad \text{for all } k_1, k_2, k_3 \in K$$

are preserved by μ^*, it induces a homomorphism $\mathcal{E}(K) \to \frac{\mathcal{E}(H)}{M}$. On the other hand, since $M \leq \ker(\hat{\phi})$, we have a homomorphism $\tilde{\phi} : \mathcal{E}(H)/M \to \mathcal{E}(K)$ such that $\tilde{\phi}(Mh) = \hat{\phi}(h)$ and $\tilde{\phi}(Mh^{\psi}) = \hat{\phi}(h^{\psi})$ for all $h \in H$. The composition of μ^* and $\tilde{\phi}$ gives us

$$\tilde{\phi}\mu^*(k) = \tilde{\phi}(M\phi^{-1}(k)) = k, \quad \tilde{\phi}\mu^*(k^{\psi}) = \tilde{\phi}(M(\phi^{-1}(k))^{\psi}) = k^{\psi}$$

for all $k \in K$. So, $\tilde{\phi}\mu^* = 1_{\mathcal{E}(K)}$, which shows that $\tilde{\phi}$ is an isomorphism.

(iii) This item follows from

$$\begin{aligned} \mathcal{D}(H) \cap \ker\hat{\phi} &= \mathcal{D}(H) \cap (\langle N, N^{\psi}\rangle [N, H^{\psi}] \cdot [H, N^{\psi}]) \\ &= [N, N^{\psi}][N, H^{\psi}][N^{\psi}, H] \\ &= [N, H^{\psi}][N^{\psi}, H]. \end{aligned}$$

□

Proposition 4. *Let H and K be normal subgroups of a group G such that*

$$G = H \oplus K.$$

Then

$$\begin{aligned} &\mathcal{E}(G) = \langle H, H^{\psi}\rangle [H, K^{\psi}][H^{\psi}, K]\langle K, K^{\psi}\rangle, \\ &\mathcal{E}(H) \cong \langle H, H^{\psi}\rangle, \\ &\mathcal{E}(K) \cong \langle K, K^{\psi}\rangle. \end{aligned}$$

Proof. By the previous proposition, the projection $\phi : G \to H$ can be extended to an epimorphism $\hat{\phi} : \mathcal{E}(G) \to \mathcal{E}(H)$ such that

$$\ker(\hat{\phi}) = \langle K, K^{\psi}\rangle [HK, K^{\psi}][H^{\psi}K^{\psi}, K].$$

It follows that

$$[HK, K^{\psi}] = [H, K^{\psi}][K, K^{\psi}],$$

since by Proposition 2 the subgroups K, K^{ψ}, H and H^{ψ} normalize $[H, K^{\psi}]$. Similarly, it follows that

$$[H^{\psi}K^{\psi}, K] = [H^{\psi}, K][K, K^{\psi}].$$

Thus,

$$\mathcal{E}(G) \cong \langle K, K^{\psi}\rangle [H, K^{\psi}][H^{\psi}, K]\mathcal{E}(H).$$

But as $\hat{\phi}(\langle H, H^{\psi}\rangle) = \mathcal{E}(H)$, and $\mathcal{E}(H)$ maps onto $\langle H, H^{\psi}\rangle$, it follows that

$$\ker(\hat{\phi}) \cap \langle H, H^{\psi}\rangle = 1, \quad \mathcal{E}(H) \cong \langle H, H^{\psi}\rangle.$$

We conclude

$$\mathcal{E}(G) \cong \langle K, K^{\psi}\rangle [H, K^{\psi}][H^{\psi}, K]\langle K, K^{\psi}\rangle.$$

The isomorphism $\mathcal{E}(K) \cong \langle K, K^{\psi}\rangle$ follows in a similar manner. □

5 The augmentation ideal of $\mathbb{Z}(H/H')$

Let $\tilde{G} = \mathbb{Z}(H/H') \rtimes H$ denote the semidirect product of the group ring $\mathbb{Z}(H/H')$ of the abelianization of H by the group H, where H acts on $\mathbb{Z}(H/H')$ by right multiplication

$$\left(\sum x_{H'h} H'h\right)^{h_1} = \sum x_{H'h}\, H'hh_1$$

for all $h, h_1 \in H$. Denote the elements of $\tilde{G}$ as pairs (w, h) with $w \in \mathbb{Z}(H/H')$, $h \in H$. Identify the complement H with $\{0\} \times H$, write $\overline{h}$ for the coset $H'h$ for any $h \in H$ and let $u = (\bar{1}, 1) \in \tilde{G}$. Also, let $\mathcal{A}_{\mathbb{Z}}(H/H')$ be the augmentation ideal of the group ring $\mathbb{Z}(H/H')$ and let G be the subgroup of $\tilde{G}$ generated by $\mathcal{A}_{\mathbb{Z}}(H/H')$ and H. Note that H' acts trivially on $\mathbb{Z}(H/H')$.

Proposition 5. *The map $\varepsilon : H \cup H^{\psi} \to \tilde{G}$ defined by $h \to h$, $h^{\psi} \to h^{u}$ for all $h \in H$ extends to an epimorphism $\varepsilon : \mathcal{E}(H) \to \tilde{G}$. Moreover, if H is abelian, then* $\ker(\varepsilon) = \mathcal{L}(H)'$, *the commutator subgroup of $\mathcal{L}(H)$.*

Proof. Consider

$$[H, u] = \langle [h, u] \mid h \in H \rangle \leqslant \tilde{G}.$$

It is easy to see that

$$[h, u] = h^{-1}h^{u} = u^{-h}u = (\bar{1} - \overline{h}, 1).$$

Therefore, $[H, u] = \mathcal{A}_{\mathbb{Z}}(H/H') \times \{1\}$ and $G = \langle H, H^{u} \rangle$. Since $\mathbb{Z}(H/H')$ is abelian and H' acts trivially on $\mathbb{Z}(H/H')$, we find that

$$\begin{aligned}
[[h_1, h_2^{u}], [h, u]] &= [h_1^{-1}u^{-1}h_2^{-1}uh_1u^{-1}h_2u, [h, u]] \\
&= [u^{-h_1}(h_1^{-1}h_2^{-1}uh_1h_2)u^{-h_2}u, [h, u]] \\
&= [u^{-h_1}u^{h_2h_1}[h_1, h_2]u^{-h_2}u, u^{-h}u] \\
&= 1.
\end{aligned}$$

Thus ε extends to an epimorphism $\mathcal{E}(H) \to \mathcal{A}_{\mathbb{Z}}(H/H') \rtimes H$.

By [9, Theorem 2.1.1], $G = \mathcal{A}_{\mathbb{Z}}(H) \cdot H$ is isomorphic to the group

$$\langle H, H^{\psi} \mid [H, \psi]' = 1 \rangle$$

and so, when H is abelian, we obtain $\ker(\varepsilon) = \mathcal{L}(H)'$. □

Next, we specialize H to a finite cyclic group.

Corollary 1. *Let $H = \langle x \rangle$ be a cyclic group of order n. Then*

$$\mathcal{E}(H) \cong \mathbb{Z}^{n-1} \rtimes H.$$

Proof. We need to show $\ker(\varepsilon) = 1$; that is, $\mathcal{L}(H)$ is abelian. We note that

$$1 = [x^n, \psi] = [x, \psi]^{x^{n-1}} [x, \psi]^{x^{n-2}} \cdots [x, \psi],$$
$$[x, \psi]^{x^{n-1}} = [x, \psi]^{-x^{n-2}} \cdots [x, \psi]^{-1};$$

therefore,

$$\mathcal{L}(H) = \langle [x, \psi], [x, \psi]^{x^2}, \dots, [x, \psi]^{x^{n-2}} \rangle.$$

We compute

$$[x^i, x^\psi] = [x^i, x.x^{-1}x^\psi] = [x^i, [x, \psi]] = [x, \psi]^{-x^i}[x, \psi] \quad \text{for all } i \in \mathbb{Z}.$$

Since $[\mathcal{D}(H), \mathcal{L}(H)] = 1$, it follows that

$$1 = [[x^i, x^\psi], [x, \psi]] = [[x, \psi]^{-x^i}, [x, \psi]]$$

and thus $\mathcal{L}(H)$ is an abelian group. □

6 $\mathcal{E}$ of solvable groups

Proof of Theorem 1 (i). Since H is a solvable group with derived length k, we apply [9, Corollary 4.1.8] to conclude $\chi(H)^{(k+1)} = 1$. As $\ker(\delta) = \Delta(H)$ is abelian, we conclude further that $\mathcal{E}(H)^{(k+2)} = 1$ and this ends the proof. □

6.1 Abelian groups

Given a group G we denote its center by $Z(G)$.

Proposition 6. *Let H be an abelian group. Then:*

(i) *we have*

$$\mathcal{D}(H) = \mathcal{W}(H) = [\mathcal{L}(H), H] = [\mathcal{L}(H), H^\psi],$$
$$\mathcal{R}(H) = [\mathcal{D}(H), H] = [\mathcal{L}(H), H, H],$$

(ii) *$\mathcal{L}(H)$ is nilpotent of class at most* 2,

$$\mathcal{L}(H)' \subseteq \mathcal{D}(H) \subseteq Z(\mathcal{L}(H)), \quad \mathcal{L}(H)' \subseteq Z(\mathcal{E}(H)).$$

Proof. (i) By Lemma 2,

$$\mathcal{L}_1(H) = [\mathcal{L}(H), H] = \mathcal{L}(H) \cap \mathcal{D}(H)H',$$
$$\mathcal{L}_2(H) = [\mathcal{L}(H), H^\psi] = \mathcal{L}(H) \cap \mathcal{D}(H)H'^\psi.$$

Since H is abelian group, we have that

$$\mathcal{W}(H) = \mathcal{L}(H) \cap \mathcal{D}(H) = \mathcal{L}_1(H) = \mathcal{L}_2(H).$$

Since $\mathcal{D}(H) \subseteq [\mathcal{L}(H), H]H'$ and H abelian, we obtain

$$\mathcal{D}(H) \subseteq [\mathcal{L}(H), H] \subseteq \mathcal{L}(H) \cap \mathcal{D}(H) = \mathcal{W}(H);$$

therefore, $\mathcal{D}(H) = [\mathcal{L}(H), H] = [\mathcal{L}(H), H^{\psi}]$. From this last equality, we obtain

$$\mathcal{R} = [H, \mathcal{L}(H), H^{\psi}] = [\mathcal{D}(H), H^{\psi}] = [\mathcal{D}(H), H] = [\mathcal{L}(H), H, H].$$

(ii) It is clear that H centralizes $\mathcal{L}(H)$ modulo $[\mathcal{L}(H), H](= \mathcal{D}(H))$ and H^{ψ} centralizes $\mathcal{L}(H)$ modulo $[\mathcal{L}(H), H^{\psi}](= \mathcal{D}(H))$; therefore,

$$\mathcal{L}(H)' \subseteq [\mathcal{E}(H), \mathcal{L}(H)] \subseteq \mathcal{D}(H).$$

On the other hand

$$[\mathcal{D}(H), \mathcal{L}(H)] = [\mathcal{L}(H), H, \mathcal{L}(H)] = [H, \mathcal{L}(H), \mathcal{L}(H)] = 1;$$

therefore,

$$[\mathcal{L}(H), \mathcal{L}(H), H] = [\mathcal{L}(H)', H] = 1.$$

Proceeding in the same way, we obtain

$$1 = [\mathcal{D}(H), \mathcal{L}(H)] = [\mathcal{L}(H), H^{\psi}, \mathcal{L}(H)] = [H^{\psi}, \mathcal{L}(H), \mathcal{L}(H)]$$

and

$$[\mathcal{L}(H), \mathcal{L}(H), H^{\psi}] = [\mathcal{L}(H)', H^{\psi}] = 1.$$

We have reached the desired conclusion:

$$\mathcal{L}(H)' \subseteq Z(\mathcal{E}(H)).$$ □

Proposition 7. *Let H be an abelian group. Then*

$$\mathcal{L}(H)' = \langle [a^{\psi}, b][a, b^{\psi}] \text{ for all } a, b \in H \rangle.$$

Proof. Consider $l_1, l_2 \in \mathcal{L}(H)$. Then

$$l_1 = [h_1, \psi][h_2, \psi] \cdots [h_s, \psi],$$

$$l_2 = [h'_1, \psi][h'_2, \psi] \cdots [h'_r, \psi]$$

for some $s, r \in \mathbb{N}$ and $h_1, h_2, \ldots, h_s, h'_1, h'_2, \ldots, h'_r \in H$.

Since $\mathcal{L}(H)' \le \mathcal{D}(H)$ and $[\mathcal{L}(H), \mathcal{D}(H)] = 1$ we have

$$[l_1, l_2] = \prod_{i=1}^{s} \prod_{j=1}^{r} [[h_i, \psi], [h'_j, \psi]].$$

However,

$$\begin{aligned}
[[h_i, \psi], [h'_j, \psi]] &= [h_i{}^{-1} h_i{}^{\psi}, h'_j{}^{-1} h'_j{}^{\psi}] \\
&= [h_i{}^{-1}, h'_j{}^{-1} h'_j{}^{\psi}]^{h_i{}^{\psi}} [h_i{}^{\psi}, h'_j{}^{-1} h'_j{}^{\psi}] \\
&= [h_i{}^{-1}, h'_j{}^{\psi}]^{h_i{}^{\psi}} [h_i{}^{\psi}, h'_j{}^{-1}]^{h'_j{}^{\psi}} \\
&= [h_i{}^{-1}, h'_j{}^{\psi}]^{h_i} [h_i{}^{\psi}, h'_j{}^{-1}]^{h'_j} \\
&= [h_i, h'_j{}^{\psi}]^{-1} [h_i{}^{\psi}, h'_j]^{-1} \\
&= [h'_j{}^{\psi}, h_i][h'_j, h_i{}^{\psi}],
\end{aligned}$$

which proves the assertion. □

Proposition 8. *Let H be an abelian group and let $a, b \in H$, have orders n and m, respectively. Then the element $[a, b^{\psi}][a^{\psi}, b]$ has order a divisor of* $\gcd(n, m)$.

Proof. From

$$1 = [a^n, b^{\psi}] = [a, b^{\psi}]^{a^{n-1}} [a, b^{\psi}]^{a^{n-2}} \cdots [a, b^{\psi}],$$

we obtain

$$1 = [a^{\psi}, b]^{(a^{n-1})^{\psi}} [a^{\psi}, b]^{(a^{n-2})^{\psi}} \cdots [a^{\psi}, b],$$

since$[\mathcal{D}(H), \mathcal{L}(H)] = 1$. Also, since $\mathcal{D}(H) = \mathcal{W}(H)$, by multiplying together these two equations, we obtain

$$\begin{aligned}
1 &= [a, b^{\psi}]^{a^{n-1}} [a, b^{\psi}]^{a^{n-2}} \cdots [a, b^{\psi}][a^{\psi}, b]^{(a^{n-1})} [a^{\psi}, b]^{(a^{n-2})} \cdots [a^{\psi}, b] \\
&= ([a, b^{\psi}][a^{\psi}, b])^{(a^{n-1})} ([a, b^{\psi}][a^{\psi}, b])^{(a^{n-2})} \cdots [a, b^{\psi}][a^{\psi}, b] \\
&= ([a, b^{\psi}][a^{\psi}, b])^{n},
\end{aligned}$$

since $\mathcal{L}' \le Z(\mathcal{E}(H))$. Similarly, $([a, b^{\psi}][a^{\psi}, b])^m = 1$; therefore, the order of $[a, b^{\psi}][a^{\psi}, b]$ divides $\gcd(n, m)$. □

The next corollary follows directly from the above proposition.

Corollary 2. *If H is a finite abelian group, then so is $\mathcal{L}(H)'$.*

6.2 Polycyclic groups

A. McDermott defined in his thesis [6] the subgroup of the free product $H * H^{\psi}$,

$$K = \langle [h_1, h_2^{\psi}]^{h_3}[{h_1}^{h_3}, {h_2}^{{h_3}^{\psi}}]^{-1} \mid h_1, h_2, h_3 \in H \rangle^{H*H^{\psi}}$$

which is clearly related to our group $\mathcal{R}(H)$. The following result is a consequence of [6, Theorem 2.2.8].

Theorem 3. *Let H be a polycyclic group with polycyclic generating sequence $S = \{a_i \mid 1 \leq a_i \leq n\}$. Then the set of generators of K can be reduced:*

$$K = \langle [a_i, a_j^{\psi}]^{a_k}[{a_i}^{a_k}, {a_j}^{{a_k}^{\psi}}]^{-1} \mid a_i, a_j, a_k \in S \rangle^{H*H^{\psi}}.$$

Proposition 9. *Let T be a transversal of H' in H. Then*

$$\mathcal{R}(H) = \langle [h_1, h_2^{\psi}]^{h_3}[{h_1}^{h_3}, {h_2}^{{h_3}^{\psi}}]^{-1} \mid h_1, h_2, h_3 \in H \rangle^{T}.$$

Proof. Define the subgroup

$$\mathcal{J}(H) = \langle [h_1, h_2^{\psi}]^{h_3}[{h_1}^{h_3}, {h_2}^{{h_3}^{\psi}}]^{-1} \mid h_1, h_2, h_3 \in H \rangle^{\mathcal{E}(H)}$$

of $\mathcal{R}(H)$, normal in $\mathcal{E}(H)$. We recall the epimorphism $\xi : \mathcal{E}(H) \to \nu(H)$ defined by $h \mapsto h$, $h^{\psi} \mapsto h^{\psi}$ for all $h \in H$, with kernel $\mathcal{R}(H)$. Then ξ induces

$$\bar{\xi} : \frac{\mathcal{E}(H)}{\mathcal{J}(H)} \to \nu(H).$$

On the other hand, the map $\sigma : \nu(H) \to \frac{\mathcal{E}(H)}{J(H)}$ defined by

$$h \mapsto \mathcal{J}(H)h, \quad h^{\psi} \mapsto \mathcal{J}(H)h^{\psi}$$

extends to an epimorphism $\bar{\sigma} : \nu(H) \to \frac{\mathcal{E}(H)}{J(H)}$, and the relations

$$[h_1, h_2^{\psi}]^{h_3^{\psi}} = [h_1, h_2^{\psi}]^{h_3} = [{h_1}^{h_3}, (h_2^{h_3})^{\psi}] \quad \text{for all } h_1, h_2, h_3 \in H$$

hold in $\frac{\mathcal{E}(H)}{\mathcal{J}(H)}$. Since $\bar{\sigma}\bar{\xi}$ is the identity on $\frac{\mathcal{E}(H)}{\mathcal{J}(H)}$, we have

$$\mathcal{R}(H) = \mathcal{J}(H).$$

By Lemma 1 (ii),

$$\mathcal{R}(H) \leq \mathcal{W}(H) \leq Z(\mathcal{D}(H));$$

therefore,

$$\mathcal{R}(H) = \langle [h_1, h_2^{\psi}]^{h_3}[{h_1}^{h_3}, ({h_2}^{h_3})^{\psi}]^{-1} \mid h_1, h_2, h_3 \in H \rangle^{H},$$

and

$$[\mathcal{R}(H), H'] = [\mathcal{R}(H), \mathcal{D}(H)] = 1.$$

Thus

$$\mathcal{R}(H) = \left\langle [h_1, h_2^{\psi}]^{h_3} [{h_1}^{h_3}, {h_2}^{h_3 \psi}]^{-1} \mid h_1, h_2, h_3 \in H \right\rangle^T. \qquad \square$$

Proposition 10. *Let H be a polycyclic group, with polycyclic generating sequence $S = \{a_1, a_2, \ldots, a_n\}$ and let T be a transversal of H' in H. Then*

$$\mathcal{R}(H) = \left\langle [a_i, a_j^{\psi}]^{a_k} [{a_i}^{a_k}, {a_j}^{a_k \psi}]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^T.$$

Proof. Considering the natural epimorphism $\phi : H * H^{\psi} \to \mathcal{E}(H)$, we have that $\phi(K) = \mathcal{R}(H)$. By the above proposition,

$$\begin{aligned} \mathcal{R}(H) &= \left\langle [a_i, a_j^{\psi}]^{a_k} [{a_i}^{a_k}, {a_j}^{a_k \psi}]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^{\mathcal{E}(H)} \\ &= \left\langle [a_i, a_j^{\psi}]^{a_k} [{a_i}^{a_k}, {a_j}^{a_k \psi}]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^T. \qquad \square \end{aligned}$$

Proof of Theorem 1 (ii). In one direction, let $\mathcal{E}(H)$ be polycyclic. Then, on recalling the epimorphism $\varepsilon : \mathcal{E}(H) \to A_{\mathbb{Z}}(H/H')H$, we conclude that $A_{\mathbb{Z}}(H/H')$ is finitely generated and therefore H/H' is finite. In the other direction, on letting H/H' be finite, we obtain that $R(H)$ is a finitely generated abelian group. Since by [1], $\nu(H)$ is polycyclic and since $\frac{\mathcal{E}(H)}{\mathcal{R}(H)} \cong \nu(H)$, we conclude that $\mathcal{E}(H)$ is polycyclic.

This ends the proof of part (ii) of Theorem 1. $\square$

7 $\mathcal{E}$ of perfect groups

When H is a perfect group, one finds a complete description of $\chi(H)$ in [9, Section 4.4], where it is shown that $\chi(H)$ is a certain stem extension of $H \times H \times H$ by the Schur multiplier $M(H)$ of H.

Proof of Theorem 1 (iii). (a) Since $\frac{\mathcal{E}(H)}{\mathcal{D}(H)L(H)} \cong \frac{H}{H'}$, it follows that

$$\mathcal{E}(H) = \mathcal{D}(H)\mathcal{L}(H)$$

and therefore, $\mathcal{W}(H) \leq Z(\mathcal{E}(H))$. From the formula

$$\mathcal{E}(H)' = \mathcal{D}(H)'\mathcal{L}(H)'[\mathcal{D}(H), \mathcal{L}(H)],$$

we derive

$$\mathcal{E}(H)' = \mathcal{D}(H)'\mathcal{L}(H)'.$$

By [9, Lemma 4.4.6], $\chi(H)$ and its subgroups $D = [H, H^\psi]$, $L = [H, \psi]$ are all perfect groups. We conclude

$$\begin{aligned}
&\Delta(H) \le \mathcal{W}(H) \le Z(\mathcal{E}(H)),\\
&\mathcal{D}(H) = \Delta(H)\mathcal{D}(H)',\\
&\mathcal{L}(H) = \Delta(H)\mathcal{L}(H)',\\
&\mathcal{E}(H) = \mathcal{D}(H)\mathcal{L}(H) = \Delta(H)\mathcal{D}(H)'\mathcal{L}(H)' = \mathcal{E}(H)'.
\end{aligned}$$

(b) Consider $\overline{\mathcal{E}(H)} = \frac{\mathcal{E}(H)}{\mathcal{D}(H)'}$. Then

$$\overline{\mathcal{D}(H)}(= \overline{[H, H^\psi]}) = \overline{\mathcal{W}(H)},$$

which is contained in the center of $\overline{\mathcal{E}(H)}$. Therefore,

$$[\overline{H^\psi}, \overline{H}, \overline{H^\psi}] = [\overline{H}, \overline{H^\psi}, \overline{H^\psi}] = 1$$

and by the Witt identity,

$$[\overline{H^\psi}, \overline{H^\psi}, \overline{H}] = 1,$$
$$[\overline{(H')^\psi}, \overline{H}] = [\overline{H^\psi}, \overline{H}] = 1.$$

Thus,

$$\mathcal{D}(H) = \mathcal{D}(H)', \text{ a stem extension of } H.$$

(c) From the structure of $\chi(H)$, we have that $W(H)$ is central in $D(H)$,

$$\frac{D(H)}{W(H)} \cong H \quad \text{and} \quad W(H) \cong M(H);$$

that is, $D(H)$ is a total covering of H. As H is perfect, $D(H)$ is the total covering of H. Therefore the map γ from the generating set $\{[h, h^\psi] \mid h \in H\}$ of $D(H)$ into the generating set $\{[h, h^\psi] \mid h \in H\}$ of $D(H)$ defined by

$$[h, h^\psi] \mapsto [h, h^\psi],$$

extends to an epimorphism $\gamma : D(H) \to D(H)$. Hence, the composition

$$\delta\gamma : D(H) \to D(H)$$

is the identity and $\Delta(H) = 1$ follows.

This ends the proof of part (iii) of Theorem 1. □

Proof of Theorem 1 (iv). Assume that H is a finite group. Since $\mathcal{E}(H)$ maps onto $A_{\mathbb{Z}}(H/H') \rtimes H$, on supposing $\mathcal{E}(H)$ finite, we obtain $H = H'$. Reciprocally, on supposing H perfect, we obtain by the previous theorem, $\mathcal{E}(H) \cong \chi(H)$.

This proves the final part of Theorem 1. □

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Author information

Bruno C. R. Lima, Instituto Federal de Educação, Ciência e Tecnologia de Goiás, Formosa-Go, Brazil.
E-mail: bruno.crlima84@gmail.com

Said N. Sidki, Departamento de Matemática, Universidade de Brasília, 70910 Brasília-DF, Brazil.
E-mail: ssidki@gmail.com