

Certain irreducible characters over a normal subgroup

Gabriel Navarro and Noelia Rizo

Communicated by Robert M. Guralnick

Abstract. We extend the Howlett–Isaacs theorem on the solvability of groups of central type taking into account actions by automorphisms. Then we study certain induced characters whose constituents have all the same degree.

1 Introduction

The celebrated Howlett–Isaacs [3] theorem on groups of central type solved a conjecture proposed by Iwahori and Matsumoto in 1964: if Z is a normal subgroup of a finite group G , $\lambda \in \text{Irr}(Z)$ is a G -invariant complex irreducible character of Z , and the induced character λ^G is a multiple of a single $\chi \in \text{Irr}(G)$, then G/Z is solvable. (In this case, it is said that λ is *fully ramified* in G/Z and that G is of *central type* if furthermore $Z = \mathbf{Z}(G)$.) This theorem, proved in 1982, is one of the first applications of the Classification of Finite Simple Groups to Representation Theory. Fully ramified characters are essential in both Ordinary and Modular Representation Theory.

Our first main result in this note is the following generalization.

Theorem A. *Suppose that $Z \triangleleft G$, and let $\lambda \in \text{Irr}(Z)$. Assume that if $\chi, \psi \in \text{Irr}(G)$ are irreducible constituents of the induced character λ^G , then there exists an automorphism $a \in \text{Aut}(G)$ stabilizing Z , such that $\chi^a = \psi$. If T is the stabilizer of λ in G , then T/Z is solvable.*

In a different language of projective representations, Theorem A was obtained by R. J. Higgs under some solvability conditions [2]. His proof is mostly sketched, among other reasons because he uses some of the arguments in [3] or [8] (where, as a matter of fact, the Iwahori–Matsumoto conjecture was wrongly proven) or in some other papers by the author. Here, we choose to present a complete proof of

Theorem A, in the language of character theory, and by doing so we shall adapt several arguments in all these papers. We would like to acknowledge this now.

Theorem A is only one case of a more general problem, which seems intractable by now: if all the irreducible characters of G over some G -invariant $\lambda \in \text{Irr}(Z)$ have the same degree, then G/Z is solvable. (See [9, Conjecture 11.1].)

In the second main result of this note, we study this latter situation under some special hypothesis.

Theorem B. *Suppose that G is π -separable and let $N = \mathbf{O}_\pi(G)$. Let $\theta \in \text{Irr}(N)$ be G -invariant. Then all members of $\text{Irr}(G|\theta)$ have equal degrees if and only if G/N is an abelian π' -group.*

As the reader will see, the proof of Theorem B uses a lot of deep machinery. The proof that we present here is an improvement by M. Isaacs of an earlier version which we reproduce here with his kind permission.

2 Transitive actions

In general, we follow the notation in [6]. If G is a finite group, then $\text{Irr}(G)$ is the set of the irreducible complex characters of G . If $Z \triangleleft G$ and $\lambda \in \text{Irr}(Z)$, then $\text{Irr}(G|\lambda)$ is the set of the irreducible constituents of the induced character λ^G . By Frobenius reciprocity, this is the set of characters $\chi \in \text{Irr}(G)$ such that the restriction χ_Z contains λ .

Lemma 2.1. *Suppose that $Z \triangleleft G$, and let $\lambda \in \text{Irr}(Z)$ be G -invariant. Assume that all characters in $\text{Irr}(G|\lambda)$ have the same degree $d\lambda(1)$. Let $P/Z \in \text{Syl}_p(G/Z)$. Then $d_p\lambda(1)$ is the minimum of $\{\delta(1) \mid \delta \in \text{Irr}(P|\lambda)\}$ and $|\text{Irr}(P|\lambda)| \leq |\text{Irr}(G|\lambda)|_p$.*

Proof. By Character Triple Isomorphisms (see [6, Chapter 11]), we may assume that $\lambda(1) = 1$. Write $\text{Irr}(G|\lambda) = \{\chi_j \mid 1 \leq j \leq s\}$, and observe that the multiplicity of χ_j in λ^G is $\chi_j(1)$. Since by hypothesis, all of the degrees $\chi_j(1)$ are equal, we can write

$$\lambda^G = d \sum_j \chi_j,$$

where $d = \chi_j(1)$ for all j . Also, we have $sd^2 = |G : Z|$. Write

$$\text{Irr}(P|\lambda) = \{\delta_i \mid 1 \leq i \leq t\},$$

and observe that because $\lambda(1) = 1$, we have

$$\lambda^P = \sum_i d_i \delta_i,$$

where $d_i = \delta_i(1)$ and $\sum d_i^2 = |P : Z|$. We can write

$$\delta_i^G = \sum_{j=1}^s d_{ij} \chi_j,$$

and it follows that $\delta_i^G(1)$ is a multiple of the common degree d of the χ_j . Then d divides $|G : P|d_i$, and hence the p -part d_p of d divides d_i for all i . We conclude that d_p divides the greatest common divisor e of the d_i .

We also have that

$$(\chi_j)_P = \sum_{i=1}^t d_{ij} \delta_i$$

by Frobenius reciprocity, and thus e divides $\chi_j(1) = d$. Since e is a p -power, we see that e divides d_p , and thus $e = d_p$. Then we have that

$$|P : Z| = \sum_{i=1}^t d_i^2 \geq e^2 t = (d_p)^2 t.$$

Taking p -parts in $sd^2 = |G : Z|$, we obtain that $s_p \geq t$. □

The following is a character-theoretical version of [2, Theorem 1.2].

Theorem 2.2. *Suppose that $Z \triangleleft G$, $\lambda \in \text{Irr}(Z)$ is G -invariant, p is a prime and $P/Z \in \text{Syl}_p(G/Z)$. Let $\mathcal{A} = \text{Irr}(G|\lambda)$, $\mathcal{B} = \text{Irr}(P|\lambda)$. Suppose that A is a finite group acting on $\mathcal{A}$ and $\mathcal{B}$ in such a way that*

$$[(\chi^a)_P, \delta^a] = [\chi_P, \delta]$$

for all $\chi \in \mathcal{A}$, $\delta \in \mathcal{B}$ and $a \in A$. Assume further that $\chi^a(1) = \chi(1)$ for $\chi \in \mathcal{A}$ and $a \in A$. Let $B \in \text{Syl}_p(A)$. If A acts transitively on $\mathcal{A}$, then B acts transitively on $\mathcal{B}$ and $|\mathcal{A}|_p = |\mathcal{B}|$.

Proof. Write $\mathcal{A} = \{\chi_1, \dots, \chi_s\}$ and $\mathcal{B} = \{\delta_1, \dots, \delta_t\}$. By hypothesis, we have that all the characters in $\mathcal{A}$ have the same degree $d\lambda(1)$. Hence

$$|G : Z| = d^2 s.$$

By Lemma 2.1, we have that $d_p \lambda(1)$ is the minimum of the degrees in $\mathcal{B}$ and that $t \leq s_p$. Write

$$(\chi_i)_P = \sum_{j=1}^t d_{ij} \delta_j$$

so that

$$(\delta_j)^G = \sum_{i=1}^s d_{ij} \chi_i$$

by Frobenius reciprocity. Let B be a Sylow p -subgroup of A . Let δ_j be such that $\delta_j(1) = d_p \lambda(1)$.

Now, let $S = B_{\delta_j}$ be the stabilizer of δ_j in B . We have that S acts on the set $\text{Irr}(G|\delta_j)$ of irreducible constituents of δ_j^G . Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be the set of S -orbits. Let $\psi_i \in \mathcal{O}_i$. We may write

$$(\delta_j)^G = \sum_{k=1}^r b_k \left(\sum_{\xi \in \mathcal{O}_k} \xi \right).$$

Hence

$$|G : P| d_p \lambda(1) = \sum_{k=1}^r b_k |\mathcal{O}_k| \psi_k(1) = d \lambda(1) \sum_{k=1}^r b_k |\mathcal{O}_k|$$

and therefore p does not divide

$$\sum_{k=1}^r b_k |\mathcal{O}_k|.$$

Therefore there is some k such that $b_k |\mathcal{O}_k|$ is not divisible by p . In particular, we see that there is an irreducible constituent ψ_k of $(\delta_j)^G$ that is S -fixed. Hence $S = B_{\delta_j} \subseteq B_{\psi_k} \subseteq B$. Also $B_{\psi_k} \subseteq R \in \text{Syl}_p(A_{\psi_k})$ for some Sylow p -subgroup R of A_{ψ_k} . Since A acts transitively on $\text{Irr}(G|\lambda)$, we have that $s = |A : A_{\psi_k}|$. Thus

$$\begin{aligned} s_p &= \frac{|A|_p}{|A_{\psi_k}|_p} = \frac{|A|_p}{|R|} = \frac{|B|}{|R|} \\ &\leq \frac{|B|}{|B_{\psi_k}|} \leq \frac{|B|}{|B_{\delta_j}|} = |B : B_{\delta_j}| \leq t \leq s_p. \end{aligned}$$

Thus $t = s_p$, $|B : B_{\delta_j}| = t$ and everything follows. $\square$

3 Auxiliary results

Of course, if A acts by automorphisms on G , then A also acts on $\text{Irr}(G)$. If $\chi \in \text{Irr}(G)$ and $a \in A$, then $\chi^a \in \text{Irr}(G)$ is the unique character satisfying that $\chi^a(g^a) = \chi(g)$ for $g \in G$.

Hypotheses 3.1. Suppose that $Z \subseteq N \triangleleft G$, where $Z \triangleleft G$. Let $\lambda \in \text{Irr}(Z)$. Suppose that if $\tau_i \in \text{Irr}(N|\lambda)$ for $i = 1, 2$, then there exists $g \in G$ such that $\tau_1^g = \tau_2$.

We say in this case that (G, N, λ) satisfies Hypothesis 3.1.

If $N \triangleleft G$ and $\tau \in \text{Irr}(N)$, then we denote by $I_G(\tau)$, or by G_τ , the stabilizer of τ in G . Recall that induction defines a bijection

$$\text{Irr}(I_G(\theta)|\theta) \rightarrow \text{Irr}(G|\theta)$$

by the Clifford correspondence [6, Theorem (6.11)].

Lemma 3.2. *Suppose that (G, N, λ) satisfies Hypotheses 3.1. Let $Z \subseteq K \subseteq N$, where $K \triangleleft G$. Then the following hold.*

- (a) *Let $\tau_i \in \text{Irr}(K|\lambda)$ for $i = 1, 2$. Then there exists $g \in G$ such that $\tau_1^g = \tau_2$.*
- (b) *Suppose that $L \triangleleft G$ is contained in K . Let $\epsilon \in \text{Irr}(L)$, and suppose that $\gamma_i \in \text{Irr}(I_K(\epsilon)|\epsilon)$ are such that $(\gamma_i)^K$ lie over λ for $i = 1, 2$. Then there is some $g \in I_G(\epsilon)$ such that $\gamma_1^g = \gamma_2$.*
- (c) *Let $\tau \in \text{Irr}(K|\lambda)$. Let $\gamma_i \in \text{Irr}(I_N(\tau)|\tau)$ for $i = 1, 2$. Then there exists some $g \in I_G(\tau)$ such that $\gamma_1^g = \gamma_2$.*

Proof. (a) Let $\gamma_i \in \text{Irr}(N)$ over τ_i . By hypothesis, we have that $\gamma_1^x = \gamma_2$ for some $x \in G$. We have that τ_1^x and τ_2 are under γ_2 , so by Clifford's theorem there is some $n \in N$ such that $\tau_1^{xn} = \tau_2$. Set $g = xn$.

(b) By part (a), there is some $g \in G$ such that $((\gamma_1)^K)^g = (\gamma_2)^K$. Now, ϵ^g and ϵ are under $(\gamma_2)^K$, so by replacing g by gk for some $k \in K$, we may assume also that $\epsilon^g = \epsilon$. Then $g \in I_G(\epsilon)$. By the uniqueness in the Clifford correspondence, we deduce that $(\gamma_1)^g = \gamma_2$.

(c) By the Clifford correspondence, we have that $(\gamma_i)^N \in \text{Irr}(N)$ lie over λ . By hypotheses, there is some $g \in G$ such that $((\gamma_1)^N)^g = (\gamma_2)^N$. Now, τ^g and τ are N -conjugate by Clifford's theorem, so by replacing g by gn , for some $n \in N$, we may assume that $\tau^g = \tau$. Notice now that $g \in I_G(\tau)$. Also, $\gamma_1^g = \gamma_2$, by the uniqueness in the Clifford correspondence. $\square$

Theorem 3.3. *Assume Hypotheses 3.1, with $Z \subseteq \mathbf{Z}(N)$. Let $U \subseteq N$, with $U \triangleleft G$. Suppose that q is a prime dividing $|U|$. Then q divides $|Z \cap U|$.*

Proof. Let $K = UZ \triangleleft G$. If q does not divide $|K : Z| = |U : U \cap Z|$, we are done. Let $1 \neq Q/Z \in \text{Syl}_q(K/Z)$. By Lemma 3.2 (a), we know that $G_\lambda = I_G(\lambda)$ acts transitively on $\text{Irr}(K|\lambda)$. By the Frattini argument, we have that $G_\lambda = K N_{G_\lambda}(Q)$. Notice then that $A = N_{G_\lambda}(Q)$ acts transitively on $\text{Irr}(K|\lambda)$. Also A acts on $\text{Irr}(Q|\lambda)$ and $[(\chi^a)_Q, \delta^a] = [\chi_Q, \delta]$ for $a \in A$, $\chi \in \text{Irr}(K|\lambda)$ and $\delta \in \text{Irr}(Q|\lambda)$. By Theorem 2.2, we have that A acts transitively on $\text{Irr}(Q|\lambda)$.

Suppose now that q does not divide $|Z \cap U|$. Let $\nu = \lambda_{Z \cap U}$. Then ν has a canonical extension $\hat{\nu} \in \text{Irr}(Q \cap U)$ of q' -order. By [4, Corollary (4.2)], we

know that restriction defines a natural bijection

$$\mathrm{Irr}(Q|\lambda) \rightarrow \mathrm{Irr}(Q \cap U|v).$$

Let $\rho \in \mathrm{Irr}(Q|\lambda)$ be such that $\rho_{Q \cap U} = \hat{v}$. In particular, ρ is linear. Also $\rho_Z = \lambda$. Let $a \in A$. Then a fixes λ , and therefore v . Now, a normalizes Q and U , so a normalizes $U \cap Q$. By uniqueness, we have that $(\hat{v})^a = \hat{v}$. Thus $\rho^a = \rho$ by uniqueness. Since A acts transitively on $\mathrm{Irr}(Q|\lambda)$, it follows that $\mathrm{Irr}(Q|\lambda) = \{\rho\}$. Since $\rho_Z = \lambda$, by Gallagher's Corollary (6.17) of [6], we know that

$$|\mathrm{Irr}(Q|\lambda)| = |\mathrm{Irr}(Q/Z)|.$$

We conclude that $Q = Z$, and this is the final contradiction. $\square$

As in [3], we shall only use the Classification of Finite Simple Groups in the following result. If X is a finite group, recall that $M(X)$ is the Schur multiplier of X .

Theorem 3.4. *Let X be a non-abelian simple group. Then there exists a prime p such that p divides $|X|$, p does not divide $|M(X)|$, and there is no solvable subgroup of X having p -power index.*

Proof. This is [3, Theorem (2.1)]. $\square$

4 The Glauberman correspondence

The idea to use the Glauberman correspondence in the Iwahori–Matsumoto conjecture appears in [3]. As we shall see in the proof of our main theorem, we need to do the same here, in a more sophisticated way. For the definition and properties of the Glauberman correspondence, we refer the reader to [6, Chapter 13].

We remark now the following. If Q is a p -group that acts by automorphisms on a p' -group L , then the Glauberman correspondence is a bijection

$$* : \mathrm{Irr}_Q(L) \rightarrow \mathrm{Irr}(C),$$

where $\mathrm{Irr}_Q(L)$ is the set of Q -invariant irreducible characters of L and $C = \mathbf{C}_L(Q)$ is the fixed point subgroup. Furthermore, for $\chi \in \mathrm{Irr}_Q(L)$, we have that

$$\chi_C = e\chi^* + p\Delta,$$

where p does not divide the integer e and Δ is a character of C (or zero). In particular, we easily check that the Glauberman correspondence $*$ commutes with the action of $\mathrm{Gal}(\mathbb{Q}/\mathbb{Q})$ and with the action of the group of automorphisms of the semidirect product LQ that fix Q . In particular, we have that $\mathbb{Q}(\chi) = \mathbb{Q}(\chi^*)$.

The next deep result is key in character theory. Its proof, in the case where $Z = 1$, is due to E. C. Dade. (Other proofs are due to L. Puig.) The following useful strengthening is due to A. Turull, who we thank for useful conversations on this subject.

Theorem 4.1. *Suppose that G is a finite group, $LQ \triangleleft G$, where $L \triangleleft G$, $(|L|, |Q|)$ is equal to 1, and Q is a p -group for some prime p . Suppose that $LQ \subseteq N \triangleleft G$, and $Z \triangleleft G$, is contained in Q and in $\mathbf{Z}(N)$. Let $\lambda \in \text{Irr}(Z)$. Let $H = \mathbf{N}_G(Q)$ and $C = \mathbf{C}_L(Q)$. Then for every $\tau \in \text{Irr}_Q(L)$ there is a bijection*

$$\pi(N, \tau) : \text{Irr}(N|\tau) \rightarrow \text{Irr}(N \cap H|\tau^*),$$

where $\tau^* \in \text{Irr}(C)$ is the Q -Glauberman correspondent of τ , such that:

(a) For $\gamma \in \text{Irr}(N|\tau)$, $h \in H$ we have that

$$\pi(N, \tau^h)(\gamma^h) = (\pi(N, \tau)(\gamma))^h.$$

(b) $\rho \in \text{Irr}(N|\tau)$ lies over λ if and only if $\pi(N, \tau)(\rho)$ lies over λ .

Proof. It follows from the proofs of [12, Theorem 7.12] and [11, Theorem 6.5]. Specifically, we make $\psi = \theta$ in [12, Theorem 7.12], and G, H, θ in [12, Theorem 7.12], correspond to G, L, τ ; while G', H', θ' correspond to H, C and τ^* , respectively. Now, parts (1) and (2) of [12, Theorem 7.12] predict a bijection

$$' : \bigcup_{x \in H} \text{Irr}(N|\tau^x) \rightarrow \bigcup_{x \in H} \text{Irr}(N \cap H|(\tau^*)^x),$$

which commutes with the action of H (part (7) of [12, Theorem 7.12]). By parts (4), (1) and (2) of the same theorem, writing $R = L$ and $S = N$, we have that $\gamma \in \text{Irr}(N|\tau)$ if and only if $\gamma' \in \text{Irr}(N \cap H|\tau^*)$. We call $\pi(N, \tau)$ the restriction of the map $'$ to $\text{Irr}(N|\tau)$. Part (b) follows from [13, Theorem 10.1]. $\square$

The following is easy.

Lemma 4.2. *Suppose that $LQ \triangleleft G$, where $L \triangleleft G$, $(|L|, |Q|)$ is equal to 1, and Q is a p -group for some prime p . Suppose that $\tau \in \text{Irr}_Q(L)$, and let $\tau^* \in \text{Irr}(C)$ be the Glauberman correspondent, where $C = \mathbf{C}_L(Q)$. Suppose that $Z \triangleleft G$ is contained in C . Let $\lambda \in \text{Irr}(Z)$ be L -invariant. Let $H = \mathbf{N}_G(Q)$. Suppose that*

$$\lambda^L = f(\tau^{h_1} + \cdots + \tau^{h_s})$$

for some $h_i \in H$ and some integer f . Then

$$\lambda^C = f^*((\tau^*)^{h_1} + \cdots + (\tau^*)^{h_s})$$

for some integer f^* .

Proof. We know by [6, Theorem (13.29)] that if $\nu \in \text{Irr}_Q(L)$, then ν^* lies above λ if and only if ν lies above λ . Let $\rho \in \text{Irr}(C|\lambda)$. Then $\rho = \nu^*$ for some $\nu \in \text{Irr}(L|\lambda)$. Thus $\nu = \tau^h$ for some $h \in H$, by hypothesis. Then

$$\rho = \nu^* = (\tau^h)^* = (\tau^*)^h,$$

because H commutes with Glauberman correspondence. Since λ is C -invariant, we easily conclude the proof of the lemma. $\square$

5 Proof of Theorem A

In this section, we prove Theorem A.

Theorem 5.1. *Assume Hypothesis 3.1. Then $I_N(\lambda)/Z$ is solvable.*

Proof. We argue by induction on $|N : Z|$. Let S/Z be the largest solvable normal subgroup of N/Z . Let $T = I_G(\lambda)$ be the stabilizer of λ in G .

Step 1. *We may assume that λ is G -invariant.*

We claim that the triple $(I_G(\lambda), I_N(\lambda), \lambda)$ satisfies Hypothesis 3.1. Indeed, let $\tau_i \in \text{Irr}(I_N(\lambda)|\lambda)$ for $i = 1, 2$.

By Lemma 3.2(c) (with $K = Z$), there is an element $g \in I_G(\lambda)$ such that $(\tau_1)^g = \tau_2$. Hence, by working in $I_G(\lambda)$, we see that it is no loss to assume that λ is invariant in G . Hence, we wish to prove that N/Z is solvable, that is, that $S = N$.

Step 2. *If $Z \leq K < N$, with $K \triangleleft G$, then K/Z is solvable. Also N/S is isomorphic to a direct product of a non-abelian simple group X .*

By Lemma 3.2(a) and induction, we have that if $Z \leq K < N$, with $K \triangleleft G$, then K/Z is solvable. Then N/S is a chief factor of G/Z , and it is isomorphic to a direct product of a non-abelian simple group X .

Step 3. *We may assume that Z is cyclic and that λ is faithful.*

This follows by using character triple isomorphisms.

Step 4. *If $Z < K \subseteq N$ is a normal subgroup of G and $\tau \in \text{Irr}(K|\lambda)$, then $I_N(\tau)/K$ is solvable. Also $S > Z$.*

The first part is a direct consequence of Lemma 3.2(c) and induction. If $S = Z$, then by Step 2, we have that N/Z is a minimal normal non-abelian subgroup of G/Z . Then N/Z is a direct product of non-abelian simple groups isomorphic

to X , and $Z = \mathbf{Z}(N)$. Also, $N'Z = N$. By Theorem 3.4, there is a prime p dividing $|X|$ such that p does not divide $|M(X)|$. By [3, Corollary 7.2], we have that p does not divide $|N' \cap Z|$. Since p divides $|N'|$, this contradicts Theorem 3.3 with $U = N'$.

Step 5. We have $F = \mathbf{F}(N) = S$.

Otherwise, let R/F be a solvable chief factor of G inside N . Thus R/F is a q -group for some prime q . Let L be the Sylow q -complement of F , and let $Z_{q'} = L \cap Z$. Let Q be a Sylow q -subgroup of R , so that $R = LQ$, and let $Z_q = Q \cap Z$, so that $Z = Z_{q'} \times Z_q$. We have that $G = LH$, where $H = \mathbf{N}_G(Q)$, by the Frattini argument. Let $C = \mathbf{C}_L(Q)$.

Write $\lambda = \lambda_{q'} \times \lambda_q$, where $\lambda_{q'} = \lambda_{Z_{q'}}$, and $\lambda_q = \lambda_{Z_q}$. By coprime action and counting, we see that Q fixes some $\tau_{q'} \in \text{Irr}(L|\lambda_{q'})$. Let $\tau = \tau_{q'} \times \lambda_q \in \text{Irr}(LZ)$. By hypothesis and Lemma 3.2 (a), we can write

$$\lambda^{LZ} = f(\tau^{h_1} + \cdots + \tau^{h_s}),$$

where $h_i \in H$, and $\lambda^{h_i} = \lambda$, because λ is G -invariant. Hence

$$\lambda_{q'}^L = f(\tau_{q'}^{h_1} + \cdots + \tau_{q'}^{h_s}).$$

By Lemma 4.2, we have that

$$\lambda_{q'}^C = f^*((\tau_{q'}^*)^{h_1} + \cdots + (\tau_{q'}^*)^{h_s}).$$

By Theorem 4.1, we know that there is a bijection

$$\pi(N, \tau_{q'}) : \text{Irr}(N|\tau_{q'}) \rightarrow \text{Irr}(\mathbf{N}_N(Q)|\tau_{q'}^*)$$

that commutes with H -action.

We claim that $(\mathbf{N}_G(Q), \mathbf{N}_N(Q), \lambda)$ satisfies Hypothesis 3.1. If this is the case, since $\mathbf{N}_N(Q) < N$, we will have $|\mathbf{N}_N(Q) : Z| < |N : Z|$, and by induction, we will conclude that $\mathbf{N}_N(Q)/Z$ is solvable. This implies that N/Z is solvable, and the proof of the theorem would be complete. Suppose now that $\psi_i \in \text{Irr}(\mathbf{N}_N(Q)|\lambda)$ for $i = 1, 2$. We are going to show that there exists $x \in H$ such that $\psi_1^x = \psi_2$. Since ψ_i lies over $\lambda_{q'}$, we have that ψ_1 lies over some $(\tau_{q'}^*)^{h_j}$, and ψ_2 lies over some $(\tau_{q'}^*)^{h_k}$ for some $h_j, h_k \in H$. Conjugating by h_j^{-1} and by h_k^{-1} , we may assume that ψ_1 and ψ_2 lie over $\tau_{q'}^*$.

Now, we know that there exists $\mu_i \in \text{Irr}(N|\tau_{q'})$ such that $\pi(N, \tau_{q'})(\mu_i) = \psi_i$. In fact, since ψ_i lies over λ_q , we have that $\mu_i \in \text{Irr}(N|\lambda_q)$ by Theorem 4.1 (b) (with the role of λ in that theorem being played now here by λ_q), and therefore $\mu_i \in \text{Irr}(N|\tau) \subseteq \text{Irr}(N|\lambda)$. By hypothesis, there is some $h \in H$ such that

$\mu_1^h = \mu_2$. Now, $\tau_{q'}^h$ and $\tau_{q'}$ are below μ_2 , so there is some element $h_1 \in N \cap H$ such that $\tau_{q'}^{hh_1} = \tau_{q'}$. Replacing h by hh_1 , we may assume that $(\tau_{q'})^h = \tau_{q'}$. Now

$$\psi_1^h = \pi(N, \tau_{q'}) (\mu_1)^h = \pi(N, \tau_{q'}^h) (\mu_1^h) = \pi(N, \tau_{q'}) (\mu_2) = \psi_2,$$

as desired. By induction, $N \cap H$ is solvable, so N is solvable. This proves Step 5.

Step 6. *If p divides $|F : Z|$, then N has a solvable subgroup of p -power index. Therefore, so do the simple groups factors in the direct product of N/S .*

Suppose that Q/Z is a non-trivial normal p -subgroup of G/Z , where Q is contained in N . Then the irreducible constituents of λ^Q all have the same degree by Lemma 3.2 (a), for instance. So we can write $\lambda^Q = f(\tau_1 + \cdots + \tau_k)$, where $\tau_i \in \text{Irr}(Q|\lambda)$ are all the different constituents. Write $\tau = \tau_1$. Notice that $f = \tau(1)$. Thus we deduce that k is a power of p . Now, since G acts on $\Omega = \{\tau_1, \dots, \tau_k\}$ transitively by conjugation by Lemma 3.2 (a), we have that $|G : I_G(\tau)| = k$ is a power of p . Hence, $|N : I_N(\tau)|$ is a power of p . If $Q > Z$, then we know by induction that $I_N(\tau)/Q$ is solvable. In this case, we deduce that N has a solvable subgroup with p -power index. The same happens for factors of N .

Step 7. *Final contradiction.*

We know by Step 2 that N/S is isomorphic to a direct product of a non-abelian simple group X . By [3, Theorem 2.1], there exists a prime q dividing $|X|$, such that q does not divide the order of the Schur multiplier of X , and such that no solvable subgroup of X has q -power index. By Step 6, we have that q does not divide $|F : Z|$. Let W be the normal q -complement of F . Hence $F = WZ$. Also $F/W = Z(N/W)$. By [3, Corollary 7.2], we have that q does not divide $|(N/W)' \cap F/W|$. But F/W is a q -group, so $(N/W)' \cap F/W = W/W$. In particular, $N' \cap F \subseteq W$. Thus q does not divide $|N' \cap F|$. Thus q does not divide $|N' \cap Z|$. Since N/F is perfect, we have that $N'F = N$, so that q divides $|N'|$. But this contradicts Theorem 3.3 with $U = N'$. $\square$

Next is Theorem A of the introduction.

Corollary 5.2. *Suppose that $Z \triangleleft G$, and let $\lambda \in \text{Irr}(Z)$. Assume that if we have $\chi, \psi \in \text{Irr}(G|\lambda)$, then there exists $a \in \text{Aut}(G)$ stabilizing Z such that $\chi^a = \psi$. If T is the stabilizer of λ in G , then T/Z is solvable.*

Proof. Let $A = \text{Aut}(G)_Z$ be the group of automorphisms of G that stabilize Z . Let $\Gamma = GA$ be the semidirect product. We have $Z \triangleleft \Gamma$. By hypothesis, (Γ, G, λ) satisfies Hypothesis 3.1. By Theorem 5.1, we have that T/Z is solvable. $\square$

6 Proof of Theorem B

We begin by giving another proof of a result of U. Riese [10] that we shall need later.

Lemma 6.1. *Let $H \subseteq G$ and $\alpha \in \text{Irr}(H)$. Suppose that $\alpha^G = \chi \in \text{Irr}(G)$ and that every irreducible constituent of χ_H has degree equal to $\alpha(1)$. Then χ vanishes on $G - H$.*

Proof. By hypothesis, χ_H is the sum of $\chi(1)/\alpha(1) = |G : H|$ irreducible characters, and thus $[\chi_H, \chi_H] \geq |G : H|$. Then $|H|[\chi_H, \chi_H] \geq |G|[\chi, \chi]$, and so χ vanishes on $G - H$, as claimed. $\square$

Next is the proof of Riese's theorem (by M. Isaacs).

Theorem 6.2. *Let $A \subseteq G$, where A is abelian, and assume that λ^G is irreducible, where $\lambda \in \text{Irr}(A)$. Then $A \triangleleft\triangleleft G$.*

Proof. We prove the theorem by induction on $|G|$. Write $\chi = \lambda^G \in \text{Irr}(G)$, and let V be the subgroup of G generated by the elements $g \in G$ with $\chi(g) \neq 0$. Since A is abelian, each irreducible constituent of χ_A has degree 1 $= \lambda(1)$, and thus by Lemma 6.1, we have $V \subseteq A$. Also, writing $Z = \mathbf{Z}(G)$, we have $Z \subseteq V$.

If $A \subseteq H < G$, then since λ^H is irreducible, the inductive hypothesis yields $A \triangleleft\triangleleft H$. Assuming that A is not subnormal in G , then Wielandt's zipper lemma [7, Theorem 2.9] guarantees that there is a unique maximal subgroup M of G with $A \subseteq M$. Also, if the normal closure $A^G < G$, then $A \triangleleft\triangleleft A^G \triangleleft G$, and we are done. We can thus suppose that $A^G = G$, and so $A^g \not\subseteq M$ for some element $g \in G$. By the uniqueness of M , therefore, we have $\langle A, A^g \rangle = G$. But $V \triangleleft G$ and $V \subseteq A$, and thus $V \subseteq A \cap A^g \subseteq Z$, and we have $V = Z = A \cap A^g$. Thus χ vanishes off Z , and so χ is fully ramified with respect to Z . In particular, $|G : A|^2 = \chi(1)^2 = |G : Z|$, and we have $|G : A| = |A : Z|$. Thus $|G : A|$ equals $|A^g : A^g \cap A|$, and it follows that $AA^g = G$. This implies that $A = A^g$, and thus $A = G$. This is a contradiction since A was assumed to be not subnormal. $\square$

Corollary 6.3. *Let $N \triangleleft G$ and let $\theta \in \text{Irr}(N)$ be G -invariant. Let $N \subseteq A \subseteq G$, where A/N is abelian, and suppose that θ has an extension $\phi \in \text{Irr}(A)$ such that ϕ^G is irreducible. Then $A \triangleleft\triangleleft G$.*

Proof. By character triple isomorphisms (see [6, Chapter 11]), we can assume that θ is linear and faithful. Then ϕ is linear and $A' \subseteq N \cap \ker(\phi) = \ker(\theta) = 1$. Then A is abelian, and since ϕ^G is irreducible, Theorem 6.2 yields the result. $\square$

Now, we prove an extension of Theorem B. We should mention that the Classification of Finite Simple Groups is implicitly used in the proof. Specifically, in the Howlett–Isaacs theorem on central type groups [3].

Theorem 6.4. *Let $N \triangleleft G$. Suppose that $\theta \in \text{Irr}(N)$ is G -invariant and $o(\theta)\theta(1)$ is a π -number. Assume that G/N is π -separable and that $\mathbf{O}_\pi(G/N) = 1$. Then all members of $\text{Irr}(G|\theta)$ have equal degrees if and only if G/N is an abelian π' -group.*

Proof. If G/N is an abelian π' -group, then θ extends to G by [6, Corollary 8.16], and we are done by Gallagher’s Corollary 6.17 of [6]. To prove the converse, we argue by induction on $|G/N|$ and assume that $|G/N| > 1$. We argue first that the common degree d of the characters in $\text{Irr}(G|\theta)$ is a π -number. To see this, let $q \in \pi'$ and let $Q/N \in \text{Syl}_q(G/N)$. Then θ extends to Q , and the induction to G of such an extension has degree $\theta(1)|G : Q|$, which is a q' -number. Since this degree is a multiple of d , it follows that d is a q' -number, and since $q \in \pi'$ was arbitrary, we see that d is a π -number.

Let $U/N = \mathbf{O}_{\pi'}(G/N)$ and note that $U > N$. All degrees of characters in $\text{Irr}(U|\theta)$ divide d , and so are π -numbers. But since U/N is a π' -number, it follows that all degrees of characters in $\text{Irr}(U|\theta)$ equal $\theta(1)$, and so all of these characters extend θ . It follows that U/N is abelian by Gallagher’s Corollary 6.17 of [6]. If $U = G$, we are done, and so we suppose that $U < G$ and we let $V/U = \mathbf{O}_\pi(G/U)$. Note that $V > U$. By [6, Corollary 8.16], there exists a unique extension $\hat{\theta} \in \text{Irr}(U)$ of θ with determinantal π -order. By uniqueness, $\hat{\theta}$ is G -invariant. Now, let $\phi \in \text{Irr}(V|\hat{\theta})$. Since V/U is a π -group, ϕ_U is a multiple of $\hat{\theta}$ and $o(\hat{\theta})$ is a π -number, we easily have that $o(\phi)$ is a π -number. Write $T = G_\phi$ for the stabilizer of ϕ in G . Then all members of $\text{Irr}(T|\phi)$ induce irreducibly to G , yielding characters of degree d , and thus these characters all have degree $d/|G : T|$. We claim that T satisfies the hypotheses of the theorem with respect to the character ϕ and the normal subgroup V . To see this, we need to check that $\mathbf{O}_\pi(T/V)$ is trivial.

Let $W/V = \mathbf{O}_{\pi'}(G/V)$. We argue that W stabilizes ϕ . This is because the G/V -orbit of ϕ has size dividing d , and so is a π -number, and W/V is a normal π' -subgroup of G/V . Thus $W \subseteq T$ and $\mathbf{O}_\pi(T/V)$ centralizes the normal π' -subgroup $W/V = \mathbf{O}_{\pi'}(G/V)$. But $\mathbf{O}_\pi(G/V)$ is trivial, and Hall–Higman’s Lemma 1.2.3 applies to show that $\mathbf{O}_\pi(T/V) = 1$, as wanted.

By the inductive hypothesis, we conclude that T/V is a π' -group. Also, by the Clifford correspondence, $|G : T|$ divides d , which we know is a π -number. Thus T/V is a full Hall π' -subgroup of G/V . Also, ϕ extends to T , and so $\phi(1) = d/|G : T| = d/|G/V|_\pi$ is constant for $\phi \in \text{Irr}(V|\theta)$. It follows that the

hypotheses are satisfied in the group V with respect to θ . If $V < G$, the inductive hypothesis yields that V/N is a π' -group, and this is a contradiction.

It follows that $V = G$ and G/U is a π -group. Also, G/U acts faithfully on U/N because $\mathbf{O}_\pi(G/N)$ is trivial. Now let $\lambda \in \text{Irr}(U/N)$, so that λ is linear. Let $S = G_\lambda$, and note that λ extends to S since S/U is a π -group. Write $a = |G : S|$.

Note that S is the stabilizer of $\lambda\hat{\theta}$ in G , and thus all characters in $\text{Irr}(S|\lambda\hat{\theta})$ have degree d/a . If r is the number of such characters, this yields

$$r(d/a)^2 = |S : U|\theta(1)^2.$$

Also, since λ extends to S , by [6, Theorem 6.16] there is a degree-preserving bijection between $\text{Irr}(S|\lambda\hat{\theta})$ and $\text{Irr}(S|\hat{\theta})$, and hence the latter set contains exactly r characters, and each has degree d/a . Each of these must therefore induce irreducibly to G , and it follows that each member of $\text{Irr}(G|\hat{\theta})$ is induced from a member of $\text{Irr}(S|\hat{\theta})$.

Note that the number of different members of $\text{Irr}(S|\hat{\theta})$ that can have the same induction to G is at most $|G : S| = a$.

Now let $t = |\text{Irr}(G|\hat{\theta})|$ so that $td^2 = |G : U|\theta(1)$. If we divide this equation by our previous one, we get $ta^2/r = |G : S| = a$, and so $t = r/a$. It follows that each of the t members of $\text{Irr}(G|\hat{\theta})$ is induced from exactly a characters in $\text{Irr}(S|\hat{\theta})$. In other words, if $\chi \in \text{Irr}(G|\hat{\theta})$, then χ_S has exactly a distinct irreducible constituents, each with degree d/a , and so by Lemma 6.1, it follows that χ vanishes on $G - S$. In other words, the only elements of G on which χ can have a nonzero value lie in the stabilizer of λ for every linear character λ of U/N . But G/U acts faithfully on this set of linear characters, and thus χ vanishes on $G - U$. In other words, $\hat{\theta}$ is fully ramified in G . It follows that $d = \theta(1)|G : U|^{1/2}$.

Also, $a\theta(1)$ divides d , and so a must divide $|G : U|^{1/2}$. Write $s = |S : U|$, so that $as = |G : U|$. Then a^2 divides as , and thus a divides s . In particular, we have $a \leq s$, so $|G : S| \leq |S : U|$. Thus

$$|G : U| = |G : S||S : U| \leq |S : U|^2.$$

Now, by the Howlett–Isaacs theorem we have that G/U is solvable. This group acts faithfully on the group of linear characters of U/N , and so by the main result in [1], there exist character stabilizers T and R such that $T \cap R = U$. By the result of the previous paragraph, each of T/U and R/U has order at least $|G : U|^{1/2}$. Now

$$|G : U| = |G : T||T : U| \geq |R : U||T : U| \geq |G : U|.$$

Then $TR = G$, and that each of $|T : U|$ and $|R : U|$ has order $|G : U|^{1/2}$. Therefore all characters in $\text{Irr}(T|\hat{\theta})$ are extensions of $\hat{\theta}$ and induce irreducibly to G . In particular, T/U is abelian, and similarly R/U is abelian.

By Corollary 6.3, it follows that $R \triangleleft G$, and since R/U is abelian, we have $R/U \subseteq \mathbf{F}(G/U)$. Similarly, $T/U \subseteq \mathbf{F}(G/U)$ and thus G/U is nilpotent. But then, since G/U acts faithfully on the group of linear characters of U/N , it follows that if G/U is nontrivial, then some character $\lambda \in \text{Irr}(U/N)$ has a stabilizer S in G such that

$$|S : U| < |G : U|^{1/2}$$

by [5, Theorem B]. But then

$$|G : U| = |G : S||S : U| \leq |S : U|^2 < |G : U|.$$

This contradiction completes the proof. $\square$

Acknowledgments. We thank the referee for a very careful reading of this manuscript.

Bibliography

- [1] S. Dolfi, Large orbits in coprime actions of solvable groups, *Trans. Amer. Math. Soc.* **360** (2008), no. 1, 135–152.
- [2] R. J. Higgs, Groups which act transitively on the projective characters of a normal subgroup, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), no. 3, 429–434.
- [3] R. B. Howlett and I. M. Isaacs, On groups of central type, *Math. Z.* **179** (1982), 555–569.
- [4] I. M. Isaacs, Characters of π -separable groups, *J. Algebra* **86** (1984), 98–128.
- [5] I. M. Isaacs, Large orbits in actions of nilpotent groups, *Proc. Amer. Math. Soc.* **127** (1999), 45–50.
- [6] I. M. Isaacs, *Character Theory of Finite Groups*, AMS-Chelsea, Providence, 2006.
- [7] I. M. Isaacs, *Finite Group Theory*, Grad. Stud. Math. 92, American Mathematical Society, Providence, 2008.
- [8] R. Liebler and J. Yellen, In search of nonsolvable groups of central type, *Pacific J. Math.* **82** (1979), 485–492.
- [9] G. Navarro, Problems in character theory, in: *Character theory of finite groups*, Contemp. Math. 524, American Mathematical Society, Providence (2010), 97–125.
- [10] U. Riese, A subnormality criterion in finite groups related to character degrees, *J. Algebra* **201** (1998), 357–362.
- [11] A. Turull, Above the Glauberman correspondence, *Adv. Math.* **217** (2008), no. 5, 2170–2205.
- [12] A. Turull, The Brauer–Clifford group, *J. Algebra* **321** (2009), no. 12, 3620–3642.
- [13] A. Turull, Endoisomorphisms and character triple isomorphisms, preprint (2016).

Received June 8, 2016; revised August 4, 2016.

Author information

Gabriel Navarro, Departament de Matemàtiques, Universitat de València,
46100 Burjassot, València, Spain.

E-mail: gabriel.navarro@uv.es

Noelia Rizo, Departament de Matemàtiques, Universitat de València,
46100 Burjassot, València, Spain.

E-mail: noelia.rizo@uv.es