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Expansive automorphisms of totally disconnected, locally compact groups

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Abstract. We study topological automorphisms α of a totally disconnected, locally compact group G which are *expansive* in the sense that

$$\bigcap_{n\in\mathbb{Z}} \alpha^n(U) = \{1\}$$

for some identity neighbourhood $U \subseteq G$. Notably, we prove that the automorphism induced by an expansive automorphism α on a quotient group G/N modulo an α-stable closed normal subgroup N is always expansive. Further results involve the contraction groups

$$U_\alpha := \{g \in G : \alpha^n(g) \to 1 \text{ as } n \to \infty\}.$$

If α is expansive, then $U_\alpha U_{\alpha^{-1}}$ is an open identity neighbourhood in G. We give examples where $U_\alpha U_{\alpha^{-1}}$ fails to be a subgroup. However, $U_\alpha U_{\alpha^{-1}}$ is an α-stable, nilpotent open subgroup of G if G is a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. Further results are devoted to the divisible and torsion parts of U_α, and to the so-called "nub" $\mathrm{nub}(\alpha) = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}}$ of an expansive automorphism.

Introduction and statement of results

We consider automorphisms $\alpha\colon G \to G$ (thus α is a group automorphism such that both α and α^{-1} are continuous) of a totally disconnected locally compact topological group G which are *expansive* in the sense that

$$\bigcap_{n\in\mathbb{Z}} \alpha^n(U) = \{1\}$$

for some identity neighbourhood $U \subseteq G$. Expansive automorphisms of totally disconnected, *compact* groups were studied in [14, 22] and recently in [31]. The importance of expansive automorphisms for the theory of general automorphisms is highlighted by the fact that every automorphism α of a totally disconnected

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compact group G is a projective limit

$$\alpha = \varprojlim \alpha_j$$

of expansive automorphisms α_j of certain Hausdorff quotient groups G/N_j of G such that $G = \varprojlim G/N_j$ (see [31]).

Our goal is to improve the understanding in the case of non-compact groups. Special cases of expansive automorphisms are automorphisms $\alpha\colon G \to G$ which are *contractive* in the sense that $\alpha^n(g) \to 1$ as $n \to \infty$ for each $g \in G$ (see Remark 1.10). The structure of totally disconnected, locally compact groups admitting contractive automorphisms was elucidated in [10] (building on earlier works like [24] and [26]), and the results obtained there can also be used as tools in the investigation of expansive automorphisms (as we shall see). Further tools come from the structure theory of totally disconnected, locally compact groups ([29, 30]), in which contractive automorphisms play an important role (as worked out in [1]).

Note that every automorphism α of a discrete group G (e.g., $\alpha = \mathrm{id}_G$) is expansive (as we may choose $U = \{1\}$ then). Therefore discrete groups and their automorphisms are part of the theory of expansive automorphisms. As a consequence, groups permitting expansive automorphisms need not have any particular algebraic properties. However, there are topological implications: a totally disconnected, locally compact group admitting an expansive automorphism α is always metrizable (cf. [15]) and has a second countable, α-stable open subgroup (Lemma 1.1 (a)).

Our first main result generalizes [31, Proposition 6.1] (devoted to the case of compact groups). As usual, a subset $H \subseteq G$ is called *α-stable* if $\alpha(H) = H$.

Theorem A. *Let $\alpha\colon G \to G$ be an automorphism of a totally disconnected, locally compact group, let $N \subseteq G$ be an α-stable closed normal subgroup and let $\bar{\alpha}\colon G/N \to G/N$, $gN \mapsto \alpha(g)N$ be the automorphism of G/N induced by α. Then α is expansive if and only if both $\alpha|_N$ and $\bar{\alpha}$ are expansive.*

With a view towards our next result, recall that a topological group G is called *topologically perfect* if its commutator group $[G, G]$ is dense in G.

Composition series play a central role in the study of contractive automorphisms [10]. In the case of expansive automorphisms, composition series need not exist. However, certain substitutes are available.

Theorem B. *If $\alpha\colon G \to G$ is an expansive automorphism of a totally disconnected, locally compact group G, then there exist α-stable closed subgroups*

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{1\}$$

of G such that G_j is normal in G_{j-1} for $j \in \{1, \ldots, n\}$ and every α_j-stable *closed normal subgroup of G_{j-1}/G_j is discrete or open, where α_j denotes the induced automorphism $G_{j-1}/G_j \to G_{j-1}/G_j$, $gG_j \mapsto \alpha(g)G_j$. Moreover, one can achieve this in such a way that each of the quotient groups G_{j-1}/G_j is discrete, abelian or topologically perfect.*

In addition, one may assume that all abelian, non-discrete factors G_{j-1}/G_j are simple contraction groups with respect to the automorphism α_j or its inverse, or isomorphic to an infinite power $C_p^{\mathbb{Z}}$ of a cyclic group of prime order, endowed with the right-shift (cf. Remark 6.1 and Proposition 6.2). For second countable groups, more detailed information on the perfect factors is available (see Remark 4.2). The proofs of Theorem A and Theorem B hinge on the fact that there is a bound on the number of non-discrete factors in series for (G, α) (see Proposition 2.7).

According to [30], a compact open subgroup $V \subseteq G$ is called *tidy for* α if it has the following properties:

(T1) $V = V_+V_-$, where $V_+ := \bigcap_{n=0}^{\infty} \alpha^n(V)$ and $V_- := \bigcap_{n=0}^{\infty} \alpha^{-n}(V)$,

(T2) The α-stable subgroups $V_{++} := \bigcup_{n\in\mathbb{N}_0} \alpha^n(V_+)$, $V_{--} := \bigcup_{n\in\mathbb{N}_0} \alpha^{-n}(V_-)$ are closed in G.

Note that

$$V_+ \subseteq \alpha(V_+) \subseteq \alpha^2(V_+) \subseteq \cdots$$

and

$$V_- \subseteq \alpha^{-1}(V_-) \subseteq \alpha^{-2}(V_-) \subseteq \cdots$$

here. The index

$$s(\alpha) := [\alpha(V_+) : V_+] \in \mathbb{N}$$

is called the *scale of* α; it is independent of the choice of the tidy subgroup V (see [30]). Following [31], the intersection nub(α) of all subgroups V which are tidy for α is called the *nub* of α; it is a compact, α-stable subgroup of G.

If $\alpha: G \to G$ is an automorphism of a totally disconnected, locally compact group G, then

$$U_\alpha := \{g \in G : \alpha^n(g) \to 1 \text{ as } n \to \infty\}$$

is a subgroup of G called the associated *contraction group*. In general, U_α need not be closed. However, if α is expansive, then the topology on U_α *can be made locally compact*, i.e., it can be refined to a totally disconnected, locally compact group topology τ^* with respect to which $\alpha|_{U_\alpha}$ is contractive (see [25, Proposition 9] for this fact, or our Lemma 2.3). In this way, the structure theory of locally compact contraction groups (see [10, 24, 26]) becomes available. In particular, the set

$$T_\alpha := \mathrm{tor}(U_\alpha)$$

of all torsion elements in the group U_α and the set

$$D_\alpha := \operatorname{div}(U_\alpha)$$

of divisible elements are α-stable closed subgroups of (U_α, τ^*), and

$$(U_\alpha, \tau^*) = D_\alpha \times T_\alpha$$

internally as a topological group, if we endow D_α and T_α with the topology induced by (U_α, τ^*) (see [10, Theorem B]).

Recall that the closure $\overline{U_\alpha}$ of the contraction group U_α in G plays a role in the structure theory of totally disconnected, locally compact groups; for example, the scale $s(\alpha^{-1})$ can be calculated as the module of the restriction of α^{-1} to $\overline{U_\alpha}$ (see [1, Proposition 3.21 (3)]). Our next theorem provides information on $\overline{U_\alpha}$, and on the divisible part D_α of U_α.

Theorem C. *Let G be a totally disconnected, locally compact group and let $\alpha: G \to G$ be an automorphism such that U_α can be made locally compact (for example, any expansive automorphism). Then $\overline{U_\alpha} = D_\alpha \times \overline{T_\alpha}$ (internally) as a topological group, and $\overline{T_\alpha} = T_\alpha \operatorname{nub}(\alpha)$. In particular, D_α is an α-stable closed subgroup of G, and both the closure $\overline{T_\alpha}$ of T_α in G and* nub(α) *centralize D_α.*

We mention that the nub of an expansive automorphism α need not have an open normalizer in G (see Remark 5.2), in which case not both of U_α and $U_{\alpha^{-1}}$ normalize nub(α).

Classes of examples are also considered. An analytic automorphism α of a Lie group G over a totally disconnected local field $\mathbb{K}$ is expansive if and only if the associated Lie algebra automorphism $L(\alpha)$ is expansive, which means that none of its eigenvalues in an algebraic closure has absolute value 1. The Lie algebra $L(G)$ of G is then nilpotent (see Proposition 7.1). In the case of p-adic Lie groups for a prime number p, we obtain:

Theorem D. *Let G be a p-adic Lie group which is* linear *in the sense that there exists an injective continuous homomorphism $G \to \mathrm{GL}_n(\mathbb{Q}_p)$ for some $n \in \mathbb{N}$. Let $\alpha: G \to G$ be an expansive automorphism. Then G has an open α-stable subgroup which is nilpotent.*

If $\alpha: G \to G$ is an expansive automorphism of a totally disconnected, locally compact group G, then $U_\alpha U_{\alpha^{-1}}$ is an open subset of G (see Proposition 1.1). This is essential for our studies; for instance, it allows finiteness properties of locally compact contraction groups (viz. bounds for the length of series of α-stable closed subgroups) to be exploited in the proof of Theorem B.

In many examples, $U_\alpha U_{\alpha^{-1}}$ happens to be a subgroup of G (for instance, for all expansive automorphisms of closed subgroups $G \subseteq \mathrm{GL}_n(\mathbb{Q}_p)$, see Proposition 7.8). However, this is not always so, as can be seen from Remark 7.7. We also consider the localized completion $G_{p,q}$ of a Baumslag–Solitar group

$$\mathrm{BS}(p,q) = \langle a, t \mid ta^p t^{-1} = a^q \rangle$$

with primes $p \neq q$, as recently studied in [4]. Then $\mathrm{BS}(p,q) \subseteq G_{p,q}$. We show the following theorem.

Theorem E. *Let $\alpha\colon G_{p,q} \to G_{p,q}$ be the conjugation by t. Then α is expansive but $U_\alpha U_{\alpha^{-1}}$ is not a subgroup of $G_{p,q}$.*

1 Preliminaries and basic facts

We write $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z} := \mathbb{N}_0 \cup -\mathbb{N}$. If J is a finite set, we let $\#J$ be its cardinality. We write $X \subseteq Y$ for inclusion of sets, while $X \subset Y$ means that X is a proper subset of Y. As usual, we write $N \lhd G$ if N is a normal subgroup of G. All topological groups considered in this article are assumed Hausdorff, and locally compact topological groups are simply called locally compact groups. Totally disconnected, locally compact non-discrete topological fields (like the field of p-adic numbers) will be called local fields (see [27] for further information). See [3] and [23] for basic information on Lie groups over local (and more general complete ultrametric) fields (which we always assume finite-dimensional). If we say that α is an automorphism of a topological group, then we assume that both α and α^{-1} are continuous; similarly, both α and α^{-1} are assumed analytic if α is an automorphism of an analytic Lie group over a local field. We write $\mathrm{Aut}(G)$ for the group of all automorphisms of a topological group G. If a subgroup $N \subseteq G$ is stable under all $\alpha \in \mathrm{Aut}(G)$, then N is called *topologically characteristic*. A topological group G is called *topologically perfect* if its commutator group $[G,G]$ is dense in G. If F is a finite group and X a set, we write $F^X := \prod_{x \in X} F$ for the direct power endowed with the compact product topology. By contrast, $F^{(X)} \subseteq F^X$ is the subgroup of all $(g_x)_{x \in X} \in F^X$ such that $g_x = 1$ for all but finitely many $x \in X$. We shall always endow $F^{(X)}$ with the discrete topology. Surjective, open, continuous homomorphisms between topological groups are called *quotient morphisms*. A topological space X is called *σ-compact* if it is a union $X = \bigcup_{n \in \mathbb{N}} K_n$ of a sequence of compact sets $K_n \subseteq X$.

If $\alpha\colon G \to G$ is an automorphism of a locally compact group, choose a Haar measure λ on G and denote the module of α by $\Delta_G(\alpha)$; thus

$$\Delta_G(\alpha) = \lambda(\alpha(K))/\lambda(K)$$

for any compact subset $K \subseteq G$ with non-empty interior. If $\alpha\colon G \to G$ is an automorphism of a totally disconnected, locally compact group, we define $\mathrm{nub}(\alpha)$, the contraction group U_α and its subgroups T_α and D_α as explained in the introduction. If $V \subseteq G$ is a compact open subgroup, we shall use the subgroups V_+, V_{++}, V_- and V_{--} defined there, and abbreviate

$$V_0 := V_+ \cap V_- = \bigcap_{n\in\mathbb{Z}} \alpha^n(V). \tag{1.1}$$

We shall also need the so-called *Levi factor*

$$M_\alpha := \{g \in G : \{\alpha^n(g) : n \in \mathbb{Z}\} \text{ is relatively compact}\};$$

it is known that M_α is an α-stable closed subgroup of G (see [1, p. 224]). The following lemma compiles basic facts concerning expansive automorphisms.

Lemma 1.1. *If α is an expansive automorphism of a totally disconnected, locally compact group G, then the following holds:*

(a) *G is metrizable and has an α-stable, σ-compact open subgroup,*

(b) *$V_{--} = U_\alpha$ and $V_{++} = U_{\alpha^{-1}}$ for each compact open subgroup $V \subseteq G$ such that $V_0 = \{1\}$,*

(c) *G has a compact open subgroup V such that $V = V_+V_-$ and $V_0 = \{1\}$,*

(d) *$U_\alpha U_{\alpha^{-1}}$ is open in G,*

(e) *$\alpha|_H$ is expansive, for each α-stable subgroup $H \subseteq G$.*

Proof. (a) Since α is expansive, there exists an identity neighbourhood V such that $\bigcap_{n\in\mathbb{Z}} \alpha^n(V) = \{1\}$. After replacing V with a smaller compact identity neighbourhood if necessary, we may assume that V is compact. Since V is compact and $(\bigcap_{k=-n}^{n} \alpha^k(V))_{n\in\mathbb{N}}$ a decreasing sequence of closed identity neighbourhoods in V with intersection $\{1\}$, the members of the sequence form a basis of identity neighbourhoods. Hence G is metrizable. The subgroup of G generated by $\bigcup_{n\in\mathbb{Z}} \alpha^n(V)$ is α-stable, open and σ-compact.

(b) By [1, Proposition 3.16], we have $V_{--} = U_\alpha V_0$ and thus $V_{--} = U_\alpha$. Likewise, $V_{++} = U_{\alpha^{-1}}$.

(c) and (d) Using expansiveness and van Dantzig's Theorem [11, Theorem 7.7], we find a compact open subgroup $W \subseteq G$ such that

$$\bigcap_{n\in\mathbb{Z}} \alpha^n(W) = \{1\}.$$

By [29, Lemma 1], there exists $m \in \mathbb{N}$ such that $V := \bigcap_{k=1}^{m} \alpha^k(W)$ satisfies $V = V_+V_-$. Then we have $V_0 \subseteq W_0 = \{1\}$ and thus $V_0 = \{1\}$, proving (c). The latter entails $U_\alpha = V_{--}$ and $U_{\alpha^{-1}} = V_{++}$, by (b). In particular, $V_- \subseteq U_\alpha$ and $V_+ \subseteq U_{\alpha^{-1}}$, entailing that $V = V_+V_- \subseteq U_\alpha U_{\alpha^{-1}}$. Thus $U_\alpha U_{\alpha^{-1}}$ is an identity neighbourhood. Given $g \in U_\alpha$ and $h \in U_{\alpha^{-1}}$, the map $G \to G$ given by $x \mapsto gxh$ is a homeomorphism which takes $U_\alpha U_{\alpha^{-1}}$ onto itself and 1 to gh. Hence $U_\alpha U_{\alpha^{-1}}$ has gh in its interior and thus $U_\alpha U_{\alpha^{-1}}$ is open.

(e) If $V \subseteq G$ is an identity neighbourhood with $\bigcap_{n\in\mathbb{Z}} \alpha^n(V) = \{1\}$, then $V \cap H$ is an identity neighbourhood in H and $\bigcap_{n\in\mathbb{Z}} \alpha^n(V \cap H) = \{1\}$. □

The first statement of Lemma 1.1 (a) also follows from [15, Lemma 2.4].

1.2. Let α be an automorphism of a totally disconnected, locally compact group G. The following facts are useful:

(a) The closure of U_α in G is $\overline{U_\alpha} = U_\alpha \operatorname{nub}(\alpha)$ (see [1, Corollary 3.30] if G is metrizable; the general case follows with [13]).

(b) $\operatorname{nub}(\alpha) = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}}$ (see [1, Corollary 3.27] if G is metrizable; the general case follows with [13]).

(c) Both $\operatorname{nub}(\alpha) \cap U_\alpha$ and $\operatorname{nub}(\alpha) \cap U_{\alpha^{-1}}$ are dense in $\operatorname{nub}(\alpha)$ (see [31, Theorem 4.1 (v) and Proposition 5.4 (i)]).

(d) If $N \subseteq G$ is a closed α-stable normal subgroup, $q\colon G \to G/N$ the canonical quotient morphism and $\bar{\alpha}$ the automorphism of G/N induced by α, then $q(U_\alpha) = U_{\bar{\alpha}}$ (see [1, Theorem 3.8] if G is metrizable, [13] in the general case).

(e) U_α is closed if and only if $U_{\alpha^{-1}}$ is closed, if and only if G has small subgroups tidy for α, i.e., every identity neighbourhood of G contains some tidy subgroup (see [1, Theorem 3.32] if G is metrizable; the general case can be deduced using the techniques from [13]).

(f) The so-called *parabolic subgroup*

$$P_\alpha := \{g \in G : \{\alpha^n(g) : n \in \mathbb{N}_0\} \text{ is relatively compact}\}$$

normalizes U_α (see [1, Proposition 3.4]). Hence also its subgroups $M_\alpha = P_\alpha \cap P_{\alpha^{-1}}$ and $\operatorname{nub}(\alpha) \subseteq M_\alpha$ normalize U_α.

The following results help to show that certain automorphisms are expansive.

Proposition 1.3. *Let α be an automorphism of a totally disconnected, locally compact group G. Then the following holds:*

(a) *α is expansive if and only if its restriction $\alpha|_{M_\alpha}$ to the Levi factor is expansive.*

(b) *If U_α is closed, then α is expansive if and only if $U_\alpha U_{\alpha^{-1}}$ is open in G, if and only if M_α is discrete.*

Proof. (a) In view of Lemma 1.1 (e), we only need to show that if $\alpha|_{M_\alpha}$ is expansive, then so is α. Let $P \subseteq M_\alpha$ be a compact, open identity neighbourhood such that $\bigcap_{n\in\mathbb{Z}} \alpha^n(P) = \{1\}$. There is a compact identity neighbourhood $Q \subseteq G$ such that $Q \cap M_\alpha = P$. If $g \in I := \bigcap_{n\in\mathbb{Z}} \alpha^n(Q)$, then $\alpha^n(g) \in Q$ for each $n \in \mathbb{Z}$, whence $\alpha^{\mathbb{Z}}(g)$ is relatively compact and thus $g \in M_\alpha$. Hence $I \subseteq M_\alpha$. Since I is α-stable and $I \subseteq Q \cap M_\alpha = P$, we deduce that

$$I = \bigcap_{n\in\mathbb{Z}} \alpha^n(I) \subseteq \bigcap_{n\in\mathbb{Z}} \alpha^n(P) = \{1\}.$$

Thus $\bigcap_{n\in\mathbb{Z}} \alpha^n(Q) = \{1\}$ and thus α is expansive.

(b) If α is expansive, then $U_\alpha U_{\alpha^{-1}}$ is open by Lemma 1.1 (d). If U_α is closed, then the set $U_\alpha M_\alpha U_{\alpha^{-1}}$ is open in G and the product map

$$U_\alpha \times M_\alpha \times U_{\alpha^{-1}} \to U_\alpha M_\alpha U_{\alpha^{-1}}, \quad (x, y, z) \mapsto xyz$$

is a homeomorphism (by 1.2 (e) above and part (f) from the theorem in [5]). Hence $U_\alpha U_{\alpha^{-1}} \cap M_\alpha = \{1\}$. If $U_\alpha U_{\alpha^{-1}}$ is open, this implies that M_α is discrete. If M_α is discrete, then $\alpha|_{M_\alpha}$ is expansive and hence also α, by (a). □

1.4. If α is an automorphism of a totally disconnected, *compact* group G, then U_α and $U_{\alpha^{-1}}$ are normal in G (since $G = M_\alpha = M_{\alpha^{-1}}$ in 1.2 (f)), and hence so is $\mathrm{nub}(\alpha) = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}}$.

1.5. Recall that a group G is called a torsion group *of finite exponent* if there exists $m \in \mathbb{N}$ such that $g^m = 1$ for all $g \in G$. If such a group is a subgroup of a topological group H, then also $g^m = 1$ for all g in the closure $\overline{G}$ of G in H, and thus also $\overline{G}$ is a torsion group of finite exponent.

1.6. Let α be an expansive automorphism of a totally disconnected, compact group G. Then the following holds:

(a) $\mathrm{nub}(\alpha)$ is open in G (see [31, Lemma 5.1]), whence also $\overline{U_\alpha}$ and $\overline{U_{\alpha^{-1}}}$ are open in G (by 1.2 (b)).

(b) G is a torsion group of finite exponent. [By (a) and 1.4, $\mathrm{nub}(\alpha)$ is a normal subgroup and $G/\mathrm{nub}(\alpha)$ is finite, hence a torsion group of finite exponent. It therefore suffices to show that $\mathrm{nub}(\alpha)$ is a torsion group of finite exponent. This is immediate from [31, Proposition 4.4, Theorem 6.2, Proposition 6.3]).

(c) Since U_α and $U_{\alpha^{-1}}$ are normal in G, the set $U_\alpha U_{\alpha^{-1}}$ is an open α-stable subgroup of G contained in $\mathrm{nub}(\alpha)$. Thus $\mathrm{nub}(\alpha) = U_\alpha U_{\alpha^{-1}}$ by [31, Corollary 4.3].

Lemma 1.7. *Let α be an automorphism of a totally disconnected, locally compact group G. Let $H \subseteq G$ be a closed, α-stable subgroup. Then the following holds:*

(a) *Then nub of $\alpha|_H$ is contained in* $\operatorname{nub}(\alpha)$.

(b) *If* $\operatorname{nub}(\alpha) \subseteq H$, *then* $\operatorname{nub}(\alpha|_H) = \operatorname{nub}(\alpha)$.

Proof. (a) Using 1.2 (b) twice, we deduce that

$$\operatorname{nub}(\alpha|_H) = \overline{U_{\alpha|_H}} \cap \overline{U_{(\alpha|_H)^{-1}}} \subseteq \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}} = \operatorname{nub}(\alpha).$$

(b) Since $\operatorname{nub}(\alpha) \subseteq H$, we have $U_\alpha \cap \operatorname{nub}(\alpha) \subseteq U_\alpha \cap H = U_{\alpha|_H}$ and thus

$$\operatorname{nub}(\alpha) = \overline{\operatorname{nub}(\alpha) \cap U_\alpha} \subseteq \overline{U_{\alpha|_H}}$$

(using 1.2 (c)). Likewise, $\operatorname{nub}(\alpha) \subseteq \overline{U_{(\alpha|_H)^{-1}}}$. Hence

$$\operatorname{nub}(\alpha) \subseteq \overline{U_{\alpha|_H}} \cap \overline{U_{(\alpha|_H)^{-1}}} = \operatorname{nub}(\alpha|_H).$$

Since $\operatorname{nub}(\alpha|_H) \subseteq \operatorname{nub}(\alpha)$ by (a), equality follows. □

We shall also need certain facts concerning contractive automorphisms.

1.8. Let α be a contractive automorphism of a topological group G.

(a) If $G \neq \{1\}$, then G is infinite and non-discrete. [If $x \in G \setminus \{1\}$, then we have $\alpha^n(x) \neq 1$ for all n and $\alpha^n(x) \to 1$, entailing that the topological group G is not discrete and hence infinite.]

(b) If G is locally compact, then α is *compactly contractive*, i.e., for each identity neighbourhood $U \subseteq G$ and compact set $K \subseteq G$ there exists $m \in \mathbb{N}$ such that $\alpha^n(K) \subseteq U$ for all $n \geq m$ (see [24, Lemma 1.4 (iv)]). Moreover, G is non-compact (unless $G = \{1\}$); see [24, Section 3.1].

1.9. Let α be a contractive automorphism of a totally disconnected, locally compact group G. Then the following holds:

(a) If $G \neq \{1\}$, then $\Delta_G(\alpha^{-1})$ is an integer ≥ 2 (see [10, Proposition 1.1 (e)]).

(b) If $G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$ is a series of α-stable closed subgroups of G such that G_j is a proper normal subgroup of G_{j-1} for all $j \in \{1, \ldots, n\}$, then n is bounded by the number of prime factors of $\Delta_G(\alpha^{-1})$ (see [10, Lemma 3.5]).

(c) $G = \operatorname{div}(G) \times \operatorname{tor}(G)$ as a topological group for a divisible torsion free group $\operatorname{div}(G)$ and a torsion group $\operatorname{tor}(G)$ of finite exponent (cf. [10, Theorem B]).

(d) If $N \subseteq G$ is an α-stable closed normal subgroup, $q\colon G \to G/N$ the canonical quotient morphism and $\bar{\alpha}$ the automorphism of G/N induced by α, then

$$q(\mathrm{div}(G)) = \mathrm{div}(G/N) \quad \text{and} \quad q(\mathrm{tor}(G)) = \mathrm{tor}(G/N).$$

[The inclusions $q(\mathrm{div}(G)) \subseteq \mathrm{div}(G/N)$ and $q(\mathrm{tor}(G)) \subseteq \mathrm{tor}(G/N)$ are clear. Since we have $G/N = q(\mathrm{div}(G)\,\mathrm{tor}(G)) = q(\mathrm{div}(G))q(\mathrm{tor}(G))$ and $G/N = \mathrm{div}(G/N) \times \mathrm{tor}(G/N)$, equality follows.]

Remark 1.10. If an automorphism $\alpha\colon G \to G$ of a totally disconnected, locally compact group is contractive, then it is also expansive. To see this, let $V \subseteq G$ be any compact neighbourhood of the identity. If $g \in G \setminus \{1\}$, then $G \setminus \{g\}$ is an identity neighbourhood, whence there exists $n \in \mathbb{N}$ such that $\alpha^n(V) \subseteq G \setminus \{g\}$ (see 1.8 (b)). Thus $g \notin \alpha^n(V)$ and we have shown that $\bigcap_{n\in\mathbb{Z}} \alpha^n(V) = \{1\}$.

2 Making contraction groups locally compact

The problem of refining group topologies on contraction groups was studied by Siebert [25]. The following special case is useful for our purposes.

Definition 2.1. Let G be a topological group, with topology τ, and let

$$\alpha\colon (G,\tau) \to (G,\tau)$$

be a contractive automorphism. We say that (G,α) (or simply G) *can be made locally compact* if there exists a locally compact topology τ^* on G making it a topological group such that $\tau \subseteq \tau^*$ and $\alpha\colon (G,\tau^*) \to (G,\tau^*)$ is a contractive automorphism. We write G^* for G, endowed with the topology τ^*.

If τ is totally disconnected, then also τ^* is totally disconnected (as the inclusion map $G^* \to G$ is continuous).

The topology τ^* is unique (if it exists): see [25, Corollary 8] for a discussion in a more general framework. We give a streamlined proof in our setting.

Lemma 2.2. *The topology τ^* is uniquely determined by the properties from Definition* 2.1.

Proof. Assume that $\hat{\tau}$ is a group topology on G with the same properties as τ^*. We show that the identity map

$$\phi\colon (G,\hat{\tau}) \to (G,\tau^*), \quad x \mapsto x$$

is continuous. Reversing the roles of $\hat{\tau}$ and τ^*, also ϕ^{-1} will be continuous and thus $\hat{\tau} = \tau^*$. Because ϕ is a homomorphism, we need only prove its continuity at 1. By local compactness, there exists a compact identity neighbourhood $V \subseteq (G, \hat{\tau})$. After replacing V with the closure of its interior V^0 in $(G, \hat{\tau})$, we may assume that V^0 is dense in V. Then V is also compact in (G, τ). Let $U \subseteq (G, \tau^*)$ be an arbitrary identity neighbourhood, and let $W \subseteq (G, \tau^*)$ be a compact identity neighbourhood such that $WW^{-1} \subseteq U$. Then W is also compact in (G, τ), hence closed in (G, τ) and hence closed in $(G, \hat{\tau})$. Since $\alpha\colon (G, \tau^*) \to (G, \tau^*)$ is contractive, we have $G = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(W)$. Hence V is the countable union of the closed subsets $V \cap \alpha^{-n}(W)$, for $n \in \mathbb{N}_0$. By the Baire Category Theorem, there exists $m \in \mathbb{N}_0$ such that $V \cap \alpha^{-m}(W)$ has non-empty interior in V. Since V^0 is dense in V, we deduce that $\alpha^{-n}(W) \cap V^0$ has non-empty interior in V^0, and so W has non-empty interior W^0 in $(G, \hat{\tau})$. Then $W^0(W^0)^{-1}$ is an identity neighbourhood in $(G, \hat{\tau})$ and hence U is an identity neighbourhood in $(G, \hat{\tau})$. Since $U = \phi^{-1}(U)$, we see that ϕ is continuous at 1. □

See also [25, Proposition 9] for the following fact.

Lemma 2.3. *Let G be a totally disconnected, locally compact group. If an automorphism $\alpha\colon G \to G$ is expansive, then $(U_\alpha, \alpha|_{U_\alpha})$ and $(U_{\alpha^{-1}}, \alpha^{-1}|_{U_{\alpha^{-1}}})$ can be made locally compact.*

Proof. Let $V \subseteq G$ be a compact open subgroup such that $\bigcap_{n \in \mathbb{Z}} \alpha^n(V) = \{1\}$. Then $\alpha^n(V_-) = V_- \cap \bigcap_{k=1}^{n} \alpha^k(V)$ is open in V_- for each $n \in \mathbb{N}_0$. Since V_- is compact and $\bigcap_{n \in \mathbb{N}_0} \alpha^n(V_-) = \bigcap_{n \in \mathbb{Z}} \alpha^n(V) = \{1\}$, it follows that the open subgroups $\alpha^n(V_-)$ form a basis of identity neighbourhoods in V_-, for $n \in \mathbb{N}_0$. If $n \in \mathbb{N}$ and $g \in U_\alpha = V_{--} = \bigcup_{k \in \mathbb{N}_0} \alpha^{-k}(V_-)$ (see Proposition 1.1 (c)), then $g \in \alpha^{-k}(V_-)$ for some $k \in \mathbb{N}_0$ and $\alpha^n(V_-)$ is an open subgroup of the topological group $\alpha^{-k}(V_-)$. Hence, there exists $m \in \mathbb{N}_0$ such that $g\alpha^m(V_-)g^{-1} \subseteq \alpha^n(V_-)$. By the preceding, there exists a group topology τ^* on U_α for which the set $\{\alpha^n(V_-) : n \in \mathbb{N}_0\}$ is a basis of identity neighbourhoods. Thus V_- is an open subgroup of (U_α, τ^*), and the latter group induces the given compact topology on V_- (as $\{\alpha^n(V_-) : n \in \mathbb{N}_0\}$ is a basis of identity neighbourhoods for both topologies on the subgroup V_-). Thus (U_α, τ^*) is a totally disconnected, locally compact group and α is still continuous (being continuous on the open subgroup V_-) as well as α^{-1} (being continuous on the subgroup $\alpha(V_-)$ which is open in V_-). Since $U_\alpha = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(V_-)$ and $(\alpha^n(V_-))_{n \in \mathbb{N}_0}$ is a basis of identity neighbourhoods, it readily follows that the automorphism α of (U_α, τ^*) is contractive. □

Remark 2.4. The group U_α can also be made locally compact if α is an arbitrary (not necessarily expansive) analytic automorphism of a Lie group G over a local field (see Proposition 13.3 (b) in the extended preprint version of [7]).

Example 2.5. The right-shift α is an automorphism of the compact group $G := (\mathbb{Z}_p)^{\mathbb{Z}}$, where $\mathbb{Z}_p$ is the additive group of p-adic integers. The contraction group U_α is non-trivial, as it is the group of all $(z_n)_{n\in\mathbb{Z}}$ such that $z_n \to 0$ as $n \to -\infty$. Then U_α cannot be locally compact, because $\operatorname{tor}(U_\alpha) \subseteq \operatorname{tor}(G) = \{0\}$ and $\operatorname{div}(U_\alpha) \subseteq \operatorname{div}(G) = \{0\}$. Thus, if U_α could be made locally compact, then we would have $U_\alpha = D_\alpha + T_\alpha = \{0\}$ (using 1.9 (c)), contrary to $U_\alpha \neq \{0\}$.

Lemma 2.6. *Let G be a topological group and $\alpha\colon G \to G$ a contractive automorphism such that (G, α) can be made locally compact. Then we have:*

(a) *$(H, \alpha|_H)$ can be made locally compact for each closed α-stable subgroup H of G (or G^*), and H^* carries the topology induced by G^*.*

(b) *If $\phi\colon G \to H$ is a continuous homomorphism to a topological group H admitting an automorphism $\beta\colon H \to H$ such that $\beta \circ \phi = \phi \circ \alpha$, then $\beta|_{\phi(G)}$ is contractive, $\phi(G)$ can be made locally compact and*

$$G^* \to \phi(G)^*, \quad x \mapsto \phi(x)$$

is a topological quotient map.

Proof. (a) Let τ and τ^* be as in Definition 2.1. Since $\tau \subseteq \tau^*$, H is closed in G^* (in either case) and hence H is a locally compact group in the topology σ on H induced by G^*, which is finer than the topology induced by G and turns $\alpha|_H$ into a contractive automorphism of (H, σ). Thus $H^* = (H, \sigma)$.

(b) Let σ be the topology on $\phi(G)$ turning $G^* \to (\phi(G), \sigma)$, $x \mapsto \phi(x)$ into a quotient map. Then σ is finer than the topology induced on $\phi(G)$ by H. Moreover, $(\phi(G), \sigma) \cong G^*/\ker\phi$ is locally compact. Since

$$\beta|_{\phi(G)} \circ \phi = \phi \circ \alpha\colon G^* \to (\phi(G), \sigma)$$

is continuous, the map $\beta|_{\phi(G)}$ is continuous with respect to the quotient topology σ. Also $(\beta|_{\phi(G)})^{-1}$ is continuous, by an analogous argument. Finally, the map $\beta|_{\phi(G)}\colon (\phi(G), \sigma) \to (\phi(G), \sigma)$ is contractive: Since $\beta^n \circ \phi = \phi \circ \alpha^n$, this follows from the facts that $\alpha\colon G^* \to G^*$ is contractive and $\phi\colon G^* \to \phi(G)^*$ is continuous. □

The following observation is crucial for many of our arguments.

Proposition 2.7. *Let α be an expansive automorphism of a totally disconnected, locally compact group G and let*

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n$$

be α-stable closed subgroups of G such that G_j is normal in G_{j-1} for all indices $j \in \{1, \ldots, n\}$. Let J be the set of all $j \in \{1, \ldots, n\}$ such that G_{j-1}/G_j is not *discrete. Then*

$$\#J \le \ell_\alpha + \ell_{\alpha^{-1}},$$

where ℓ_α *is the number of prime factors of* $\Delta_{U_\alpha^*}(\alpha^{-1}|_{U_\alpha^*})$ *and* $\ell_{\alpha^{-1}}$ *is the number of prime factors of* $\Delta_{U_{\alpha^{-1}}^*}(\alpha|_{U_{\alpha^{-1}}^*})$.

Proof. Let J_α (resp., $J_{\alpha^{-1}}$) be the set of all $j \in \{1, \dots, n\}$ such that

$$U_\alpha \cap G_j \subset U_\alpha \cap G_{j-1}$$

(resp., $U_{\alpha^{-1}} \cap G_j \subset U_{\alpha^{-1}} \cap G_{j-1}$). If $j \in \{1, \dots, n\} \setminus (J_\alpha \cup J_{\alpha^{-1}})$, then

$$U_\alpha \cap G_j = U_\alpha \cap G_{j-1} \quad \text{and} \quad U_{\alpha^{-1}} \cap G_j = U_{\alpha^{-1}} \cap G_{j-1}.$$

Since α is expansive, it follows that $(U_\alpha \cap G_{j-1})(U_{\alpha^{-1}} \cap G_{j-1})$ is open in G_{j-1} (see Lemma 1.1 (d) and (e)). We deduce that G_j is open in G_{j-1} and thus $j \notin J$. Hence $J \subseteq J_\alpha \cup J_{\alpha^{-1}}$, whence $\#J \le \#J_\alpha + \#J_{\alpha^{-1}} \le \ell_\alpha + \ell_{\alpha^{-1}}$ (using 1.9 (b) in the last step). □

We shall use a simple fact.

Lemma 2.8. *Let K be a compact group and* $D \subseteq K$ *a subgroup which is divisible. Then also the closure* $\overline{D}$ *is divisible. If K is totally disconnected, then* $D = \{1\}$.

Proof. For each $m \in \mathbb{N}$, the map $f_m \colon \overline{D} \to \overline{D}$ given by $g \mapsto g^m$ is continuous and hence has compact image. As the image contains D by hypothesis, we see that $f_m(\overline{D}) = \overline{D}$. Thus $\overline{D}$ is divisible.

If K is totally disconnected, then K is a pro-finite group. In particular, the homomorphisms $f \colon K \to F$ to finite groups F separate points on $\overline{D}$. But each $f(\overline{D})$ is both finite and divisible and therefore the trivial group. Hence also $\overline{D} = \{1\}$. □

Lemma 2.9. *Let* α *be an automorphism of a totally disconnected, locally compact group* G. *If* U_α *can be made locally compact (e.g., if* α *is expansive), then the following holds:*

(a) $U_\alpha \cap \mathrm{nub}(\alpha) = T_\alpha \cap \mathrm{nub}(\alpha)$;

(b) $\mathrm{nub}(\alpha) = \overline{T_\alpha \cap \mathrm{nub}(\alpha)}$;

(c) $\overline{T_\alpha} \cap U_\alpha = T_\alpha$;

(d) $\overline{T_\alpha} = T_\alpha \, \mathrm{nub}(\alpha)$.

If both U_α *and* $U_{\alpha^{-1}}$ *can be made locally compact, then we also have:*

(e) $\mathrm{nub}(\alpha) = \overline{T_\alpha} \cap \overline{T_{\alpha^{-1}}}$.

Proof. (a) By Lemma 2.6 (a), $U_\alpha \cap \text{nub}(\alpha)$ can be made locally compact. Thus α restricts to a contractive automorphism β of $(U_\alpha \cap \text{nub}(\alpha))^*$, enabling us to write $(U_\alpha \cap \text{nub}(\alpha))^* = D_\beta T_\beta$. Since $\text{nub}(\alpha)$ is compact and totally disconnected, its divisible subgroup D_β has to be trivial, by Lemma 2.8. Thus

$$U_\alpha \cap \text{nub}(\alpha) = U_\beta = T_\beta \subseteq T_\alpha$$

and hence $U_\alpha \cap \text{nub}(\alpha) = T_\alpha \cap \text{nub}(\alpha)$.

(b) As $\text{nub}(\alpha) = \overline{U_\alpha \cap \text{nub}(\alpha)}$ (see 1.2 (c)), the assertion is immediate from (a).

(c) The subgroup $T_\alpha \subseteq \overline{T_\alpha} \cap U_\alpha$ is trivial. Because T_α is a torsion group of finite exponent (see 1.9 (c)), also $\overline{T_\alpha}$ is a torsion group, see 1.5. Let β the restriction of α to the closed α-stable subgroup $\overline{T_\alpha} \cap U_\alpha^*$ of U_α^*. Then $\overline{T_\alpha} \cap U_\alpha = D_\beta T_\beta$. Since D_β is torsion-free (see 1.9 (c)) and $\overline{T_\alpha}$ a torsion group, $D_\beta = \{1\}$ follows. Thus

$$\overline{T_\alpha} \cap U_\alpha = T_\beta \subseteq T_\alpha.$$

(d) We have $\overline{T_\alpha} \subseteq \overline{U_\alpha} = U_\alpha \, \text{nub}(\alpha)$ (see 1.2 (a)). Since $\text{nub}(\alpha) \subseteq \overline{T_\alpha}$ by (b), we deduce that

$$\overline{T_\alpha} = (U_\alpha \cap \overline{T_\alpha}) \, \text{nub}(\alpha) = T_\alpha \, \text{nub}(\alpha)$$

(using (c) for the last equality).

(e) We have

$$\text{nub}(\alpha) = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}} \supseteq \overline{T_\alpha} \cap \overline{T_{\alpha^{-1}}}$$

(see 1.2 (b)) and

$$\text{nub}(\alpha) = \overline{T_\alpha \cap \text{nub}(\alpha)} \cap \overline{T_{\alpha^{-1}} \cap \text{nub}(\alpha)} \subseteq \overline{T_\alpha} \cap \overline{T_{\alpha^{-1}}}$$

(using (b)). □

Lemma 2.10. *Let α be an automorphism of a locally compact group G and let $H \subseteq G$ be an α-stable subgroup such that $\alpha|_H$ is contractive. Then the closure $\overline{H} \subseteq G$ is σ-compact.*

Proof. Let $K \subseteq \overline{H}$ be a compact identity neighbourhood. Then $\bigcup_{n\in\mathbb{Z}} \alpha^n(K)$ is a σ-compact subset of $\overline{H}$ and generates a σ-compact subgroup S of $\overline{H}$. Since S is an α-stable open subgroup of $\overline{H}$ and $\alpha|_H$ is contractive, we have $H \subseteq S$ and thus $S = \overline{H}$ (since S is closed). Hence $\overline{H} = S$ is σ-compact. □

3 Proof of Theorem A

We now prove Theorem A.

Proof of Theorem A. Let G be a totally disconnected, locally compact group, α an expansive automorphism of G and $N \subseteq G$ an α-stable, closed normal subgroup.

Let $q: G \to G/N$ be the canonical quotient morphism and $\overline{\alpha}$ the automorphism of G/N induced by α (determined by $\overline{\alpha} \circ q = q \circ \alpha$). Then $\alpha|_N$ is expansive, by Lemma 1.1 (e). We show that $\overline{\alpha}$ is also expansive. By Proposition 1.3 (a), we need only show that $\overline{\alpha}$ restricts to an expansive automorphism of $M_{\overline{\alpha}}$. After replacing G with $q^{-1}(M_{\overline{\alpha}})$ (which is closed since $M_{\overline{\alpha}}$ is closed), we may assume that $G/N = M_{\overline{\alpha}}$. Let U be a subgroup of G/N tidy for $\overline{\alpha}$. Then $U = U_+ = U_-$ as $\overline{\alpha}^{\mathbb{Z}}(g)$ is relatively compact for each $g \in U$ (cf. [29, Lemma 9]) and thus U is an $\overline{\alpha}$-stable, compact open subgroup of G/N. After replacing G with $q^{-1}(U)$, we may assume that G/N is compact. Using [31, Proposition 5.1] and the metrizability of G/N, we find a descending sequence $(H_n)_{n\in\mathbb{N}}$ of $\overline{\alpha}$-stable closed normal subgroups H_n of G/N such that $\overline{\alpha}$ induces an expansive automorphism α_n on $(G/N)/H_n$ for each $n \in \mathbb{N}$ and G/N is the projective limit

$$G/N = \varprojlim (G/N)/H_n.$$

Set $L_n := q^{-1}(H_n)$; then $(L_n)_{n\in\mathbb{N}}$ is a descending sequence of α-stable closed normal subgroups of G, with $\bigcap_{n\in\mathbb{N}} L_n = N$.

There exists $m \in \mathbb{N}$ such that L_n is open in L_m for all $n \geq m$. Indeed, if this were false, we could find a subsequence $(L_{n_k})_{k\in\mathbb{N}}$ such that, for each $k \in \mathbb{N}$, the normal subgroup $L_{n_{k+1}}$ is not open in L_{n_k}. This contradicts Proposition 2.7.

After passing to a subsequence, we may assume that L_n is open in L_1 for each $n \in \mathbb{N}$. Hence L_n contains both $U_\alpha \cap L_1$ and $U_{\alpha^{-1}} \cap L_1$. As a consequence, $N = \bigcap_{n\in\mathbb{N}} L_n$ contains both $U_\alpha \cap L_1$ and $U_{\alpha^{-1}} \cap L_1$. Hence N is open in L_1 (and in each L_n), using the fact that $\alpha|_{L_1}$ is expansive and thus $(U_\alpha \cap L_1)(U_{\alpha^{-1}} \cap L_1)$ an open subset of L_1 (see 1.1 (d) and (e)). This implies that the compact group $H_1 \cong L_1/N$ is discrete and hence a finite group. Since $H_1 \supseteq H_2 \supseteq \cdots$ with $\bigcap_{n\in\mathbb{N}} H_n = \{1\}$, we deduce that $H_n = \{1\}$ for some n. Since $\bar{\alpha}$ corresponds to the expansive automorphism α_n on $(G/N)/H_n \cong G/N$, we see that $\bar{\alpha}$ is expansive.

Conversely, assume that both $\alpha|_N$ and $\overline{\alpha}: G/N \to G/N$ are expansive.[1] Then there is an open identity neighbourhood $P \subseteq G/N$ such that $\bigcap_{n\in\mathbb{Z}} \overline{\alpha}^n(P) = \{1\}$, and an open identity neighbourhood $Q \subseteq N$ such that $\bigcap_{n\in\mathbb{Z}} \alpha^n(Q) = \{1\}$. After shrinking Q, we may assume that $Q \subseteq q^{-1}(P)$. Then $Q = N \cap V$ for some open identity neighbourhood $V \subseteq G$. After replacing V with $V \cap q^{-1}(P)$, we may assume that $V \subseteq q^{-1}(P)$. We have

$$N = q^{-1}\Big(\bigcap_{n\in\mathbb{Z}} \overline{\alpha}^n(P)\Big) = \bigcap_{n\in\mathbb{Z}} \alpha^n(q^{-1}(P)),$$

[1] Compare also [31, Proposition 6.1]. The compactness of G assumed there is inessential for this part of the proof of [31, Proposition 6.1].

entailing that $I := \bigcap_{n\in\mathbb{Z}} \alpha^n(V)$ is an α-stable subset of N. Since

$$I = I \cap N \subseteq V \cap N = Q,$$

we deduce that $I = \bigcap_{n\in\mathbb{Z}} \alpha^n(I) \subseteq \bigcap_{n\in\mathbb{Z}} \alpha^n(Q) = \{1\}$. Hence $I = \{1\}$ and α is expansive. □

4 Proof of Theorem B

The following lemma is useful.

Lemma 4.1. *Let α be an expansive automorphism of a totally disconnected, locally compact group G such that every α-stable, closed normal subgroup $N \subseteq G$ is open or discrete. Let $C \subseteq G$ be the topologically characteristic subgroup*[2] *of G generated by $U_\alpha \cup U_{\alpha^{-1}}$. Then C is an open normal subgroup of G, and one of the following cases occurs:*

(a) $\overline{[C,C]}$ *is open in G, in which case $C = \overline{[C,C]}$ is topologically perfect.*

(b) $\overline{[C,C]}$ *is discrete.*

Proof. Note that C is an open subgroup of G as it contains the open set $U_\alpha U_{\alpha^{-1}}$ (see Lemma 1.1 (d)). The closed subgroup $\overline{[C,C]}$ is topologically characteristic in C, whence it is topologically characteristic in G and hence α-stable and normal. Therefore $\overline{[C,C]}$ is open or discrete. If $\overline{[C,C]}$ is open, then it contains $U_\alpha \cup U_{\alpha^{-1}}$. Hence $\overline{[C,C]} = C$, using the fact that C is the smallest topologically characteristic subgroup of G which contains $U_\alpha \cup U_{\alpha^{-1}}$. □

Proof of Theorem B. Define ℓ_α and $\ell_{\alpha^{-1}}$ as in Proposition 2.7. For every series

$$\Sigma\colon G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{1\}$$

of α-stable closed subgroups of G, let J_Σ be the set of all $j \in \{1,\ldots,n\}$ such that G_{j-1}/G_j is not discrete. Then $J_\Sigma \le \ell_\alpha + \ell_{\alpha^{-1}}$, by Proposition 2.7, entailing that the maximum

$$m := \max_\Sigma J_\Sigma$$

over all series Σ exists. Let $\Sigma\colon G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{1\}$ be a series with $J_\Sigma = m$. Let $N \subseteq G_{j-1}$ be an α-stable closed normal subgroup with $G_j \subseteq N$. If neither G_{j-1}/N nor N/G_j were discrete, we would have $J_{\Sigma\cup\{N\}} = J_\Sigma + 1$, a contradiction. Thus N will be open in G_{j-1} or G_j is open in N.

[2] Thus C is the subgroup generated by $\bigcup_{\beta\in\mathrm{Aut}(G)} \beta(U_\alpha \cup U_{\alpha^{-1}})$.

For an index $j \in J_\Sigma$, let $q_j : G_{j-1} \to G_{j-1}/G_j$ be the canonical quotient morphism, let $\alpha_j : G_{j-1}/G_j \to G_{j-1}/G_j$ be the automorphism induced by α and let $C_j \subseteq G_{j-1}/G_j$ be the topologically characteristic subgroup generated by $U_{\alpha_j} \cup U_{\alpha_j^{-1}}$. If $\overline{[C_j, C_j]}$ is open in G_{j-1}/G_j, define

$$M_j := N_j := q_j^{-1}(C_j);$$

thus we have that $G_{j-1}/M_j \cong (G_{j-1}/G_j)/C_j$ and $M_j/N_j = \{1\}$ are discrete and $N_j/G_j \cong C_j$ is topologically perfect. If $\overline{[C_j, C_j]}$ is discrete, we define

$$M_j := q_j^{-1}(C_j) \quad \text{and} \quad N_j := q_j^{-1}(\overline{[C_j, C_j]})$$

(by Lemma 4.1, only these two cases can occur). So $G_{j-1}/M_j \cong (G_{j-1}/G_j)/C_j$ is discrete, $M_j/N_j \cong C_j/\overline{[C_j, C_j]}$ is abelian and $N_j/G_j \cong \overline{[C_j, C_j]}$ is discrete. Hence

$$\Sigma' := \Sigma \cup \bigcup_{j \in J_\Sigma} \{M_j, N_j\}$$

is a series of α-stable closed subnormal subgroups such that all non-discrete subfactors are abelian or topologically perfect. Since $\#J_{\Sigma'} = \#J_\Sigma$ is maximal, all non-discrete subfactors of Σ' have the property that all stable closed normal subgroups are open or discrete. □

Using the recent theory of elementary groups [28], slightly more detailed information on the factor groups can be obtained, in the case of second countable groups. Recall that the class of elementary groups is the smallest class of totally disconnected, second countable, locally compact groups that contains all countable discrete groups and all second countable pro-finite groups, and is closed under extensions as well as countable increasing unions. A totally disconnected, second countable, locally compact group G is called *elementary-free* if all of its elementary closed normal subgroups and all of its elementary Hausdorff quotient groups are trivial [28, Definition 7.14]. If α is an expansive automorphism of a totally disconnected, locally compact non-trivial group G and G does not have closed α-stable subgroups except for G and $\{1\}$, then (G, α) is called a *simple expansion group*. Note that if G is a non-trivial elementary-free group and α an expansive automorphism of G such that every α-stable closed normal subgroup of G is discrete or open, then (G, α) is a simple expansion group. We remark:

Remark 4.2. If G is second countable in Theorem B, then one can achieve there that each of the quotient groups G_{j-1}/G_j is discrete, abelian, both topologically perfect and elementary, or an elementary-free simple expansion group.

In fact, let us consider a topologically perfect factor $Q := G_{j-1}/G_j$ in a series all of whose factors are discrete, abelian, or topologically perfect, and which has a maximum number of non-discrete factors. By [28, Theorem 7.15], there are two topologically characteristic, closed subgroups D_1 and D_2 of Q such that $Q \supseteq D_1 \supseteq D_2 \supseteq \{1\}$ and, moreover, both D_2 and Q/D_1 are elementary and D_1/D_2 is elementary-free. Let N_1 and N_2 be the pre-images of D_1 and D_2, respectively, under the quotient morphism $G_{j-1} \to G_{j-1}/G_j$. Then N_1 and N_2 are α-stable closed normal subgroups of G_{j-1}, and $G_{j-1} \rhd N_1 \rhd N_2 \rhd G_j$.

Case 1: If D_1/D_2 is non-trivial, the elementary-free group $N_1/N_2 \cong D_1/D_2$ is non-discrete (as it would be elementary otherwise), whence N_2 is not open in N_1. Hence N_2 is not open in G_{j-1} and hence N_2/G_j is discrete (by maximality of the number of non-discrete factors). Again by maximality, G_{j-1}/N_1 is discrete and N_1/N_2 does not have closed normal subgroups stable under the induced expansive automorphism other than open or discrete subgroups. So, the elementary-free group N_1/N_2 is a simple expansion group.

Case 2: If D_1/D_2 is trivial, then the topologically perfect group $G_{j-1}/G_j = Q \rhd D_1 = D_2 \rhd \{1\}$ is elementary, as any extension of elementary groups is.

5 Proof of Theorem C

We prove Theorem C and record some related results.

Proof of Theorem C. Since $\operatorname{nub}(\alpha) \subseteq \overline{U_\alpha}$, we may replace G with $\overline{U_\alpha}$ without changing the nub (see 1.7), or $\overline{T_\alpha}$, or D_α. We may therefore assume that U_α is dense in G. Since $\operatorname{nub}(\alpha)$ normalizes U_α (see 1.2 (f)), U_α is a normal subgroup of $G = U_\alpha \operatorname{nub}(\alpha)$ (exploiting 1.2 (a)). Hence also the characteristic subgroups D_α and T_α of U_α are normal in G. Therefore also $\overline{T_\alpha}$ is normal in G. Since $\overline{T_\alpha}$ is a torsion group (see 1.9 (c) and 1.5) and D_α torsion-free (see 1.9 (c)), we see that $D_\alpha \cap \overline{T_\alpha} = \{1\}$. Moreover, using the fact that $T_\alpha \operatorname{nub}(\alpha) = \overline{T_\alpha}$ by Lemma 2.9 (d), we obtain

$$G = U_\alpha \operatorname{nub}(\alpha) = D_\alpha T_\alpha \operatorname{nub}(\alpha) = D_\alpha \overline{T_\alpha}.$$

Hence $G = D_\alpha \times \overline{T_\alpha}$ as an abstract group. In particular, D_α centralizes $\overline{T_\alpha}$. Thus D_α also centralizes $\operatorname{nub}(\alpha) \subseteq \overline{T_\alpha}$. Since $\overline{T_\alpha}$ is σ-compact by Lemma 2.10, also $D_\alpha^* \times \overline{T_\alpha}$ is a σ-compact locally compact group (writing D_α^* for D_α, endowed with the locally compact topology induced by U_α^*). Because also G is locally compact and the product map $\pi\colon D_\alpha^* \times \overline{T_\alpha} \to G$, $(x, y) \mapsto xy$ is a continuous isomorphism of abstract groups, we deduce from [11, Section 5.29] that π is an isomorphism of topological groups. Hence $G = D_\alpha \times \overline{T_\alpha}$ (internally). In particular, D_α is closed in G. □

Corollary 5.1. *Let G be a totally disconnected, locally compact group and α an automorphism of G such that U_α and $U_{\alpha^{-1}}$ can be made locally compact (e.g., any expansive automorphism). Then $D_\alpha \cap D_{\alpha^{-1}} = \{1\}$.*

Proof. Since D_α and $D_{\alpha^{-1}}$ are closed and α-stable, it follows that their intersection $H := D_\alpha \cap D_{\alpha^{-1}}$ is a totally disconnected, locally compact contraction group for both $\alpha|_H$ and $\alpha^{-1}|_H$. Hence $H = \{1\}$. Indeed, if $H \neq \{1\}$, then both $\Delta_H(\alpha|_H)$ and $\Delta_H(\alpha|_H^{-1}) = \Delta_H(\alpha|_H)^{-1}$ would be integers ≥ 2 (see 1.9 (a)), which is impossible. □

Remark 5.2. The nub of an expansive automorphism $\alpha\colon G \to G$ need not have an open normalizer in G. To see this, let F be a finite group which is a semidirect product $F = N \rtimes H$ of a normal subgroup N and a subgroup H which is not normal in F (e.g., F might be the dihedral group $C_3 \rtimes C_2$). Let G be the group of all $(n_k, h_k)_{k\in\mathbb{Z}} \in F^{\mathbb{Z}}$ such that $(n_k)_{k\in\mathbb{Z}} \in N^{(-\mathbb{N})} \times N^{\mathbb{N}_0} =: M$. Thus $G = M \rtimes H^{\mathbb{Z}}$ as an abstract group. Endow G with the topology making it the direct product topological space of the restricted product M and the compact group $H^{\mathbb{Z}}$. Then G is a topological group, being the ascending union of the open subgroups $H^{\{k\in\mathbb{Z}:k<-m\}} \times F^{\{k\in\mathbb{Z}:k\geq -m\}}$ for $m \in \mathbb{N}$, which are topological groups. The right-shift α is an automorphism of G. We have

$$U_\alpha = M \rtimes (H^{(-\mathbb{N})} \times H^{\mathbb{N}_0})$$

and

$$U_{\alpha^{-1}} = H^{-\mathbb{N}} \times H^{(\mathbb{N}_0)}.$$

Thus $\overline{U_\alpha} = G$, $\overline{U_{\alpha^{-1}}} = H^{\mathbb{Z}}$ and $\mathrm{nub}(\alpha) = \overline{U_\alpha} \cap \overline{U_{\alpha^{-1}}} = H^{\mathbb{Z}}$ (using 1.2 (b)). As H is not normal in F, we see that $\mathrm{nub}(\alpha) = H^{\mathbb{Z}}$ is not normal in G. If the normalizer $N_G(\mathrm{nub}(\alpha))$ was open in G, then (being α-stable), it would contain the dense subgroup U_α of G and hence coincide with G (a contradiction). Thus $N_G(\mathrm{nub}(\alpha))$ is not open.

6 Abelian expansion groups

We show that, after passing to a refinement if necessary, only abelian, non-discrete groups of a special form will occur in Theorem B.

Remark 6.1. In the situation of Theorem B, let I be the set of all indices $j \in \{1, \dots, n\}$ such that G_{j-1}/G_j is abelian and non-discrete. Let α_j be the automorphism of G_{j-1}/G_j induced by α and let $q_j\colon G_{j-1} \to G_{j-1}/G_j$ be the quotient homomorphism, for $j \in I$. Then $U_{\alpha_j}U_{\alpha_j^{-1}}$ is an open α_j-stable subgroup of G_{j-1}/G_j and hence $H_j := q_j^{-1}(U_{\alpha_j}U_{\alpha_j^{-1}})$ is an α-stable open normal subgroup of G_{j-1}. Then G_{j-1}/H_j is discrete and all stable, closed, proper subgroups of H_j/G_j are discrete. After inserting the H_j into the series for all $j \in I$, we may thus assume without loss of generality that all abelian, non-discrete subfactors G_{j-1}/G_j have the property that all of their α_j-stable, closed, proper subgroups are discrete, and that $G_{j-1}/G_j = U_{\alpha_j}U_{\alpha_j^{-1}}$.

Let G_j be a topological group and $\alpha_j \in \operatorname{Aut}(G_j)$ for $j \in \{1,2\}$. We say that (G_1, α_1) and (G_2, α_2) are isomorphic if there exists an isomorphism $\phi\colon G_1 \to G_2$ of topological groups such that $\alpha_2 \circ \phi = \phi \circ \alpha_1$.

Proposition 6.2. *Let $A \neq \{1\}$ be an abelian, totally disconnected, locally compact group and let $\alpha\colon A \to A$ be an expansive automorphism. Assume that $A = U_\alpha U_{\alpha^{-1}}$ and assume that every α-stable proper closed subgroup of A is discrete. Then there exists a prime number p such that (A, α) isomorphic to one of the following:*

(a) *$\mathbb{Q}_p^n$ for some $n \in \mathbb{N}$, together with a contractive linear automorphism $\beta\colon \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ not admitting non-trivial proper β-stable vector subspaces,*

(b) *$\mathbb{Q}_p^n$ for some $n \in \mathbb{N}$, together with β^{-1} for a contractive linear automorphism $\beta\colon \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ not admitting non-trivial proper β-stable vector subspaces,*

(c) *$C_p^{(-\mathbb{N})} \times C_p^{\mathbb{N}_0}$ with the right-shift,*

(d) *$C_p^{(-\mathbb{N})} \times C_p^{\mathbb{N}_0}$ with the left-shift,*

(e) *$C_p^{\mathbb{Z}}$ with the right-shift.*

Proof. Let D_α be the divisible part and let T_α be the torsion part of U_α, and define $D_{\alpha^{-1}}$ and $T_{\alpha^{-1}}$ analogously. If $D_\alpha \neq \{1\}$, then $D_\alpha = D_\alpha^*$ is an α-stable closed subgroup (see Theorem C) which is non-discrete (see 1.8 (a)) and thus $A = D_\alpha$. By 1.8 (a) and the hypotheses, D_α is a divisible simple contraction group and hence of the form described in (a) (see [10, Theorem A]). Likewise, A is of the form described in (b) whenever $D_{\alpha^{-1}} \neq \{1\}$.

Throughout the rest of the proof, assume that $D_\alpha = D_{\alpha^{-1}} = \{1\}$. Then we have $A = U_\alpha U_{\alpha^{-1}} = T_\alpha T_{\alpha^{-1}}$.

Since $\operatorname{nub}(\alpha)$ is an α-stable closed subgroup of A, it is either all of A or discrete. Being also compact, it is finite in the latter case, and thus $\{1\}$ is an open α-stable (normal) subgroup of $\operatorname{nub}(\alpha)$. Now [31, Corollary 4.4] (proper such do not exist) shows that $\operatorname{nub}(\alpha) = \{1\}$.

Case $\operatorname{nub}(\alpha) = \{1\}$: In this case $T_\alpha = U_\alpha = U_\alpha \operatorname{nub}(\alpha)$ and $T_{\alpha^{-1}} = U_{\alpha^{-1}} = U_{\alpha^{-1}} \operatorname{nub}(\alpha)$ are closed α-stable subgroups of A (using 1.2 (a)). If $T_\alpha \neq \{1\}$, then T_α is non-discrete. Hence $T_\alpha = A$ by the hypotheses, and this is a simple contraction group which is a torsion group and hence of the form described in (c) (see [10, Theorem A]). Likewise, A is of the form described in (d) if $T_{\alpha^{-1}} \neq \{1\}$.

Case $A = \operatorname{nub}(\alpha)$: In this case A is compact and is irreducible in the sense of [31, Definition 6.1] as all its proper α-stable closed (normal) subgroups are finite, and moreover A is infinite (as U_α or $U_{\alpha^{-1}}$ is non-trivial and hence non-discrete, being a contraction group). Hence, by [31, Proposition 6.3], (A, α) is isomorphic to the right-shift of $F^{\mathbb{Z}}$ for a finite simple group F. Since A is abelian, $F \cong C_p$ for some p and thus A is of the form described in (e). □

Remark 6.3. Let G be a totally disconnected, locally compact group and let $\alpha\colon G \to G$ be an expansive automorphism. If G is abelian, then the map

$$\pi\colon U_\alpha^* \times U_{\alpha^{-1}}^* \to G, \quad (x, y) \mapsto xy$$

is a continuous, open homomorphism with discrete kernel. For non-abelian G, the map still has open image (see Lemma 1.1 (d)), is a local homeomorphism, and equivariant with respect to the natural left and right actions of U_α and $U_{\alpha^{-1}}$, respectively.

[To see this, let $V \subseteq G$ be a compact open subgroup such that $V_+ \cap V_- = \{1\}$ and $V = V_+V_-$ (see Lemma 1.1 (c)). Then V_- and V_+ are open subgroups of U_α^* and $U_{\alpha^{-1}}^*$, respectively (see proof of Lemma 2.3). Then $\pi(V_+ \times V_-) = V$ is open in G and $\pi|_{V_+\times V_-}$ is injective, as $vw = v'w'$ for $v, v' \in V_+$, $w, w' \in V_-$ implies $v^{-1}v' = w(w')^{-1} \in V_+ \cap V_- = \{1\}$ and thus $v = v'$ and $w = w'$. Since $V_+ \times V_-$ is compact, it follows that π restricts to a homeomorphism $V_+ \times V_- \to V$. Since $\pi(gv, wh) = g\pi(v, w)h$ for all $g \in U_\alpha$, $h \in U_{\alpha^{-1}}$ and $(v, w) \in V_+ \times V_-$, also $\pi|_{gV_+\times V_-h}$ is a homeomorphism onto an open set.]

Remark 6.4. It can happen that U_α is closed for an expansive automorphism α of a totally disconnected, locally compact group G, but $U_{\bar\alpha}$ is not closed for the induced automorphism $\bar\alpha$ on G/N for some α-stable closed normal subgroup $N \subseteq G$. The following example also illustrates Remark 6.3.

Given a non-trivial finite abelian group $(F, +)$, consider the restricted products

$$H_1 := F^{(-\mathbb{N})} \times F^{\mathbb{N}_0} \quad \text{and} \quad H_2 := F^{-\mathbb{N}} \times F^{(\mathbb{N}_0)},$$

with $V_1 := F^{\mathbb{N}_0}$ and $V_2 := F^{-\mathbb{N}}$, respectively, as compact open subgroups. Let α be the right-shift on $G := H_1 \times H_2$ (i.e., on both H_1 and H_2). Then α is an automorphism and it is expansive as $\bigcap_{n\in\mathbb{Z}} \alpha^n(V_1 \times V_2) = \{0\}$. Moreover, $U_\alpha = H_1$ and $U_{\alpha^{-1}} = H_2$ are closed. Also, let $\bar\alpha$ be the right-shift on $F^{\mathbb{Z}}$. Then

$$q\colon G \to F^{\mathbb{Z}}, \quad (f, g) \mapsto f + g$$

is a continuous surjective homomorphism. Restricted to the compact open subgroup $V_1 \times V_2$, the map q is an isomorphism of topological groups. Hence q is open, has discrete kernel, and is a quotient morphism. Finally, $U_{\bar\alpha} = F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ is a dense proper subgroup in $F^{\mathbb{Z}}$. Hence $U_{\bar\alpha}$ is not closed in $F^{\mathbb{Z}} \cong G/\ker(q)$.

Another property can be observed.

Proposition 6.5. *Let G be a totally disconnected, locally compact group that is abelian, and $\alpha\colon G \to G$ be an expansive automorphism. Then the torsion subgroup* $\operatorname{tor}(G)$ *is closed in G.*

Proof. Since $V := U_\alpha U_{\alpha^{-1}}$ is an open subgroup of G, we need only show that $V \cap \text{tor}(G) = \text{tor}(V)$ is closed. After replacing G with its α-stable subgroup V, we may therefore assume that $G = U_\alpha U_{\alpha^{-1}}$. Since D_α and $D_{\alpha^{-1}}$ are torsion-free (see 1.9 (c)) and $D_\alpha \cap D_{\alpha^{-1}} = \{1\}$ by Corollary 5.1, we deduce that $D_\alpha D_{\alpha^{-1}}$ is isomorphic to $D_\alpha \times D_{\alpha^{-1}}$ as an abstract group and hence torsion-free. Hence $D_\alpha D_{\alpha^{-1}} \cap T_\alpha T_{\alpha^{-1}} = \{1\}$. Combining this with

$$G = U_\alpha U_{\alpha^{-1}} = D_\alpha D_{\alpha^{-1}} T_\alpha T_{\alpha^{-1}},$$

we see that

$$G = (D_\alpha D_{\alpha^{-1}}) \times (T_\alpha T_{\alpha^{-1}}) = D_\alpha \times D_{\alpha^{-1}} \times T_\alpha T_{\alpha^{-1}} \tag{6.1}$$

internally as an abstract group. By equation (6.1), the torsion subgroup of G is $\text{tor}(G) = T_\alpha T_{\alpha^{-1}}$. Hence $\text{tor}(G)$ has finite exponent (like T_α and $T_{\alpha^{-1}}$). Thus also $\overline{\text{tor}(G)}$ is a torsion group (by 1.5) and thus $\text{tor}(G) = \overline{\text{tor}(G)}$. □

7 Example: *p*-adic Lie groups

Let $\mathbb{K}$ be a local field and let $|\cdot|$ be an absolute value on $\mathbb{K}$ defining its topology (see [27]). We pick an algebraic closure $\overline{\mathbb{K}}$ containing $\mathbb{K}$ and use the same symbol, $|\cdot|$, for the unique extension of the absolute value on $\mathbb{K}$ to an absolute value on $\overline{\mathbb{K}}$ (see [20, Theorem 16.1]). If E is a finite-dimensional $\mathbb{K}$-vector space and $\beta\colon E \to E$ a $\mathbb{K}$-linear automorphism, we write $R(\beta)$ for the set of all absolute values $|\lambda|$ of zeros λ of the characteristic polynomial of β in $\overline{\mathbb{K}}$. We let $\widetilde{E}_\lambda \subseteq E \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ be the generalized eigenspace of $\beta \otimes_{\mathbb{K}} \text{id}_{\overline{\mathbb{K}}}$ for the eigenvalue λ. For $\rho \in R(\beta)$, we let

$$E_\rho := \left(\bigoplus_{|\lambda|=\rho} \widetilde{E}_\lambda \right) \cap E.$$

Then $E = \bigoplus_{\rho \in R(\beta)} E_\rho$ (see [17, Chapter II, Section 1]) and we recall that

$$E = U_\beta \oplus M_\beta \oplus U_{\beta^{-1}}$$

with

$$M_\beta = E_1, \quad U_\beta = \bigoplus_{\rho<1} E_\rho \quad \text{and} \quad U_{\beta^{-1}} = \bigoplus_{\rho>1} E_\rho \tag{7.1}$$

(cf. [8, Lemma 2.5]).

If G is a Lie group over $\mathbb{K}$, then its tangent space $L(G) := T_1(G)$ at the identity element carries a natural Lie algebra structure, and $L(\alpha)\colon L(G) \to L(H)$ is a Lie algebra homomorphism for each $\mathbb{K}$-analytic homomorphism $\alpha\colon G \to H$ between $\mathbb{K}$-analytic Lie groups. We abbreviate $\text{Ad}(g) := L(I_g)$, where $I_g\colon G \to G$ is given by $x \mapsto gxg^{-1}$ for $g \in G$ (cf. [23] for further information).

Proposition 7.1. *Let α be an analytic automorphism of a Lie group G over a local field. Then the following conditions are equivalent:*

(a) *α is expansive.*

(b) *$\beta := L(\alpha)\colon L(G) \to L(G)$ is expansive.*

(c) *$1 \notin R(\beta)$.*

If α is expansive, then $L(G)$ is a nilpotent Lie algebra.

Proof. (a) ⇒ (b) By contraposition. If (b) is false, then β is not expansive. To deduce that α is not expansive, let $V \subseteq G$ be an identity neighbourhood. Since U_β is a vector subspace of $L(G)$ by (7.1) and hence closed, using Proposition 1.3 (b) we deduce that M_β is not discrete and hence a non-trivial vector subspace (in view of (7.1)). But then G contains a so-called centre manifold W around the fixed point 1 of α, which can be chosen as a submanifold of G contained in V that is stable under α and satisfies $T_1(W) = M_\alpha$ (whence $W \neq \{1\}$); see Proposition 6.3 (a) and part (b) of the Local Invariant Manifold Theorem in [8]; cf. also [7]. Then $\{1\} \neq W \subseteq \bigcap_{n\in\mathbb{Z}} \alpha^n(V)$, and thus α is not expansive.

(b) ⇔ (c) Since U_β is closed, β is expansive if and only if $M_\beta = L(G)_1$ is discrete. Since $L(G)_1$ is a vector space, the latter holds if and only if $L(G)_1 = \{0\}$, i.e., $1 \notin R(\beta)$.

(c) ⇒ (a) Note that U_α and $U_{\alpha^{-1}}$ are immersed Lie subgroups of G with Lie algebras U_β and $U_{\beta^{-1}}$, respectively (see [8, Theorem D]). We write U_α^* for U_α as a Lie group; because the underlying topology is locally compact and α restricts to a contractive Lie group automorphism, this is consistent with the definition of U_α^* in Section 2. Likewise, we consider $U_{\alpha^{-1}}^*$ as a Lie group. If $1 \notin R(\beta)$, then $L(G) = U_\beta \oplus U_{\beta^{-1}}$, entailing that the product map $\pi\colon U_\alpha^* \times U_{\alpha^{-1}}^* \to G$ given by $(x, y) \mapsto xy$ has invertible differential at $(1, 1)$ (the addition map $U_\beta \times U_{\beta^{-1}} \to L(G) = U_\beta \oplus U_{\beta^{-1}}$). Thus, by the inverse function theorem [23], there exist open identity neighbourhoods $V \subseteq U_\alpha^*$ and $W \subseteq U_{\alpha^{-1}}^*$ such that VW is open in G and the restriction

$$\pi|_{V\times W}\colon V \times W \to VW \tag{7.2}$$

is an analytic diffeomorphism. Using [24, Lemma 3.2 (i)], we find compact open subgroups $P \subseteq V$ of U_α^* and $Q \subseteq W$ of $U_{\alpha^{-1}}^*$ such that $\alpha(P) \subseteq P$, $\alpha^{-1}(P) \subseteq V$, $\alpha^{-1}(Q) \subseteq Q$ and $\alpha(Q) \subseteq W$. Then PQ is an open identity neighbourhood in G and we now show that $\bigcap_{n\in\mathbb{Z}} \alpha^n(PQ) = \{1\}$. To this end, let $x \in P$ and $y \in Q$. If $x \neq 1$, then $\alpha^{-n}(x) \notin P$ for some $n \in \mathbb{N}$, which we choose minimal. Thus $\alpha^{-n+1}(x) \in P$ and hence $\alpha^{-n}(x) \in V$. Since $\alpha^{-n}(x) \in V \setminus P$ by the preceding and $\alpha^{-n}(y) \in Q$, we see that $\alpha^{-n}(xy) = \alpha^{-n}(x)\alpha^{-n}(y) \in (V \setminus P)Q$. As the map (7.2) is a bijection, we deduce that $\alpha^{-n}(xy) \notin PQ$. Likewise, $\alpha^m(xy) \notin PQ$

for some $m \in \mathbb{N}$ if $y \neq 1$. Thus $\bigcap_{n\in\mathbb{Z}} \alpha^n(PQ) = \{1\}$ indeed and thus α is expansive.

Final assertion. If (c) holds, then β is a Lie algebra automorphism of $L(G)$ and $|\lambda| \neq 1$ for all eigenvalues λ of $\beta \otimes_{\mathbb{K}} \mathrm{id}_{\overline{\mathbb{K}}}$ in $\overline{\mathbb{K}}$, entailing that none of the λ is a root of unity. Hence $L(G)$ is nilpotent (see [3, Exercise 21 (b) among the exercises for Section 4 of Part I] or [12, Theorem 2]). □

7.2. For each continuous homomorphism $\theta\colon \mathbb{Q}_p \to \mathrm{GL}_n(\mathbb{Q}_p)$, there exists a nilpotent $n\times n$-matrix $x \in \mathbb{Q}_p^{n\times n}$ such that $\theta(t) = \exp(tx)$ for all $t \in \mathbb{Q}_p$, using the matrix exponential function [19, Theorem 1.1]. Thus $\theta'(0) = x$ uniquely determines θ, and so does $\theta|_W$ for any 0-neighbourhood $W \subseteq \mathbb{Q}_p$.

Lemma 7.3. *Let α be a contractive automorphism of a p-adic Lie group G. Then the following holds:*

(a) *For each $g \in G$, there is a unique continuous homomorphism $\theta_g : \mathbb{Q}_p \to G$ such that $\theta_g(1) = g$. Moreover, $\{\theta_g'(0) : g \in G\} = L(G)$.*

(b) *If $\mathfrak{h} \subseteq L(G)$ is an $L(\alpha)$-stable Lie subalgebra, then there exists an α-stable Lie subgroup H of G with $L(H) = \mathfrak{h}$.*

Proof. (a) Let $*\colon L(G) \times L(G) \to L(G)$ be the Campbell–Hausdorff multiplication on the nilpotent Lie algebra $L(G)$. Because (G,α) and $((L(G),*), L(\alpha))$ are locally isomorphic contraction groups, they are isomorphic (see [26, Proposition 2.2]). The nilpotent group $(L(G),*)$ inherits unique divisibility from the group $(L(G),+)$, since ng (in the vector space $L(G)$) coincides with g^n (in $(L(G),*)$). It is clear from this that $\theta_g(t) = tg$ is the unique continuous homomorphism $\mathbb{Q}_p \to (L(G),*)$ with $\theta_g(1) = g$. It satisfies $g = \theta_g'(0)$.

(b) We may work with the isomorphic group $(L(G),*)$ instead of G. Now $H := \mathfrak{h}$ is an $L(\alpha)$-stable Lie subgroup of $(L(G),*)$ with Lie algebra $\mathfrak{h}$. □

Lemma 7.4. *Let G be a linear p-adic Lie group. Assume that G is generated by $\bigcup_{\theta\in\Theta} \theta(\mathbb{Q}_p)$ for a set Θ of continuous homomorphisms $\theta\colon \mathbb{Q}_p \to G$, and $L(G)$ is generated by $\{\theta'(0) : \theta \in \Theta\}$ as a Lie algebra. Then the centre of G coincides with the kernel of* $\mathrm{Ad}\colon G \to \mathrm{Aut}(L(G))$.

Proof. Let $g \in G$. For each $\theta \in \Theta$, the map $I_g \circ \theta\colon \mathbb{Q}_p \to G$, $t \mapsto g\theta(t)g^{-1}$ is a continuous homomorphism such that $(I_g \circ \theta)'(0) = \mathrm{Ad}(g)\theta'(0)$. Thus, by 7.2, $I_g \circ \theta = \theta$ if and only if $\mathrm{Ad}(g)\theta'(0) = \theta'(0)$. Since $\bigcup_{\theta\in\Theta} \theta(\mathbb{Q}_p)$ generates G, we see that $g \in Z(G)$ if and only if $\mathrm{Ad}(g)\theta'(0) = \theta'(0)$ for all $\theta \in \Theta$. The latter is equivalent to $\mathrm{Ad}(g)(x) = x$ for all $x \in L(G)$, since $\{x \in L(G) : \mathrm{Ad}(g)(x) = x\}$ is a Lie subalgebra of $L(G)$ and $L(G)$ is generated by $\theta'(0)$ for $\theta \in \Theta$ by hypothesis. □

Proof of Theorem D. After replacing G with an open subgroup, we may assume that G is generated by $U_\alpha \cup U_{\alpha^{-1}}$ (see Lemma 1.1 (d)). We prove that G is nilpotent in this case, by induction on the dimension $\dim(G)$ of G as a p-adic manifold. If $\dim(G) = 0$, then G is discrete, whence $U_\alpha = U_{\alpha^{-1}} = \{1\}$ and the group $G = \langle U_\alpha \cup U_{\alpha^{-1}} \rangle = \{1\}$ is nilpotent.

Now assume that $\dim(G) > 0$. After replacing G with an isomorphic group, we may assume that G is a subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ for some $n \in \mathbb{N}$, and that the inclusion map $G \to \mathrm{GL}_n(\mathbb{Q}_p)$ is continuous (but not necessarily a homeomorphism onto its image). Then $L(G)$ is a non-zero nilpotent Lie algebra (see (a) $\Rightarrow$ (d) in Proposition 7.1) and so it has centre $Z(L(G)) \neq \{0\}$. The centre is $L(\alpha)$-stable, and the restriction β of $L(\alpha)$ to the centre is expansive (like $L(\alpha)$). Hence $Z(L(G)) = U_\beta \oplus U_{\beta^{-1}}$. After replacing α with α^{-1} if necessary, we may assume that $U_{\beta^{-1}} \neq \{0\}$. According to Lemma 7.3 (b), there is an α-stable Lie subgroup $H \subseteq U_{\alpha^{-1}}$ with $L(H) = U_{\beta^{-1}}$. We claim that H is in the centre $Z(G)$ of G. If this is true, then $Z(G)$ has positive dimension. Thus $G/Z(G)$ is a Lie group of dimension $\dim(G/Z(G)) < \dim(G)$, and it is a linear Lie group as it injects into $\mathrm{Aut}(L(G))$, by Lemma 7.4.[3] By induction, $G/Z(G)$ is nilpotent and hence so is G.

To prove the claim, let $h \in H$ and let $\theta\colon \mathbb{Q}_p \to H \subseteq U_{\alpha^{-1}}$ be a continuous homomorphism with $\theta(1) = h$ (see Lemma 7.3 (a)). Then $x := \theta'(0) \in L(H) \subseteq Z(L(G))$, entailing that $\mathrm{ad}(x) := [x, \bullet] = 0$. Now $\theta(t) = \exp(tx)$ for all $t \in \mathbb{Q}_p$, by 7.2. For $|t|$ small, we have $\mathrm{Ad}(\theta(t)) = \mathrm{Ad}(\exp(tx)) = e^{t\,\mathrm{ad}(x)} = \mathrm{id}_{L(G)}$, using [3, Chapter III, Section 4, no. 4, Corollary 3]). Thus $\mathrm{Ad} \circ \theta = \mathrm{id}_{L(G)}$, by the uniqueness assertion of 7.2, applied to $\mathrm{Ad} \circ \theta\colon \mathbb{Q}_p \to \mathrm{Aut}(L(G))$. In particular, $\mathrm{Ad}(h) = \mathrm{Ad}(\theta(1)) = \mathrm{id}_{L(G)}$ and thus $h \in Z(G)$, by Lemma 7.4. □

7.5. If G is a totally disconnected, locally compact group which is a nilpotent group, let $\{1\} = Z_0 \lhd Z_1 \lhd \cdots \lhd Z_n = G$ be its ascending central series defined recursively via $Z_k := q_k^{-1}(Z(G/Z_{k-1}))$, where $q_k\colon G \to G/Z_{k-1}$ is the canonical quotient morphism. Let α be an expansive automorphism of G and α_k the induced automorphism of G_k/G_{k-1}.

Proposition 7.6. *If $Z_k/Z_{k-1} = U_{\alpha_k} U_{\alpha_k^{-1}}$ for all $k \in \{1, \ldots, n\}$ in the situation of 7.5, then $G = U_\alpha U_{\alpha^{-1}}$. In particular, $U_\alpha U_{\alpha^{-1}}$ is a subgroup of G.*

Proof. If $n = 0$, then $G = \{1\} = U_\alpha U_{\alpha^{-1}}$. If $n \geq 1$, let β be the expansive automorphism of $G/Z(G)$ induced by α, and let $q\colon G \to G/Z(G)$ be the canoni-

[3] Let Θ be the set of continuous homomorphisms from $\mathbb{Q}_p$ to U_α or $U_{\alpha^{-1}}$. By $G = \langle U_\alpha \cup U_{\alpha^{-1}} \rangle$ and Lemma 7.3 (a), the first hypothesis of Lemma 7.4 is satisfied. Since $L(G) = L(U_\alpha) + L(U_{\alpha^{-1}})$ and $L(U_\alpha) \cup L(U_{\alpha^{-1}}) = \{\theta'(0) : \theta \in \Theta\}$ by Lemma 7.3 (a) and "(a) $\Rightarrow$ (c)" in Lemma 7.1, also the second hypothesis of Lemma 7.4 is satisfied.

cal quotient morphism. Then $Z_1 = Z(G) = (U_\alpha \cap Z(G))(U_{\alpha^{-1}} \cap Z(G))$ by the hypotheses and $G/Z(G) = U_\beta U_{\beta^{-1}}$ by induction. Since

$$q(U_\alpha U_{\alpha^{-1}}) = U_\beta U_{\beta^{-1}} = G/Z(G),$$

we have

$$\begin{aligned} G &= U_\alpha U_{\alpha^{-1}} Z(G) \\ &= U_\alpha U_{\alpha^{-1}} (U_\alpha \cap Z(G))(U_{\alpha^{-1}} \cap Z(G)) \\ &= U_\alpha (U_\alpha \cap Z(G)) U_{\alpha^{-1}} (U_{\alpha^{-1}} \cap Z(G)) \\ &= U_\alpha U_{\alpha^{-1}}. \end{aligned}$$

□

Note that we can easily achieve that $G/Z_{n-1} = U_{\alpha_n} U_{\alpha_n^{-1}}$ after replacing G with its open subgroup generated by $U_\alpha \cup U_{\alpha^{-1}}$. However, the hypotheses on Z_k/Z_{k-1} for $k < n$ cannot always be achieved by passing to an open subgroup (as the following example illustrates).

Remark 7.7. The following example shows that even for nilpotent p-adic Lie groups with an expansive automorphism α, the set $U_\alpha U_{\alpha^{-1}}$ may fail to be a subgroup. The example also provides a p-adic Lie group that admits expansive automorphisms but does not admit any contractive automorphism. In fact, the group has a closed discrete commutator group which is characteristic and hence would inherit a contractive automorphism (contradicting the fact that non-trivial contraction groups are non-discrete).

Let $H = \mathbb{Q}_p^3$ be the 3-dimensional p-adic Heisenberg group whose binary operation is given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in H$. Let $N = \{(0, 0, z) \in H : |z| \leq 1\}$. Then N is a compact central subgroup of H. Identify $G = H/N$ with $\mathbb{Q}_p \times \mathbb{Q}_p \times (\mathbb{Q}_p/\mathbb{Z}_p)$ as a set. Define $\alpha: G \to G$ by

$$\alpha(x, y, z + \mathbb{Z}_p) = (px, p^{-1} y, z + \mathbb{Z}_p)$$

for all $(x, y, z + \mathbb{Z}_p) \in G$. Then α is a continuous automorphism of the p-adic Lie group G with $M_\alpha = \{(0, 0, z + \mathbb{Z}_p) : z \in \mathbb{Q}_p\}$, $U_\alpha = \{(x, 0, 0) : x \in \mathbb{Q}_p\}$ and $U_{\alpha^{-1}} = \{(0, y, 0) : y \in \mathbb{Q}_p\}$. Since M_α is discrete, it follows that α is an expansive automorphism. As $[U_\alpha, U_{\alpha^{-1}}] = \{(0, 0, z + \mathbb{Z}_p) : z \in \mathbb{Q}_p\}$ and $U_\alpha U_{\alpha^{-1}} = \{(x, y, xy + \mathbb{Z}_p) : x, y \in \mathbb{Q}_p\}$, we get that $U_\alpha U_{\alpha^{-1}}$ is a not a subgroup.

Proposition 7.8. *Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$ and α an expansive automorphism of G. Then $U_\alpha U_{\alpha^{-1}}$ is an open (unipotent algebraic) subgroup of G.*

Proof. Replacing G by the group generated by U_α and $U_{\alpha^{-1}}$, we may assume by Theorem E that G is a closed nilpotent subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. Let $\mathbb{G}$ be the Zariski closure of G. Then $\mathbb{G}$ is defined over $\mathbb{Q}_p$ and $\mathbb{G}$ is nilpotent (cf. [2, Proposition 1.3 (b) and Corollary 1 in Section 2.4]). Since U_α and $U_{\alpha^{-1}}$ consists of one-parameter (unipotent) subgroups, $\mathbb{G}$ is Zariski-connected. This implies that the set of unipotent elements form a subgroup $\mathbb{G}_u$, known as the unipotent radical (cf. [2, Theorem 10.6]). Since U_α and $U_{\alpha^{-1}}$ consists of one-parameter (unipotent) subgroups, $U_\alpha, U_{\alpha^{-1}} \subseteq \mathbb{G}_u$. This implies that $\mathbb{G} = \mathbb{G}_u$, that is $\mathbb{G}$ is an unipotent algebraic group, hence $\mathbb{G}$ is defined over $\mathbb{Q}_p$ (cf. [2, Section 4.5] and the fact that $\mathbb{Q}_p$-closed and defined over $\mathbb{Q}_p$ are same as characteristic of $\mathbb{Q}_p$ is zero) and $G \subseteq \mathbb{G}(\mathbb{Q}_p)$.

For $i \geq 1$, let $D_i = [\mathbb{G}(\mathbb{Q}_p), D_{i-1}]$ with $D_0 = \mathbb{G}(\mathbb{Q}_p)$ and $G_i = \overline{[G, G_{i-1}]}$ with $G_0 = G$. Then D_{k+1} is trivial for some $k \geq 1$ as $\mathbb{G}$ is unipotent and $G_i \subset D_i$. Thus, G_k is a closed α-stable subgroup of D_k which is a vector space. Let V be the maximal vector subspace of G_k. Then V is a closed α-stable central subgroup of G. The automorphism $\beta\colon G_k/V \to G_k/V$ defined by $\beta(x + V) = \alpha(x) + V$ for $x \in G_k$ is expansive. Since V is the maximal vector subspace of G_k and G_k is a closed subgroup of the p-adic vector space D_k, we get that G_k/V is a compact subgroup of the p-adic vector space D_k/V. Since the automorphism group of a compact p-adic analytic group is compact, compact p-adic analytic groups do not admit expansive automorphisms unless finite, hence $V = G_k$. This implies that $G_k = D_k$ and $G_k = V = (U_\alpha \cap V)(U_{\alpha^{-1}} \cap V)$. Since G/G_k is a closed subgroup of $\mathbb{G}(\mathbb{Q}_p)/D_k$ which is a linear (p-adic algebraic) group, the result follows by induction. □

Remark 7.9. In the case of linear p-adic Lie groups, even if $U_\alpha U_{\alpha^{-1}}$ is an open subgroup for an expansive automorphism, the following example shows that it is not possible to have either of U_α or $U_{\alpha^{-1}}$ to normalize the other.

Let H be the 3-dimensional p-adic Heisenberg group defined as in Remark 7.7. For $i = 1, 2$, define $\alpha_i\colon H \to H$ by

$$\alpha_1(x, y, z) = (px, p^{-2}y, p^{-1}z), \quad \alpha_2(x, y, z) = (p^2x, p^{-1}y, pz)$$

for $(x, y, z) \in H$. Let $G = H \times H$ and $\alpha = \alpha_1 \times \alpha_2$. Then

$$U_\alpha = \{(x, 0, 0) : x \in \mathbb{Q}_p\} \times \{(a, 0, c) : a, c \in \mathbb{Q}_p\}$$

and

$$U_{\alpha^{-1}} = \{(0, y, z) : y, z \in \mathbb{Q}_p\} \times \{(0, b, 0) \mid b \in \mathbb{Q}_p\}.$$

Thus $U_\alpha U_{\alpha^{-1}} = G$. Since $\{(x, 0, 0) : x \in \mathbb{Q}_p\}$ and $\{(0, y, 0) : y \in \mathbb{Q}_p\}$ are not normal subgroups of H, neither U_α or $U_{\alpha^{-1}}$ normalize the other.

8 Example: Baumslag–Solitar groups

Throughout this section, we fix primes $p \neq q$. We let

$$\mathrm{BS}(p,q) := \langle a, t \mid ta^p t^{-1} = a^q \rangle$$

be the Baumslag–Solitar group. Then $\langle a \rangle \cap g\langle a \rangle g^{-1}$ has finite index in $\langle a \rangle$ for each $g \in \mathrm{BS}(p,q)$, and $\bigcap_g g\langle a \rangle g^{-1} = \{1\}$, hence the Schlichting completion $G_{p,q}$ of $\mathrm{BS}(p,q)$ can be formed, which is a certain totally disconnected, locally compact group in which $\mathrm{BS}(p,q)$ is dense, and in which $K := \overline{\langle a \rangle}$ is a compact open subgroup (see [4], cf. [9, 21]). We are interested in the inner automorphism

$$\alpha \colon G_{p,q} \to G_{p,q}, \quad x \mapsto txt^{-1}.$$

Proof of Theorem E. By [4, Proposition 8.1], K contains an open subgroup $V \cong \mathbb{Z}_p \times \mathbb{Z}_q$ and K/V is a cyclic group of order dividing $\gcd(p,q) = 1$. Thus $K = V \cong \mathbb{Z}_p \times \mathbb{Z}_q$. After multiplication with a unit, we may assume that the isomorphism takes a to $(1,1)$.

Let $G = \mathbb{Z} \ltimes (\mathbb{Q}_p \times \mathbb{Q}_q)$ be the semidirect product of $\mathbb{Z}$ and $\mathbb{Q}_p \times \mathbb{Q}_q$ given by

$$(n,u,v)(m,u',v') = (n+m, u + (q/p)^n u', v + (q/p)^n v')$$

for all $n, m \in \mathbb{Z}$, $u, u' \in \mathbb{Q}_p$ and $v, v' \in \mathbb{Q}_q$. The isomorphism $K \cong \mathbb{Z}_p \times \mathbb{Z}_q$ gives a homomorphism from $\langle a \rangle$ to G. Since

$$(1,0,0)(0,p,p)(-1,0,0) = (0,q,q),$$

sending $t \mapsto (1,0,0)$ yields a group homomorphism $\phi \colon \mathrm{BS}(p,q) \to G$. Since $\phi|_{\langle a \rangle}$ is a continuous homomorphism, ϕ extends to a continuous homomorphism of $G_{p,q}$ into G which would also be denoted by ϕ. Since $\phi|_K$ is an isomorphism, $\ker(\phi)$ is discrete. Moreover, as $\phi(G_{p,q})$ contains both $(1,0,0)$ and $\mathbb{Z}_p \times \mathbb{Z}_q$, it follows that ϕ is surjective.

Let β be the inner automorphism of G given by $(1,0,0)$. Then $\phi \circ \alpha = \beta \circ \phi$ and β is expansive. Since the kernel of ϕ is discrete, expansiveness of α follows from Theorem A. As the open subgroup $K \cong \mathbb{Z}_p \times \mathbb{Z}_q$ satisfies an ascending chain condition on closed subgroups (see, e.g., [6, Proposition 3.2]), U_α is closed by [26, Lemma 3.2].

In case $U_\alpha U_{\alpha^{-1}}$ is a group, we will now show that ϕ is an isomorphism which would lead to a contradiction as $G_{p,q}$ is not solvable but G is solvable.[4] Suppose $N := U_\alpha U_{\alpha^{-1}}$ is a group. Then $\phi|_N$ is an isomorphism of N with $\mathbb{Q}_p \times \mathbb{Q}_q$ (using the fact that $U_\alpha \cong \mathbb{Q}_q$ and $U_{\alpha^{-1}} \cong \mathbb{Q}_p$).[5] Now the group generated by t and N

[4] Alternatively, $\mathrm{BS}(p,q)$ would be a finitely generated linear group then and hence residually finite by [16]. But $\mathrm{BS}(p,q)$ is not residually finite for primes $p \neq q$, see [18].

[5] In fact, $V = K$ is tidy for α with $V_- \cong \mathbb{Z}_q$, $V_+ \cong \mathbb{Z}_p$ and $V_0 = \{1\}$ (see [4]), whence we have $U_\alpha = V_{--} \cong \mathbb{Q}_q$ and $U_{\alpha^{-1}} = V_{++} \cong \mathbb{Q}_p$ (cf. Lemma 1.1 (b)).

is an open subgroup of G containing both t and a, hence $G_{p,q} = \langle t, N\rangle = \langle t\rangle N$ (as t normalizes N). This implies that ϕ is an isomorphism. □

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