J. Group Theory **20** (2017), 169–183
DOI 10.1515/jgth-2016-0032

Groups with maximal subgroups of Sylow subgroups σ-permutably embedded

Wenbin Guo and Alexander N. Skiba

Communicated by Alexander Olshanskii

Abstract. Let $\sigma = \{\sigma_i : i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, G a finite group and $\sigma(G) = \{\sigma_i : \sigma_i \cap \pi(G) \neq \emptyset\}$. A set $1 \in \mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every non-identity group in $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma(G)$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. A subgroup H of G is called *σ-permutable* (resp. *σ-permutably embedded*) in G if G possesses a complete Hall σ-set $\mathcal{H} = \{1, H_1, \dots, H_t\}$ such that $AH_i^x = H_i^x A$ for any i and all $x \in G$ (resp. if H has a complete Hall σ-set and every Hall σ_i-subgroup of H is also a Hall σ_i-subgroup of some σ-permutable subgroup of G).

In this paper, we classify the finite groups G such that either every maximal subgroup of every Sylow subgroup of G is σ-permutable in G or every maximal subgroup of every Sylow subgroup of G is σ-permutably embedded in G.

1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i : i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; Π is always supposed to be a non-empty subset of the set σ and Π' denotes $\sigma \setminus \Pi$. A natural number n is said to be a Π-*number* if $\pi(n) \subseteq \bigcup_{\sigma_i \in \Pi} \sigma_i$.

We write $\sigma(G) = \{\sigma_i : \sigma_i \cap \pi(G) \neq \emptyset\}$, and say that G is *σ-primary* [16] provided $|\sigma(G)| \leq 1$,

A subgroup H of G is said to be a: Π-*subgroup of* G if $|H|$ is a Π-number; *Hall* Π-*subgroup* of G if H is a Π-subgroup of G and $|G : H|$ is a Π'-number; σ-Hall subgroup of G if H is a Hall Π-subgroup of G for some $\Pi \subseteq \sigma$.

A set $\mathcal{H}$ of subgroups of G with $1 \in \mathcal{H}$ is said to be a *complete Hall σ-set* of G (see [8, 17]) if every non-identity group in $\mathcal{H}$ is a Hall σ_i-subgroup of G for

Research is supported by an NNSF grant of China (grant no. 11371335) and Wu Wen-Tsun Key Laboratory of Mathematics of Chinese Academy of Sciences.

some σ_i and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. We say also, following [8], that G is *σ-full* if G possesses a complete Hall σ-set.

A subgroup H of G is said to be *σ-quasinormal* or *σ-permutable* in G (see [8]) if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^x H$ for all members $A \in \mathcal{H}$ and all $x \in G$. In particular, a subgroup H of G is said to be *$\pi(G)$-quasinormal* (Kegel [12]), *S-quasinormal* or *S-permutable* in G if HP is equal to PH for all Sylow subgroups P of G.

A subgroup H of G is said to be *S-quasinormally embedded* or *S-permutably embedded* in G (Ballester-Bolinches, M.C. Pedraza-Aguilera [3]) if, for every $p \in \pi(H)$, every Sylow p-subgroup of H is also a Sylow p-subgroup of some S-permutable subgroup of G.

In the general situation, we say the following:

Definition 1.1. A subgroup H of G is said to be *σ-quasinormally embedded* or *σ-permutably embedded* in G if H is σ-full and, for every $\sigma_i \in \sigma(H)$, every Hall σ_i-subgroup of H is also a Hall σ_i-subgroup of some σ-permutable subgroup of G.

We show (see Corollary 1.5 below) that the class of soluble groups G in which σ-permutability is a transitive relation on G (that is, every σ-permutable subgroup of a σ-permutable subgroup of G is σ-permutable in G) coincides with the class of groups in which every subgroup is σ-quasinormally embedded.

Before continuing, consider the following elementary examples.

Example 1.2. (i) G is said to be *σ-nilpotent* [7] if $G = H_1 \times \cdots \times H_t$, where $\{1, H_1, \ldots, H_t\}$ is a complete Hall σ-set of G. It is clear that every subgroup of a σ-nilpotent group is σ-permutable.

(ii) Let $p > q > r$ be primes, where q divides $p-1$ and r divides $q-1$. Let $H = Q \rtimes R$ be a non-abelian group of order qr, P a simple $\mathbb{F}_p H$-module which is faithful for H, and $G = P \rtimes H$.

Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{p, r\}$ and $\sigma_2 = \{p, r\}'$. Then G is not σ-nilpotent and $|P| > p$. Since q divides $p-1$, PQ is supersoluble. Hence for some normal subgroup L of PQ we have $1 < L < P$. Then for every Hall σ_1-subgroup V of G we have $L \leq P \leq V$, so $LV = V = VL$. On the other hand, for every Hall σ_2-subgroup Q^x of G we have $Q^x \leq PQ$, so $LQ^x = Q^x L$. Hence L is σ-permutable in G. It is also clear that L is not normal in G, and so $LR \neq RL$, which implies that L is not S-permutable in G.

(iii) Let $p > q > r$ be primes, C_q a group of order q and $H = R \rtimes C_q$, where R is a simple $\mathbb{F}_r C_q$-module which is faithful for C_q. Let $G = P \rtimes H$, where P is a simple $\mathbb{F}_p H$-module which is faithful for H. Let A be a subgroup of R of order r and $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{p\}$ and $\sigma_2 = \{p\}'$. Then $\mathcal{H} = \{1, P, H\}$

is a complete Hall σ-set of G, so PA is σ-permutable subgroup of G. This means that A is σ-permutably embedded in G. Since $C_G(P) = P$, $H_G = 1$ and so A is not σ-permutable in G by [16, Theorem B and Lemma 2.6 (7)]. It is clear that PC_q is a maximal subgroup of G. Therefore A is not S-permutably embedded in G, otherwise, it is not difficult to show that $PC_q < PAC_q < G$, which contradicts the maximality of PC_q.

The S-permutable and generalized S-permutable subgroups (in particular, S-permutably embedded subgroups) have been investigated by many authors and they have many applications (see, for example, the books [2, 6, 21]) and the recent papers [11, 13–15].

The properties of σ-permutable subgroups were analyzed in the paper [8]. Our main goal here is to prove the following:

Theorem A. *Every maximal subgroup of every Sylow subgroup of G is σ-permutably embedded in G if and only if $G = D \rtimes M$, where D and M are σ-Hall subgroups of G, $D = G^{\mathfrak{N}_\sigma}$ is nilpotent of odd order and every element of M induces a power automorphism on $D/\Phi(D)$.*

In this theorem, $G^{\mathfrak{N}_\sigma}$ denotes the *σ-nilpotent residual of G*, that is, the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N.

Corollary 1.3 (Ballester-Bolinches, Pedraza-Aguilera [3]). *If every maximal subgroup of every Sylow subgroup of G is S-permutably embedded in G, then G is supersoluble.*

Corollary 1.4 (Srinivasan [18]). *If every maximal subgroup of every Sylow subgroup of G is S-permutable in G, then G is supersoluble.*

Recall that a subgroup A of G is called *σ-subnormal* in G (see [16]) if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either A_{i-1} is normal in A_i or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all i with $i = 1, \ldots, t$.

The following result shows that, in view of [16, Theorem A], the class of all groups in which every σ-subnormal subgroup is σ-permutable coincides with the class of all groups in which each subgroup is σ-permutably embedded.

Corollary 1.5. *Every subgroup of G is σ-permutably embedded in G if and only if $G = D \rtimes M$, where D and M are σ-Hall subgroups of G, $D = G^{\mathfrak{N}_\sigma}$ is abelian of odd order and every element of M induces a power automorphism on D.*

Recall that a group G is called a *PST-group* if S-permutability is a transitive relation on G. By the well-known Agrawal theorem, a soluble group G is a PST-group G if and only if $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is a Hall abelian subgroup of G of odd order and every element of M induces a power automorphism on D (see [1] or [2, Chapter 2]). Hence the following result follows from Theorem A.

Corollary 1.6. *The group G is a soluble PST-group if and only if every subgroup of G is S-permutably embedded in G.*

A chief factor H/K is said to be *σ-central* (in G) if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary. We use $Z_\sigma(G)$ to denote the *σ-hypercentre* of G, that is, the product of all normal subgroups N of G such that every chief factor of G below N is σ-central in G.

As another application of Theorem A, we also prove the following:

Theorem B. *Every maximal subgroup of every Sylow subgroup of G is σ-permutable in G if and only if $G = A \rtimes (B \rtimes C)$, where*

(i) *A and BC are σ-Hall subgroups and C is a Hall subgroup of G,*

(ii) *A is a normal nilpotent subgroup of G of odd order, $B \leq Z_\sigma(G)$ is a normal subgroup of G and C is a σ-nilpotent subgroup of G all of whose Sylow subgroups are cyclic,*

(iii) *the generators of Sylow subgroups of C induce power automorphisms on $A/\Phi(A)$ and automorphisms of order dividing a prime on A,*

(iv) *$[V, a] = 1$ for each σ_i'-element $a \in G$, where V is the maximal subgroup of a Sylow p-subgroup $P \leq C$ and $p \in \sigma_i$.*

Example 1.7. Let $G = (C_7 \rtimes \mathrm{Aut}(C_7)) \times A_5$, where C_7 is a group of order 7 and A_5 is the alternating group of degree 5. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{2, 3, 5\}$ and $\sigma_2 = \{2, 3, 5\}'$. Then G is the group in Theorem B, where $A = C_7$, $B = A_5$ and $C = \mathrm{Aut}(C_7)$.

Corollary 1.8 (Walls [20]). *Every maximal subgroup of every Sylow subgroup of G is normal in G if and only if $G = H \rtimes \langle x \rangle$, where*

(i) *H is a normal nilpotent Hall subgroup of G,*

(ii) *the generators of Sylow subgroups of $\langle x \rangle$ induce power automorphisms on $H/\Phi(H)$ and automorphisms of order dividing a prime on H.*

2 Preliminaries

Recall that G is called a D_π-group if G has a Hall π-subgroup E and every π-subgroup of G is contained in some conjugate of E.

The group G is said to be σ-*soluble* [16] if every chief factor of G is σ-primary. In view of Theorem B in [17], every σ-soluble group is a σ-full group of *Sylow type* [16], that is, every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \sigma$. Note also that if $\sigma = \{\{2\}, \{3\}, \dots\}$ is the finest partition of $\mathbb{P}$, then any group G is σ-full group of Sylow type by the classical Sylow theorem.

Lemma 2.1 (see [16, Lemmas 2.8 and 3.2]). *Let H and $R \leq K$ be subgroups of G, where H is σ-permutable in G and R is normal in G.*

(1) *The subgroup HR/R is σ-permutable in G/R.*

(2) *If G is a σ-full group of Sylow type, then $H \cap K$ is σ-permutable in K.*

(3) *If G is a σ-full group of Sylow type and K/R is σ-permutable in G/R, then K is σ-permutable in G.*

Lemma 2.2. *Let $H \leq E$ and R be subgroups of G, where H is σ-permutably embedded in G and R is normal in G.*

(1) *The subgroup HR/R is σ-permutably embedded in G/R.*

(2) *If G is a σ-full group of Sylow type, then H is σ-permutably embedded in E.*

Proof. Let $\{1, H_1, \dots, H_t\}$ be a complete Hall σ-set of H. Let W be a σ-permutable subgroup of G such that H_i is a Hall σ_i-subgroup of W.

(1) Since H_i is a Hall σ_i-subgroup of H, we conclude that $|HR : RH_i| = |H : H_i||R \cap H_i| : |R \cap H|$ is a σ_i'-number. Hence RH_i/R is a Hall σ_i-subgroup of RH/R. Then $\{R, H_1R/R, \dots, H_tR/R\}$ is a complete Hall σ-set of HR/R. Similarly, RH_i/R is a Hall σ_i-subgroup of RW/R, where WR/R is a σ-permutable subgroup of G/R by Lemma 2.1 (1). Hence HR/R is σ-permutably embedded in G/R.

(2) By Lemma 2.1 (2), $W \cap E$ is σ-permutable in E and $H_i \leq W \cap E$, so H_i is a Hall σ_i-subgroup of $W \cap E$. The lemma is proved. □

We use $O^{\Pi}(G)$ to denote the subgroup of G generated by all its Π'-subgroups; $O_{\Pi}(G)$ to denote the subgroup of G generated by all its normal Π-subgroups.

Lemma 2.3. *Let N be a normal σ_i-subgroup of G. Then $N \leq Z_\sigma(G)$ if and only if $O^{\sigma_i}(G) \leq C_G(N)$.*

Proof. If $O^{\sigma_i}(G) \leq C_G(N)$, then, for every chief factor H/K of G below N, both H/K and $G/C_G(H/K)$ are σ_i-groups since $G/O^{\sigma_i}(G)$ is a σ_i-group, so $N \leq Z_\sigma(G)$.

Now assume that $N \leq Z_\sigma(G)$. Let $1 = Z_0 < Z_1 < \cdots < Z_t = N$ be a chief series of G below N and $C_i = C_G(Z_i/Z_{i-1})$. Let $C = C_1 \cap \cdots \cap C_t$. Then G/C is a σ_i-group. On the other hand, $C/C_G(N) \simeq A \leq Aut(N)$ stabilizes the series $1 = Z_0 < Z_1 < \cdots < Z_t = N$, so $C/C_G(N)$ is a $\pi(N)$-group by [5, Theorem 0.1]. Hence $G/C_G(N)$ is a σ_i-group, and so $O^{\sigma_i}(G) \leq C_G(N)$. The lemma is proved. □

Lemma 2.4 (see [16, Lemma 2.6]). *Let A and K be subgroups of a σ-full group G. Suppose that A is σ-subnormal in G. Then:*

(1) *$A \cap K$ is σ-subnormal in K.*

(2) *If $K \leq A$ and A is σ-nilpotent, then K is σ-subnormal in G.*

(3) *If $H \neq 1$ is a Hall Π-subgroup of G and A is not a Π'-group, then $A \cap H \neq 1$ is a Hall Π-subgroup of A.*

Lemma 2.5 (see [16, Lemma 3.1]). *Let H be a σ_i-subgroup of a σ-full group G. Then H is σ-permutable in G if and only if $O^{\sigma_i}(G) \leq N_G(H)$.*

We call the product of all normal σ-nilpotent subgroups of G the *σ-Fitting subgroup* of G and denote it by $F_\sigma(G)$. We need the following facts on the subgroup $F_\sigma(G)$.

Lemma 2.6. *The following statements hold.*

(i) *$F_\sigma(G)$ is σ-nilpotent.*

(ii) *If A is a σ-subnormal subgroup of a σ-full group G and A is σ-nilpotent, then A is contained in $F_\sigma(G)$. Hence for any two σ-nilpotent σ-subnormal subgroups A and B of any σ-full group G, the subgroup $\langle A, B\rangle$ is σ-nilpotent and it is also σ-subnormal in G.*

Proof. (i) It is enough to prove that if $G = AB$, where A and B are normal σ-nilpotent subgroups of G, then G is σ-nilpotent. Moreover, in this case, in view of [16, Proposition 2.3], it is enough to show that every chief factor H/K of G is σ-central in G. Since the hypothesis holds for G/K, it is enough to consider the case when H is a minimal normal subgroup of G. Let $D = A \cap B$, H a σ_i-group and $C = C_G(H)$. If $H \leq D$, then $A/C \cap A \simeq AC/C$ and $B/C \cap B \simeq BC/C$ are σ_i-groups, so $G/C = (AC/C)(BC/C)$ is a σ_i-group. Finally, if $H \leq A$ and $H \nleq B$, then $B \leq C$ and as above we again obtain that G/C is a σ_i-group. This shows that $H/1$ is σ-central in G. Therefore in view of the Jordan–Hölder theorem, every chief factor of G is σ-central.

(ii) It is enough to consider the case when A is a σ_i-group for some $i \in I$. By hypothesis, G has a Hall σ_i-subgroup, say H. Then, by Lemma 2.4 (3), for every $x \in G$ we have $A \leq H^x$. Hence $A^G \leq H_G \leq F_\sigma(G)$. The second assertion of (ii) is a corollary of (i). □

Lemma 2.7 (see [16, Theorem B]). *Let H be a subgroup of a σ-full group G. If H is σ-permutable in G, then H is σ-subnormal in G and H/H_G is σ-nilpotent.*

Lemma 2.8. *Let H be a normal subgroup of G. If $H/H \cap \Phi(G)$ is a Π-group, then H has a Hall Π-subgroup, say E, and E is normal in G. Hence, if the subgroup $H/H \cap \Phi(G)$ is σ-nilpotent, then H is σ-nilpotent.*

Proof. Let $D = O_{\Pi'}(H)$. Since $H/(H \cap \Phi(G))$ is a Π-group, $D \leq H \cap \Phi(G)$. Now since $H \cap \Phi(G)$ is nilpotent, D is a Hall Π'-subgroup of H. By the Schur–Zassenhaus theorem, H has a Hall Π-subgroup, say E. It is clear that H is Π'-soluble, so any two Hall Π-subgroups of H are conjugate. Now by the Frattini argument, we have $G = HN_G(E) = (E(H \cap \Phi(G)))N_G(E) = N_G(E)$. Thus E is normal in G. The lemma is proved. □

3 Proofs of Theorems A and B

Let $\sigma^0 = \{\sigma_j^0 : j \in J\}$ be a partition of $\mathbb{P}$. Then we write $\sigma^0 \leq \sigma$ provided for each $j \in J$ there is $i \in I$ such that $\sigma_j^0 \subseteq \sigma_i$.

The proof of Theorem A consists of many steps and one of them is based on the following useful fact.

Theorem 3.1. *Let σ^0 be a partition of $\mathbb{P}$ such that $\sigma^0 \leq \sigma$. Suppose that G has a complete Hall σ_0-set $\mathcal{H}_0 = \{1, H_1, \dots, H_t\}$ such that every maximal subgroup of every member of $\mathcal{H}_0$ is σ-permutably embedded in G. If G is a σ^0-full group of Sylow type, then G is σ-soluble.*

Proof. Suppose that this theorem is false and let G be a counterexample of minimal order. Then $t > 1$ since $\sigma^0 \leq \sigma$. Let $\sigma^0 = \{\sigma_i^0 : i \in J \subseteq \mathbb{N}\}$. We can assume without loss of generality that H_i is a σ_i^0-group for all $i = 1, \dots, t$. Assume also that H_i is a σ_{j_i}-group for all $i = 1, \dots, t$.

Let R be a minimal normal subgroup of G. Then the hypothesis holds for G/R by Lemma 2.2 (1), so G/R is σ-soluble by the choice of G. It is easy to see that the class of all σ-soluble groups is closed under taking direct products, homomorphic images and subgroups, and that the extension of a σ-soluble group by a σ-soluble group is a σ-soluble group. Hence R is the unique minimal normal subgroup of G and R is not σ-primary.

(a) *For all i, either $H_i \leq R$ or $R \cap H_i \leq \Phi(H_i)$.*

Indeed, suppose that $H_i \nleq R$. Assume that $R \cap H_i \nleq \Phi(H_i)$. Let V be a maximal subgroup of H_i such that $R \cap H_i \nleq V$. Then $H_i = (R \cap H_i)V$. Let W be a σ-permutable subgroup of G such that V is a Hall σ_{j_i}-subgroup of W. If $W_G = 1$, then $W \simeq W/W_G$ is σ-nilpotent and so $1 < W \leq F_\sigma(G)$ by Lemmas 2.6 and 2.7. But then $R \leq F_\sigma(G)$, so R is σ-primary. This contradiction shows that $W_G \neq 1$. Hence $R \leq W$ and so $H_i = (R \cap H_i)V \leq W$, which implies that $|H_i| = |V|$, a contradiction. Hence we have (a).

(b) *If $H_i \leq R$, then H_i is of prime order. Hence a Sylow 2-subgroup G_2 of G is not contained in R.*

Let V be a maximal subgroup of H_i and W a σ-permutable subgroup of G such that V is a Hall σ_{j_i}-subgroup of W. Then $R \nleq W$. Hence $W_G = 1$ and so W is σ-subnormal in G and W is σ-nilpotent by Lemma 2.7. But then V is σ-subnormal in G by Lemma 2.4 (2). Hence by Lemma 2.4 (3), $V \cap H^x = V$ and so $V \leq H^x$ for all $x \in G$. This implies that $V^G \leq H_i$. If $V \neq 1$, then $R \leq V^G \leq H_i$, a contradiction. Hence $V = 1$ and H_i is of prime order.

(c) *There exists i such that $H_i \nleq R$.* (Since R is not σ-primary, this follows from Claim (b) and [9, Chapter IV, Section 2.8].)

Without loss of generality we can assume that $H_i \nleq R$ for all $i = 1, \ldots, r$, and $H_j \leq R$ for all $j > r$. Let $\Pi = \{\sigma_1^0, \ldots, \sigma_r^0\}$ and $\pi = \bigcup_{\sigma_i \in \Pi} \sigma_i$.

(d) *Any supplement N to R in G possesses a σ-soluble Hall Π-subgroup L such that some conjugate of H_i is contained in L for all $i = 1, \ldots, r$. Hence $RL = G$ and 2 divides $|L|$.*

Let L be a minimal supplement to R in G contained in N. Then $L \cap R \leq \Phi(L)$, so L is σ-soluble since $G/R \simeq L/L \cap R$ is σ-soluble. Let $1 \leq j \leq r$. Then we have $H_j \nleq R$, so $\sigma_j^0 \in \sigma^0(L)$. Since G is a σ^0-full group of Sylow type, L possesses a Hall σ_j^0-subgroup L_j and for some $x \in G$ we have $L_j \leq H_j^x$. Suppose that $L_j < H_j^x$. Then $|H_j^x|$ is not prime, so $H_j^x \cap R \leq \Phi(H_j)$ by Claims (a) and (b). Since

$$H_j^x \cap L_j R = L_j(H_j^x \cap R) = (H_j^x \cap L)(H_j^x \cap R)$$

and, clearly, $|RL : (H_j^x \cap L)(H_j^x \cap R)|$ is a $(\sigma_j^0)'$-number, we have that

$$H_j^x = H_j^x \cap RL = (H_j^x \cap R)(H_j^x \cap L) = (H_j \cap R)^x L_j = L_j$$

since $(H_j \cap R)^x \leq \Phi(H_j^x)$. This contradiction shows that $L_j = H_j^x \leq L$. Since $L \cap R \leq \Phi(L)$, $\pi(L/L \cap R) = \pi(L)$. Therefore $\pi(L) = \pi$, so L is a Hall Π-subgroup of G and 2 divides $|L|$. Hence we have (d).

(e) *We have $r < t$, so R is a non-abelian simple group.* (This follows from Claim (d) and the choice of G.)

Let P be a Sylow 2-subgroup of R. Then P is not of prime order. Hence there is $x \in G$ such that $P \leq L^x$ and so there is a σ-soluble Hall Π-subgroup L of G such that $P \leq L$ by Claim (d).

(f) *The group R has a Hall $\{2, p\}$-subgroup for each p dividing $|R|$.*

First assume that $p \in \pi$ and let G_p be a Sylow p-subgroup of G. The Frattini argument implies that $G = RN_G(P)$, so $N_G(P)$ possesses a σ-soluble Π-Hall subgroup L such that some conjugate of H_i is contained in L for all $i = 1, \dots, r$ by Claim (d). Hence for some $x \in G$ we have $G_p^x \leq L \leq N_G(P)$. Then $PG_p^x \cap R = P(G_p^x \cap R)$ is a Hall $\{2, p\}$-subgroup of R. Now assume that $p \in \pi(H_i)$ for some $i > r$. Then H_i is a Sylow subgroup of R by Claims (a) and (b), and there is a Hall Π-subgroup L of G such that $L \leq N_G(H_i)$ by Claim (d). But for some x we have $P^x \leq L$, so $P^x \leq N_G(H_i)$ and hence $P^x H_i \cap R = P^x(H_i \cap R)$ is a Hall $\{2, p\}$-subgroup of R.

(g) *A Sylow 2-subgroup R_2 of R is non-abelian.*

Assume this is false. Then by Claims (e) and [10, Chapter XI, Theorem 13.7], R is isomorphic to one of the following groups:

(a) $\mathrm{PSL}(2, 2^f)$,

(b) $\mathrm{PSL}(2, q)$, where 8 divides $q - 3$ or $q - 5$,

(c) the Janko group J_1,

(d) a Ree group.

But with respect to each of these groups it is well known that the group has no Hall $\{2, q\}$-subgroup for at least one odd prime q dividing its order (see, for example [19, Theorem 1]), which contradicts (f). Hence we have (g).

(h) *If $k > r$ and H_k is a p-group, $p \neq 2$, then p does not divide $|R : N_R((P')|$. Hence $G = N_G((P')L$ for each Hall Π-subgroup L of G.*

Claim (d) implies that for some element $x \in G$ we have $P \leq N_G(H_k^x)$. Let $W = H_k^x \rtimes P$. Then from Claims (a) and (b) we have that $|H_k^x| = p$, and so $W/C_W(H_k^x) \simeq P/P \cap C_W(H_k^x)$ is abelian. Hence $H_k^x \leq N_G(P')$. This implies that p does not divide $|G : N_G((P')|$.

Final contradiction. In view of Claim (d), there is a σ-soluble Hall Π-subgroup L of G such that $P \leq L$. On the other hand, $G = N_G(P')L$ by Claim (h). Hence

$$(P')^G = (P')^{LN_G(P')} = (P')^L \leq L.$$

But by Claim (g), $P' \neq 1$. Hence $R \leq L$. But then R is σ-primary. The final contradiction completes the proof. □

Proof of Theorem A. Let $D = G^{\mathfrak{N}_\sigma}$ and $\mathcal{H} = \{1, H_1, \dots, H_t\}$ a complete Hall σ-set of G. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \dots, t$.

Necessity. Suppose that this is false and let G be a counterexample of minimal order. Then $D \neq 1$, and so $t > 1$.

(1) *The hypothesis holds on every quotient of G.* (This directly follows from Lemma 2.2 (1).)

(2) *G is σ-soluble.* (This directly follows from Theorem 3.1.)

(3) *D is soluble.*

Let R be a minimal normal subgroup of G. Then $D/D \cap R \simeq DR/R = (G/R)^{\mathfrak{N}_\sigma}$ is nilpotent by Claim (1) and the choice of G. Therefore $R \leq D$ and R is the unique minimal normal subgroup of G. Assume that R is non-abelian. Since G is σ-soluble by Claim (2), R is σ-primary. Let $R \leq H_i$ and R_p be a Sylow p-subgroup of R, where $p \in \pi(R)$. Then $R_p = R \cap P \nleq \Phi(P)$, where P is a Sylow p-subgroup of G containing R_p by the Tate theorem [9, Chapter IV, Section 4.7]. Let V be a maximal subgroup of P such that $P = R_pV$. If P is a σ_i-group, then by hypothesis there is a σ-permutable subgroup W of G such that V is a Hall σ_i-subgroup of W. Hence $P \nleq W$, and so $R \nleq W$. It follows that $W_G = 1$. Therefore V is σ-subnormal in G by Lemma 2.7.

We show that V is σ-permutable in G. First note that $V \leq H_i^x$ for all $x \in G$ by Lemma 2.4 (3), so $VH_i^x = H_i^xV$. Now let $j \neq i$. Then V is a σ-Hall subgroup of WH_j^x and V is σ-subnormal in WH_j^x by Lemma 2.4 (1). Hence V is normal in WH_j^x by Lemma 2.4 (3), so $VH_j^x = H_j^xV$. This shows that V is σ-permutable in G. Therefore $R \leq N_G(V)$ by Lemma 2.5 since $R \leq D \leq O^{\sigma_i}(G)$, and so $V \cap R_p = V \cap R \cap P = V \cap R$ is normal in R, which implies that $V \cap R = 1$, Thus $|R_p| = p$. This shows that every Sylow subgroup of R is cyclic, and so R is abelian by [9, Chapter IV, Section 2.11]. This contradiction completes the proof of (3).

(4) *D is a σ-Hall subgroup of G.*

Suppose that this is false and let U be a Hall σ_i-subgroup of D such that $1 < U < H_i$. Without loss of generality, we can assume that $i = 1$. Then:

(a) *Let R be a minimal normal subgroup of G contained in D. Then $R = U$ is a Sylow p-subgroup of D for some prime $p \in \sigma_1$ and a p-complement of D is a σ-Hall subgroup of G. Hence R is the unique minimal normal subgroup of G contained in D and $R = H_1 \cap D = G_p \cap D$, where G_p is a Sylow p-subgroup of G contained in H_1.*

Since D is soluble by Claim (3), R is a p-group for some prime p. Moreover, $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a σ-Hall subgroup of G/R by Claim (1) and the choice of G. Suppose that $UR/R \neq 1$, then UR/R is a Hall σ_1-subgroup of G/R. If $p \notin \sigma_1$, then U is a Hall σ_1-subgroup of G by order considerations. This contradicts the fact that $U < H_1$. If $p \in \sigma_1$, then $R \leq U$ and so U/R is a Hall σ_1-subgroup of G/R. It follows that U is a Hall σ_1-subgroup of G, which contradicts that $U < H_1$. Therefore $UR/R = 1$. Consequently, $U \leq R$ and $U = R$. But, clearly, we have $H_1 \nleq UR \leq D$. Thus $R = U = H_1 \cap D$ is a Sylow p-subgroup of D. It is also clear that a p-complement of D is a σ-Hall subgroup of G.

(b) $R \nleq \Phi(G)$*, so for some maximal subgroup* M *of* G *we have* $G = R \rtimes M$.

Assume that $R \leq \Phi(G)$. Then $D \neq R$ by Lemma 2.8. On the other hand, D/R is a σ_1'-group by Claim (a). Hence $O_{\sigma_1'}(D) \neq 1$ by Lemma 2.8. But $O_{\sigma_1'}(D)$ is characteristic in D and so it is normal G, which contradicts (a). Thus $R \nleq \Phi(G)$. The second assertion of (b) follows from Claim (3).

(c) *If* G *has a minimal normal subgroup* $L \neq R$*, then* $H_1 = R \times L$ *and* $G_p = R \times (L \cap G_p)$*. Hence* $O_{\sigma_1'}(G) = 1$.

Indeed, $L \nleq D$ by Claim (a). On the other hand, $DL/L \simeq D$ is a σ-Hall subgroup of G/L by Claim (1) and the choice of G. Hence $L \leq H_1$ and $1 < RL/L \leq (H_1L/L) \cap (DL/L)$. Consequently, $H_1/L \leq DL/L$ and so $H_1 = L(H_1 \cap D) = L \times R$, which implies that $G_p = R \times (L \cap G_p)$.

(d) $C_G(R) = R \times V$*, where* $V = C_G(R) \cap M \leq H_1$.

In view of Claims (3) and (b), $C_G(R) = R \times V$, where $V = C_G(R) \cap M$ is a normal subgroup of G. By Claim (a), $V \cap D = 1$. Hence $V \simeq DV/D$ is σ-nilpotent. Let W be a σ_1-complement of V. Then W is characteristic in V and so it is normal in G. Therefore we have (d) by Claim (c).

(e) $|\pi(H_1)| > 1$.

Assume that $H_1 = G_p$. Claim (b) implies that $R \nleq \Phi(H_1)$. Let V be a maximal subgroup of H_1 such that $H_1 = RV$. Let W be a σ-permutable subgroup of G such that V is a Hall σ_1-subgroup of W. Then $H_1 \nleq W$, so $V = H_1 \cap W$. Hence $R \cap V = R \cap H_1 \cap W = R \cap W$ is σ-permutable in G by [16, Theorem C]. It is clear that $H_1 \leq N_G(R \cap V)$, so $G = H_1 O^{\sigma_1}(G) \leq N_G(R \cap V)$ by Lemma 2.5. The minimality of R implies that $R \cap V = 1$, so $H_1 = R \rtimes V$. First assume that $W_G \neq 1$ and let L be a minimal normal subgroup of G contained in W_G. Then $H_1 = R \times L$ by Claim (c), so $|V| = |L|$ and hence $V = L$ is normal in G. If $W_G = 1$, then arguing as in the proof of Claim (3), one can show that V is σ-permutable in G and so V is normal in G by Lemma 2.5. Hence $H_1 = R \times V$ is an elementary abelian p-group, where $|R| = p$ and V is a minimal normal subgroup of G. Note that since $H_1 \leq D$, we have that $V \nleq D$. The G-isomorphism $DV/D \simeq V$ implies that $V \leq Z_\sigma(G)$. Hence $G = H_1 O^{\sigma_1}(G) \leq C_G(V)$, and so $|V| = p$. Now let $R = \langle a \rangle$, $V = \langle b \rangle$ and $L = \langle ab \rangle$. Then, arguing as above, one can get that L is normal in G. Clearly, $L \nleq D$, so in view of the G-isomorphisms $DL/D \simeq L$ we get that $L \leq Z_\sigma(G)$. Hence $G_p = H_1 = VL \leq Z_\sigma(G)$. But then $G/C_G(R)$ is a p-group, so $G = H_1$. This contradiction shows that we have (e).

Final contradiction for (4). By [17, Theorem B], H_1 has a complement E in G such that $EG_p = G_pE$. Let $S = (G_pE)^{\mathfrak{N}_\sigma}$. By Claim (e), $EG_p \neq G$. By Lemma 2.2 (2), the hypothesis holds for G_pE, so the choice of G implies that S is a nilpotent σ-Hall subgroup of G_pE. But since $DG_pE/D \simeq G_pE/G_pE \cap D$

is σ-nilpotent, $S \le G_p E \cap D = (G_p \cap D)(E \cap D) = R(E \cap D)$ by Claim (a). Then, since $R < G_p$, it follows that S is a p'-group. Now since $R \le D \le EG_p$ by Claim (a), $S \le C_G(R) \le H_1$. Hence $S \le D \cap H_1 = R$, and so $V = 1$. Therefore EG_p is σ-nilpotent and thereby $E \le C_G(R) \le H_1$. Thus $E = 1$ and so $t = 1$, a contradiction. Hence we have (4).

(5) *D is nilpotent and every maximal subgroup of every Sylow subgroup of D is normal in G.*

Firstly, we show that D is nilpotent. Assume that this is false and let R be a minimal normal subgroup of G. Then $RD/R = (G/R)^{\mathfrak{N}}$ is nilpotent by Claim (1) and the choice of G. Hence $R \le D$ and so R is a p-group for some prime p by Claim (3). Moreover, R is the unique minimal normal subgroup of G and for some H_i we have $R \le H_i \le D$ by Claim (4). Clearly $R \nleq \Phi(G)$, so $R = C_G(R) = F(G)$ by [4, Chapter A, Section 15.2]. Let P be a Sylow p-subgroup of G contained in H_i. Since H_i/R a nilpotent Hall subgroup of D/R, P/R is normal in D/R and hence P is normal in G. But then $P = F(G) = R$. Let V be a maximal subgroup of P. By hypothesis, there exists a σ-permutable subgroup W of G such that V is a Hall σ_i-subgroup of W, so $V = W \cap P = W \cap R$ is σ-permutable in G by [16, Theorem C]. Hence $G = H_i O^{\sigma_i}(G) = O^{\sigma_i}(G) \le N_G(V)$ by Lemma 2.5 since $H_i \le D \le O^{\sigma_i}(G)$. It follows that $V = 1$, consequently $|R| = p$. Therefore $G/C_G(R) = G/R$ is an abelian group. This implies that G is supersoluble and so D is nilpotent. Finally, note that we, in fact, have already proved that if P is a normal Sylow subgroup of G contained in D, then every maximal subgroup of P is normal in G.

(6) *If p is a prime such that $(p-1, |G|) = 1$, then p does not divide $|D|$. In particular, $|D|$ is odd.*

Assume that this is false. Then, by Claims (4) and (5), D has a maximal subgroup E such that $|D : E| = p$ and E is normal in G. Then $C_G(D/E) = G$. Since D is a Hall subgroup of G, it follows that $G/E = (D/E) \times (ME/E)$, where $ME/E \simeq M \simeq G/D$ is σ-nilpotent. Therefore G/E is σ-nilpotent. But then $D \le E$, a contradiction. Hence p does not divide $|D|$. In particular, $|D|$ is odd.

(7) *Every subgroup H of D satisfying $\Phi(D) \le H$ is normal in G.*

In view of Claim (5), $\Phi(D) = \Phi(P_1) \times \cdots \times \Phi(P_r)$, where $\{P_1, \ldots, P_r\}$ is the set of all different Sylow subgroups of D. Assume that for some i, $\Phi(P_i) \ne 1$ and let R be a minimal normal subgroup of G contained in $\Phi(D)$. Then $\Phi(D)/R = \Phi(D/R) \le H/R$. The choice of G implies that H/R is normal in G/R. It follows that H is normal in G. Now assume that $\Phi(P_i) = 1$ for all i. Then every subgroup of P_i is normal in G by Claim (5). But $H = (H \cap P_1) \times \cdots \times (H \cap P_r)$, so H is normal in G.

From Claims (3)–(7) we get that the necessity holds for G, which contradicts the choice of G.

Sufficiency. Let V be a maximal subgroup of P_i, where P_i is a Sylow p_i-subgroup of G. Assume that $P_i \leq D$, without loss of generality, we may assume that $i = 1$. Then $V\Phi(D) = V\Phi(P_1) \times \cdots \times \Phi(P_r) = V \times \Phi(P_2) \times \cdots \times \Phi(P_r)$ is normal in G by hypothesis. Clearly, V is characteristic in $V\Phi(D)$, so V is normal in G. Finally, suppose that $P_i \nleq D$, and let P_i is a σ_j-group. Then DV/D is a subgroup of the σ-nilpotent group G/D, so DV/D is σ-permutable in G/D. Hence DV is σ-permutable in G by Lemma 2.1 (3), where V is a Hall σ_j-subgroup of DV since D is a σ-Hall subgroup of G by hypothesis. Hence V is σ-permutably embedded in G. The theorem is proved. □

Proof of Corollary 1.5. *Sufficiency.* Let $H \leq G$. If $H \nleq D$, then $H_pD/D \leq G/D$ is σ-nilpotent, and so H_pD/D is σ-permutable in G/D. Hence H_pD is σ-permutable in G by Lemma 2.1 (3). This means that H_p is σ-permutable embedded in G. Now assume that $H \leq D = P_1 \times \cdots \times P_r$, where P_i is the Sylow subgroup of D. Then P_i is normal in G and $H = H \cap D = (H \cap P_1) \times \cdots \times (H \cap P_r)$. Hence $HP_i = P_iH$, and so $HD_{\sigma_i} = D_{\sigma_i}H$, where D_{σ_i} is a σ_i-Hall subgroup of G contained in D. Moreover, since every element of M induces a power automorphism on D, we see that H is σ-permutable in G. Thus the sufficiency holds.

Necessity. In view of Theorem A, it is enough to show that every p-subgroup H of D, for any prime p dividing $|D|$, is normal in G and D is abelian. By hypothesis, there is a σ-permutable subgroup W of G such that H is a Hall σ_i-subgroups of W, for some σ_i. Let H_i be a Hall σ_i-subgroup of G. Since D is a nilpotent σ-Hall subgroup of G, we have that $H_i \leq D$. Hence H_i is normal in G, and so $H = W \cap H_i$ is σ-permutable in G by Theorem C in [16]. But then $G = H_iO^{\sigma_i}(G) = O^{\sigma_i}(G) \leq N_G(H)$ by Lemma 2.5. Thus H is normal in G. It follows also that D is a Dedekind group. But as D is of odd order, D is abelian. The corollary is proved. □

Proof of Theorem B. Let $A = G^{\mathfrak{N}_\sigma}$ and let $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ be a complete Hall σ-set of G. We can assume without loss of generality that H_i is a σ_i-group for all $i = 1, \ldots, t$.

Necessity. Suppose that this is false and let G be a counterexample of minimal order. Then $A \neq 1$, so $t > 1$.

By Theorem A, $G = A \rtimes M$, where A and M are σ-Hall subgroups of G, $A = G^{\mathfrak{N}_\sigma}$ is nilpotent of odd order and every element of M induces a power automorphism on $A/\Phi(A)$. It is clear that $A \neq G$. We can assume without loss of generality that $H_i \leq A$ for all $i = 1, \ldots, r$ and $H_i \nleq A$ for all $i > r$.

Let $i > r$, let P be a Sylow subgroup of H_i and V a maximal subgroup of P. Since V is σ-permutable in G, $V^x \leq H_i$ for all $x \in G$. Hence $V^G \leq H_i$. In view of the G-isomorphism $AV^G/A \simeq V^G$, $V^G \leq Z_i$, where $Z_i = H_i \cap Z_\sigma(G)$ is a Hall σ_i-subgroup of $Z_\sigma(G)$. Hence by Lemma 2.3, $O^{\sigma_i}(G) \leq C_G(V)$ and so $[V, a] = 1$ for each σ_i'-element $a \in G$.

Now, for every $i > r$, we write E_i to denote the product $V_1^G \cdots V_n^G$, where $\{V_1, \ldots, V_n\}$ is the set of all maximal subgroups of all Sylow subgroups of H_i. Then $E_i \leq H_i$, so $E_i < H_i$ for some $i > r$ (otherwise $G = A \times H_{r+1} \times \cdots \times H_t$ is σ-nilpotent, contrary to our assumption on G). Let $\pi_i = \pi(|H_i : E_i|)$. We show that E_i possesses a normal π_i-complement K_i. Indeed, a Sylow p-subgroup P of H_i, where $p \in \pi_i$, is cyclic and $P \not\leq E_i$, so by the Tate theorem [9, Chapter IV, Section 4.7], E_i is p-nilpotent for all $p \in \pi$. It follows that E_i is π_i'-closed, as required. Note that the subgroup K_i is characteristic in E_i, so it is normal in G.

Now, let $B = K_{r+1} \times \cdots \times K_t$. Then $B \leq Z_\sigma(G)$. Since B is a Hall subgroup of G, B has a complement C in G by the Schur–Zassenhaus theorem. From above proof, we see that G/B is an extension of the nilpotent group AB/B by a group C/B whose the Sylow subgroups are cyclic. Now it is clear that $G = A \rtimes (B \rtimes C)$ and the necessity holds.

Sufficiency. Let V be a maximal subgroup of a Sylow subgroup P of G. Suppose that P is a σ_i-group. If $P \leq B$ or $P \leq C$, then $O^{\sigma\sigma_i}(G) \leq C_G(V)$ by Lemma 2.3 and the condition. Hence V is σ-permutable in G by Lemma 2.5. Finally, assume that $V \leq P_1 \leq A$. Since A is nilpotent, it follows that $A = P_1 \times \cdots \times P_r$, where P_i is the Sylow p_i-subgroup of A. Then $V\Phi(A) = V\Phi(P_1) \times \cdots \times \Phi(P_r) = V \times \Phi(P_2) \times \cdots \times \Phi(P_r)$ is normal in G, where V is characteristic in $V\Phi(D)$. Hence V is normal in G. The theorem is proved. □

Acknowledgments. The authors cordially thank the referees for their careful reading and helpful comments.

Bibliography

[1] R. K. Agrawal, Finite groups whose subnormal subgroups permute with all Sylow subgroups, *Proc. Amer. Math. Soc.* **47** (1975), 77–83.

[2] A. Ballester-Bolinches, R. Esteban-Romero and M. Asaad, *Products of Finite Groups*, Walter de Gruyter, Berlin, 2010.

[3] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, Sufficient conditions for supersolvebility of finite groups, *J. Pure Appl. Algebra* **127** (1998), 113–118.

[4] K. Doerk and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin, 1992.

[5] T. M. Gagen, *Topics in Finite Groups*, London Math. Soc. Lecture Note Ser. 16, Cambridge University Press, Cambridge, 1976.

[6] W. Guo, *Structure Theory for Canonical Classes of Finite Groups*, Springer, New York, 2015.

[7] W. Guo and A. N. Skiba, Finite groups with permutable complete Wielandt sets of subgroups, *J. Group Theory* **18** (2015), 191–200.

[8] W. Guo and A. N. Skiba, On Π-permutable subgroups of finite groups, preprint (2016), `http://arxiv.org/abs/1606.03197`.

[9] B. Huppert, *Endliche Gruppen I*, Springer, Berlin, 1967.

[10] B. Huppert and N. Blackburn, *Finite Groups III*, Springer, Berlin, 1982.

[11] I. M. Isaacs, Semipermutable π-subgroups, *Arch. Math.* **102** (2014), 1–6.

[12] O. H. Kegel, Sylow-Gruppen und Subnormalteiler endlicher Gruppen, *Math. Z.* **78** (1962), 205–221.

[13] Y. Li and Y. Wang, On π-quasinormally embedded subgroups of finite groups, *J. Algebra* **281** (2004), 109–123.

[14] J. Lio and S. Li, CLT-groups with Hall S-quasinormally embedded subgroups, *Ukrainian Math. J.* **66** (2014), 1281–1287.

[15] A. N. Skiba, A characterization of hypercyclically embedded subgroups of finite groups, *J. Pure Appl. Algebra* **215** (2011), 257–261.

[16] A. N. Skiba, On σ-subnormal and σ-permutable subgroups of finite groups, *J. Algebra* **436** (2015), 1–16.

[17] A. N. Skiba, A generalization of a Hall theorem, *J. Algebra Appl.* **15** (2016), no. 4, Article ID 1650085.

[18] S. Srinivasan, Two sufficient conditions for supersolvability of finite groups, *Israel J. Math.* **35** (1980), no. 3, 210–214.

[19] V. N. Tyutyanov, On the Hall conjecture, *Ukrainian Math. J.* **54** (2002), no. 7, 1181–1191.

[20] G. L. Walls, Groups with maximal subgroups of Sylow subgroups normal, *Israel J. Math.* **43** (1982), no. 2, 166–168.

[21] M. Weinstein, *Between Nilpotent and Solvable*, Polygonal Publishing House, Passaic, 1982.

Received December 21, 2015; revised January 16, 2016.

Author information

Wenbin Guo, Department of Mathematics,
University of Science and Technology of China, Hefei 230026, P. R. China.
E-mail: `wbguo@ustc.edu.cn`

Alexander N. Skiba, Department of Mathematics,
Francisk Skorina Gomel State University, Gomel 246019, Belarus.
E-mail: `alexander.skiba49@gmail.com`