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Sylow permutability in generalized soluble groups

Derek J. S. Robinson

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Abstract. A PST-group is a group in which Sylow permutability is a transitive relation in the group. A classification is given of finitely generated hyperabelian groups all of whose finite quotients are PST-groups.

1 Introduction

In recent years there has been widespread interest in the phenomenon of subgroup permutability, especially in finite groups. Recall that a subgroup H is said to be *permutable* in a group G if $HK = KH$ for every subgroup K, and *Sylow permutable* in G if $HP = PH$ for every Sylow subgroup P of G. Groups in which permutability is a transitive relation have received particular attention. A group G is called a *PT-group* if H permutable in K and K permutable in G always imply that H is permutable in G. Also G is called a *PST-group* if the same holds with Sylow permutability instead of permutability. In addition recall that a *T-group* is a group in which normality is transitive.

By a well-known theorem of Kegel [7] a Sylow permutable subgroup of a finite group is always subnormal. From this result it follows quickly that a finite group G is a PT-group or a PST-group if and only if every subnormal subgroup is permutable or Sylow permutable in G respectively.

The structures of finite soluble PT-groups and PST-groups have been determined by Zacher [16] and Agrawal [1] respectively, their results following the pattern established by Gaschütz [5] for T-groups nearly sixty years ago. For finite insoluble PT-groups and PST-groups see [14]; infinite soluble T-groups and PT-groups are studied in [13] and [11, 12] respectively. An excellent general reference for permutability in finite groups is the book by Ballester-Bolinches, Estaban Romero and Asaad [3].

Here we are concerned with the role of Sylow permutability in generalized soluble groups. It has long been known that the finite quotients of a polycyclic group G carry a large amount of information about the group structure. For example, if all the finite quotients are T-groups, then G is a T-group; this follows from the theorem of Mal'cev ([10]; see also [8]) that every subgroup of a polycyclic group is closed in the profinite topology. In the same vein, if every finite quotient of G is a PT-group, then G is a PT-group: in this case the assertion follows from a result of Lennox and Wilson [9] which states that the product of any pair of subgroups in a polycyclic group is profinite closed. At this point it should be noted that a finitely generated soluble T-group is finite or abelian [13], while a finitely generated soluble PT-group is finite or nilpotent and modular [11, 12].

The results just mentioned suggest a natural question: what is the structure of a polycyclic group whose finite quotients are PST-groups?[1] The answer cannot be just that the group is a PST-group as this would say nothing about torsion-free groups. But clearly the PST condition on finite quotients must have consequences for the group structure. A key observation is that every finite quotient of the infinite dihedral group, but not the group itself, is a PST-group.

Groups of infinite dihedral type. Let us say that a group G is of *infinite dihedral type* if the hypercenter H is a finite 2-group and G/H is isomorphic with the dihedral group $\mathrm{Dih}(B)$ on a finitely generated, infinite abelian group B with no involutions, i.e.,

$$\mathrm{Dih}(B) = \langle t \rangle \ltimes B, \text{ where } b^t = b^{-1}, (b \in B), \text{ and } t^2 = 1.$$

An alternative description of these groups is given in Lemma 2 below.

With this terminology we can state our principal result, which is a complete description of the finitely generated hyperabelian groups whose finite quotients are PST-groups. (Recall that a group is *hyperabelian* if it has an ascending normal series with abelian factors).

Theorem. *Let G be a finitely generated hyperabelian group. Then every finite quotient of G is a PST-group if and only if G is one of the following:*

(i) *a finite soluble PST-group,*

(ii) *a nilpotent group,*

(iii) *a group of infinite dihedral type.*

Here finite soluble PST-groups and nilpotent groups were to be expected; it is the groups of infinite dihedral type that are the novelty.

We record some consequences of the theorem.

Corollary 1. *A finitely generated, torsion-free hyperabelian group whose finite quotients are PST-groups is nilpotent.*

[1] I am grateful to Professor J.Ċ. Beidleman for drawing my attention to this question.

This because a group of infinite dihedral type cannot be torsion-free. By Proposition 2 below a group of infinite dihedral type is not a PST-group. Therefore we have the following result.

Corollary 2. *Let G be a finitely generated hyperabelian group. If G and all its finite quotients are PST-groups, then G is finite or nilpotent.*

Notation. (i) $\gamma_\infty(G)$: the intersection of the finitary terms of the lower central series in a group G.

(ii) $Z(G)$: the centre of a group G.

(iii) $\pi(G)$: the set of prime divisors of the orders of elements of a group G.

(iv) G_π: the π-component of a nilpotent group G where π is a set of primes.

(v) $\mathbb{Q}_\pi$: the additive group of π-adic rationals where π is a set of primes.

(vi) $\bar{H}, \bar{x}$: the images of a subgroup H and a group element x in a specified quotient of a group.

2 Auxiliary results

A critical role in the proof of the main theorem is played by the structure theorem for finite soluble PST-groups, which is due to Agrawal [1]. We state it for future reference.

Proposition 1. *A finite group G is a soluble PST-group if and only if $\gamma_\infty(G)$ is abelian of odd order, $|\gamma_\infty(G)|$ and $|G : \gamma_\infty(G)|$ are relatively prime and elements of G induce power automorphisms in $\gamma_\infty(G)$.*

Notice the consequence that $\gamma_\infty(G) \cap Z(G) = 1$.

In the proof of our result Agrawal's theorem is applied to suitably chosen finite quotients groups. Another useful tool in the proof is the following result about finitely generated nilpotent groups: it will be familiar to many readers.

Lemma 1. *Let G be a finitely generated nilpotent group.*

(i) *If π is an infinite set of primes, then $\bigcap_{p\in\pi} G^p$ is finite.*

(ii) *For each prime p there is an integer $k_p > 0$ such that $G^{p^{k_p}} \cap G_p = 1$.*

Proof. (i) Clearly we may assume that G is torsion-free. It follows from the theory of basic commutators – see [6, Theorem 12.3.1] – that G^p consists of all pth powers of elements of G, provided that p is greater than c, the nilpotent class of G. Thus $G^p \cap Z(G) = Z(G)^p$ since $G/Z(G)$ is torsion-free. Writing $I = \bigcap_{p\in\pi,\, p>c} G^p$, we have

$$I \cap Z(G) = \bigcap_{p\in\pi,\, p>c} Z(G)^p = 1,$$

and consequently $I = 1$.

(ii) Let $1 \neq g \in G_p$. Since G is residually finite, there exists $L_g \lhd G$ which is maximal with respect to $g \notin L_g$ and G/L_g being finite. Now G/L_g cannot have two non-trivial primary components and gL_g is a p-element. Hence G/L_g is a p-group. Define $M(p) = \bigcap_{1\neq g\in G_p} L_g$ and note that $G/M(p)$ is a finite p-group. Hence there exists $k_p > 0$ such that $G^{p^{k_p}} \leq M(p)$. Finally,

$$G^{p^{k_p}} \cap G_p \leq M(p) \cap G_p = 1. \qquad \square$$

The next result provides an alternative description of the groups of infinite dihedral type: while it is more technical, this form is convenient in the proof of the main theorem.

Lemma 2. *A group G is of infinite dihedral type if and only if it has a normal subgroup A such that:*

(i) *A is a finitely generated, infinite abelian group containing no involutions,*

(ii) *G/A is a finite 2-group and $|G : C_G(A)| = 2$,*

(iii) *elements in $G\backslash C_G(A)$ induce inversion in A.*

Proof. Assume that G is of infinite dihedral type, so the hypercentre H of G is a finite 2-group and $G/H = \mathrm{Dih}(B)$ where B is a finitely generated, infinite abelian group without involutions. There is a nilpotent normal subgroup C such that $H \leq C < G$, where $C/H = B$, $G = \langle t, C\rangle$, t inverts in C/H and $t^2 \in H$. Next $C^{2^k} \cap H = 1$ for some $k > 0$ by Lemma 1, so we have

$$C/H \overset{\langle t\rangle}{\simeq} (C/H)^{2^k} \overset{\langle t\rangle}{\simeq} C^{2^k}.$$

Hence t inverts in $A = C^{2^k}$. Also $[A, C] \leq A \cap C' \leq A \cap H = 1$, so we have $|G : C_G(A)| = 2$. Thus G has the stated property.

Conversely, assume that there is a finitely generated, infinite abelian normal subgroup A without involutions such that the quotient G/A is a finite 2-group, $|G : C = C_G(A)| = 2$, and $t \in G\backslash C$ induces inversion in A. We can write A in the form $F \times D$, where F is free abelian and D has odd order. Then $A \leq Z(C)$, so $C/Z(C)$ is a finite 2-group, as is C'. Therefore $C' \leq C_2 = H$, say. By Lemma 1 there is a $k > 0$ such that $C^{2^k} \cap H = 1$ and also $C^{2^k} \leq A$. Hence we have

$$C/H \overset{\langle t\rangle}{\simeq} (C/H)^{2^k} \overset{\langle t\rangle}{\simeq} C^{2^k} \leq A.$$

Therefore t inverts in $B = C/H$. Observe that t is a 2-element and $t^2 \in C$, so $t^2 \in H$. This shows that $G/H \simeq \mathrm{Dih}(B)$. Notice that $[H,_r G] \leq A \cap H = 1$ for some $r > 0$, so H is contained in the hypercentre of G; indeed H coincides with the hypercentre since $Z(\mathrm{Dih}(B)) = 1$. Therefore G is of infinite dihedral type. □

Next it is shown that groups of infinite dihedral type have the property stated in the theorem.

Lemma 3. *If G is a group of infinite dihedral type, then all its finite quotients are PST-groups.*

Proof. By Lemma 2 there is a finitely generated, infinite abelian normal subgroup $A \lhd G$, without involutions, such that G/A is a finite 2-group, $|G : C_G(A)| = 2$ and $t \in G \backslash C_G(A)$ induces inversion in A. Let G/M be a finite quotient; then $A^m \leq M$ for some $m > 0$ and $\bar{G} = G/A^m$ is finite. By Proposition 1 it is enough to prove that $\bar{G}$ is a PST-group. Let $B/A^m = O_{2'}(A/A^m)$. Then G/B is a finite 2-group and $\gamma_\infty(\bar{G}) \leq \bar{B}$. Since t inverts in A and $|\bar{B}|$ is odd, $[\bar{B}, \bar{t}] = \bar{B}$, so $\bar{B} = \gamma_\infty(\bar{G})$ and by Proposition 1 we conclude that $\bar{G}$ is a PST-group. □

The next result stands in contrast to Lemma 3.

Proposition 2. *If G is a group of infinite dihedral type, then G is not a PST-group.*

Proof. There is a finitely generated, infinite abelian normal subgroup A without involutions such that G/A is a finite 2-group, $|G : C_G(A)| = 2$ and $t \in G \backslash C_G(A)$ induces inversion in A. Write $A = F \times D$, where F is free abelian of positive rank and D has odd order. Let $a \in F \backslash F^2$; then $(ta)^2 = t^2$, so t and ta are 2-elements of equal order. Let P be a Sylow 2-subgroup containing ta and put $|P| = 2^k$. Set $H = \langle t, A^{2^{k+1}} \rangle$, noting that H is subnormal in $\langle t, A \rangle$ and hence in G.

Now assume that G is a PST-group; then, since H is subnormal in G, it is Sylow-permutable and $HP = PH = J$, say. Notice that $a \in J$ since $t \in H$ and $ta \in P$. Hence $J = HP = \langle t \rangle \langle a \rangle A^{2^{k+1}} P$. Now

$$|J : H| = |P : H \cap P| \leq |P| = 2^k,$$

while on the other hand

$$\begin{aligned} |J : H| &= |\langle t \rangle \langle a \rangle A^{2^{k+1}} P : \langle t \rangle A^{2^{k+1}}| \\ &\geq |\langle t \rangle \langle a \rangle A^{2^{k+1}} : \langle t \rangle A^{2^{k+1}}| = |\langle a \rangle : \langle a \rangle \cap \langle t \rangle A^{2^{k+1}}|. \end{aligned}$$

Now $\langle t \rangle \cap A = 1$ since t is a 2-element. Hence

$$\langle a \rangle \cap (\langle t \rangle A^{2^{k+1}}) = \langle a \rangle \cap A^{2^{k+1}} = \langle a^{2^{k+1}} \rangle$$

since $a \in F \backslash F^2$. Therefore $|J : H| \geq |\langle a \rangle : \langle a^{2^{k+1}} \rangle| = 2^{k+1}$, a contradiction which shows that G is not a PST-group. □

3 Proof of the main theorem

The main part of the proof consists in establishing the result for polycyclic groups.

Proposition 3. *Let G be a polycyclic group. If every finite quotient of G is a PST-group, then G is finite soluble PST or nilpotent or a group of infinite dihedral type.*

Proof. Since finite soluble PST-groups are supersoluble by Agrawal's theorem (see Proposition 1), every finite quotient of G is supersoluble, and hence G is supersoluble by a theorem of Baer [2]. We assume that G is infinite and argue by induction on the Hirsch number. Thus G has an infinite cyclic normal subgroup $N = \langle z \rangle$ and by induction hypothesis G/N is finite or nilpotent or a group of infinite dihedral type. Each of the three possibilities will be handled separately.

(i) *Case: G/N is finite.* We show that G is either nilpotent or of infinite dihedral type. Suppose first that $N \leq Z(G)$. If G is not nilpotent, $\gamma_\infty(G/N) \neq 1$; let p be a prime dividing $|\gamma_\infty(G/N)|$. Now $\bar{G} = G/N^p$ is a finite PST-group and p divides $|\gamma_\infty(\bar{G})|$, so it cannot divide $|\bar{G} : \gamma_\infty(\bar{G})|$ by Proposition 1. Therefore $\bar{N} \leq \gamma_\infty(\bar{G})$. However, $\bar{N} \leq Z(\bar{G})$ and $\gamma_\infty(\bar{G}) \cap Z(\bar{G}) = 1$ by Proposition 1, so we have a contradiction. Hence G is nilpotent in this case. Therefore we may assume that $N \not\leq Z(G)$.

Set $C = C_G(N)$, so that $|G : C| = 2$ and $G = \langle t, C \rangle$, where t induces inversion in N. Let p be an odd prime and consider $\bar{G} = G/N^p$, which is a finite soluble PST-group. Since t inverts in $\bar{N}$, we have $1 \neq \bar{N} \leq \gamma_\infty(\bar{G})$. Therefore $\overline{G(p)} \leq \gamma_\infty(\bar{G})$, where $G(p)/N$ is a Sylow p-subgroup of G/N. By Proposition 1, $G(p) \lhd G$ and $\overline{G(p)}$ is abelian. Therefore $A/N = O_{2'}(G/N)$ is abelian and G/A is a finite 2-group. Clearly $A \leq C$. Next C/A induces a 2-group of power automorphisms in each $\overline{G(p)}$, but it also centralizes $\bar{N}$. Therefore C centralizes $\overline{G(p)}$ and hence A/N. Thus $[A, C] \leq N$. But C' is finite since $N \leq Z(C)$, so $[A, C] = 1$. It follows that $C_G(A) = C$. Thus A is abelian and clearly it does not contain involutions. Also t induces a power automorphism of order 1 or 2 in each $\overline{G(p)}$, since t inverts in $\bar{N}$. Therefore it must invert in A/N, as well as in N. A simple argument now shows that t inverts in A. By Lemma 2 the group G is of infinite dihedral type.

(ii) *Case: G/N is nilpotent.* Here we may assume that $N \not\leq Z(G)$ since otherwise G is nilpotent. Put $C = C_G(N)$, so that $|G : C| = 2$ and C is nilpotent. Also $G = \langle t, C \rangle$ where t inverts in N.

Let p be an odd prime and set $\bar{G} = G/C^p$, which is a finite PST-group. Since G/N is nilpotent, $\gamma_\infty(\bar{G}) \leq \bar{N} \leq \bar{C}$. Suppose that $\gamma_\infty(\bar{G}) \neq 1$; then p divides $|\gamma_\infty(\bar{G})|$, so it does not divide $|\bar{G}/\gamma_\infty(\bar{G})|$, which implies that $\gamma_\infty(\bar{G}) = \bar{N} = \bar{C}$, i.e., $C = NC^p$. Since C/N is finitely generated and nilpotent, it follows that C/N, and hence G/N, is finite. By case (i) either G is nilpotent or it is a group of infinite dihedral type. Thus we may assume that $\gamma_\infty(\bar{G}) = 1$, i.e., G/C^p is nilpotent for all $p > 2$. Since t inverts in N, it follows that $\bar{N} = 1$ and hence $N \leq \bigcap_{p>2} C^p$. Lemma 1 gives the contradiction that N is finite.

(iii) *Case: G/N is a group of infinite dihedral type.* In the first place G is not nilpotent, as a group of infinite dihedral type cannot be nilpotent. By Lemma 2 there exists $L \lhd G$ such that G/L is a finite 2-group, L/N is finitely generated, infinite abelian without involutions, $|G : C_G(L/N)| = 2$ and $G = \langle t, C_G(L/N)\rangle$ where t induces inversion in L/N. Write $K = C_L(N)$; then we have $|L : K| \leq 2$ and K/N has the same structure as L/N. Note also that G/K is a 2-group and $C_G(L/N) = C_G(K/N)$. Thus we can replace L by K without loss: from this point on we assume that $N \leq Z(L)$; thus L is nilpotent of class ≤ 2.

Suppose that L is non-abelian. If $\ell_1, \ell_2 \in L$, then $\ell_i^t = \ell_i^{-1} a_i$ for some $a_i \in N$ and

$$[\ell_1, \ell_2]^t = [\ell_1^{-1} a_1, \ell_2^{-1} a_2] = [\ell_1^{-1}, \ell_2^{-1}] = [\ell_1, \ell_2].$$

Thus $[L', t] = 1$ and, since $1 \neq L' \leq N$, it follows that $[N, t] = 1$. Let p be an odd prime and put $\bar{G} = G/L^p$; then $\gamma_\infty(\bar{G}) \leq \bar{L}$ since G/L is a 2-group. Now $\gamma_\infty(\bar{G})$ cannot be trivial: for if it were, since $\bar{t}$ inverts in $\bar{L}/\bar{N}$, we would have $\bar{L} = \bar{N}$, i.e., $L = L^p N$. But this implies that L/N is finite. Therefore $\gamma_\infty(\bar{G}) \neq 1$ and p divides $|\gamma_\infty(\bar{G})|$, so it cannot divide $|\bar{G} : \gamma_\infty(\bar{G})|$. It follows that $\gamma_\infty(\bar{G}) = \bar{L}$, so $\bar{L}$ is abelian by Proposition 1. Thus $L' \leq \bigcap_{p>2} L^p$, which is finite. But $L' \leq N$, so it follows that $L' = 1$ and L is abelian.

Clearly L contains no involutions. The next step is to prove that $C_G(L/N)$ is equal to $C_G(L)$; let $x \in C_G(L/N)$ and let p be an odd prime. Consider the finite PST-group $\bar{G} = G/L^p$. Now $\bar{G}$ cannot be nilpotent since t inverts in L/N, and we have seen that $\bar{L} = \bar{N}$ is impossible. Hence $\gamma_\infty(\bar{G}) \neq 1$ and p divides $|\gamma_\infty(\bar{G})|$, so it does not divide $|\bar{G} : \gamma_\infty(\bar{G})|$. As $\gamma_\infty(\bar{G}) \leq \bar{L}$, we obtain $\gamma_\infty(\bar{G}) = \bar{L}$ and consequently $\bar{x}$ induces a power automorphism of p'-order in $\bar{L}$. Commuting with $\bar{x}$ yields an endomorphism θ of $\bar{L}$ with the property $\mathrm{Im}(\theta) \leq \bar{N} < \bar{L}$. Hence we have $1 \neq \mathrm{Ker}(\theta) = C_{\bar{L}}(\bar{x})$. Therefore $[\bar{L}, \bar{x}] = 1$ and $[L, x] \leq L^p$ for all $p > 2$, showing that $[L, x] \leq \bigcap_{p>2} L^p$, which is finite. But $[L, x] \leq N$, so $[L, x] = 1$ and $x \in C_G(L)$. It follows that $C_G(L) = C_G(L/N)$ and $G = \langle t, C_G(L)\rangle$.

We know that t inverts in L/N; it will be shown next that it also inverts in N. Suppose this is false, so that $z^t = z$. Let p be an odd prime not in $\pi(L/N)$ and consider the group $\bar{G} = G/L^p$. Now $z \notin L^p$ since $p \notin \pi(L/N)$. Hence we have

$1 \neq \bar{N} \leq Z(\bar{G})$. Next, if $\gamma_\infty(\bar{G}) = 1$, then G/L^p is nilpotent and thus $L = L^p N$ since t inverts in L/N. This is impossible, so $\gamma_\infty(\bar{G}) \neq 1$. Hence $\gamma_\infty(\bar{G}) = \bar{L}$ and $1 \neq \bar{N} \leq \gamma_\infty(\bar{G}) \cap Z(\bar{G}) = 1$, a contradiction. Therefore t must invert in N and consequently $L^{(t+1)^2} = 1$.

To complete the proof observe that some power t^{2^k} with $k > 0$ belongs to L, and hence to N, from which it follows that $t^{2^k} = 1$. Also t^2 centralizes L/N and N. Hence

$$[L, t^2]^{2^{k-1}} = [L, t^{2^k}] = 1,$$

and since $[L, t^2] \leq N$, it follows that $[L, t^2] = 1$. Since $L^{(t+1)^2} = 1 = L^{t^2-1}$, we conclude that $L^{2(t+1)} = 1$ and $L^{t+1} = 1$, i.e., t inverts in L. It follows via Lemma 2 that G is a group of infinite dihedral type. □

Proof of the theorem. Let G be a finitely generated, hyperabelian group whose finite quotients are PST-groups. By Proposition 3 it is enough to prove that G is polycyclic, so assume this is false. Then, since polycyclic groups are finitely presented, there is a largest $M \lhd G$ such that G/M is not polycyclic, and we can assume that $M = 1$, so that G is *just non-polycyclic*, i.e., G is not polycyclic, but every proper quotient of G is polycyclic. By Baer's theorem G is even just non-supersoluble. Since G is hyperabelian, there is an infinite abelian normal subgroup N and G/N is supersoluble; hence G is soluble.

At this point we invoke the theory of soluble just non-supersoluble groups given in [15, Sections 3.4, 2.4]. According to this there is a normal subgroup $A \simeq \mathbb{Q}_\pi$, the additive group of π-adic rationals with π a finite set of primes, and a free abelian subgroup X of finite rank such that $|G : XA|$ finite and $X \cap A = 1$. Without loss we may assume that $G = XA$, so G is metabelian. Let $\{x_1, x_2, \dots, x_r\}$ be a basis of X and, with p be an odd prime not in π, define

$$Y = \langle x_1^{p(p-1)}, x_2^{p-1}, \dots, x_r^{p-1} \rangle.$$

Since $A/A^p \simeq \mathbb{Z}_p$, we have $[A, Y] \leq A^p$ and $YA^p \lhd G$. Consider $\bar{G} = G/YA^p$, which is a finite PST-group, and note that $\gamma_\infty(\bar{G}) \leq \bar{A}$. Suppose that $\gamma_\infty(\bar{G}) \neq 1$; then p divides $|\gamma_\infty(\bar{G})|$, so it cannot divide $|\bar{G} : \gamma_\infty(\bar{G})|$. However, $\bar{G}/\gamma_\infty(\bar{G})$ maps homomorphically onto X/Y, and hence onto $\mathbb{Z}_p$. By this contradiction $\gamma_\infty(\bar{G}) = 1$ and, since $|\bar{A}| = p$, it follows that $[\bar{A}, \bar{G}] = 1$. Hence

$$[A, G] \leq \bigcap_{p>2,\, p\notin\pi} A^p = 1$$

and G is nilpotent. But then A is finitely generated, which is a contradiction.

The converse statement follows from Lemma 3. □

For a final application we return to polycyclic groups. Many algorithms have been found which are able to perform standard operations in polycyclic groups: we refer to [4] for details. Here we record a new algorithm.

Corollary 3. *There is an algorithm which, when a finite presentation of a polycyclic group G is given, decides if all the finite quotients of G are PST-groups.*

Proof. By the theorem it is enough to decide if G is finite or nilpotent or a group of infinite dihedral type. Algorithms to decide finiteness and nilpotence were given in [4]. Thus it remains to decide if G is of infinite dihedral type and for this purpose the original definition of these groups is convenient. Using other algorithms in [4], we can find the hypercentre F of G and check to see if it is a 2-group. If this is the case, pass to the group $\bar{G} = G/F$ and compute its Fitting subgroup $\bar{F}$. Next determine whether $\bar{F}$ is an infinite abelian group without involutions. If this is the case, check to see if $|\bar{G} : \bar{F}| = 2$. If so, choose an element t in $\bar{G} \setminus \bar{F}$ and determine whether t inverts each generator of $\bar{F}$. If this happens, then G is a group of infinite dihedral type. □

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Author information

Derek J. S. Robinson, Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA.
E-mail: dsrobins@illinois.edu