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Character degree graphs of solvable groups with diameter three

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Abstract. Let G be a finite solvable group and $\mathrm{cd}(G)$ the set of character degrees of G. The character degree graph $\Delta(G)$ is the graph whose vertices are the primes dividing the degrees in $\mathrm{cd}(G)$ and there is an edge between two distinct primes p and q if their product pq divides some degree in $\mathrm{cd}(G)$. When $\Delta(G)$ has diameter three, we can partition the vertices $\rho(G)$ into four non-empty disjoint subsets $\rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$ where no prime in ρ_1 is adjacent to any prime in $\rho_3 \cup \rho_4$; no prime in ρ_4 is adjacent to any prime in $\rho_1 \cup \rho_2$; every prime in ρ_2 is adjacent to some prime in ρ_3; every prime in ρ_3 is adjacent to some prime in ρ_2; and $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$.

We will show the following: If G is a solvable group where $\Delta(G)$ has diameter three, then ρ_3 has at least three vertices and G has a normal non-abelian Sylow p-subgroup where $p \in \rho_3$. If $\rho_1 \cup \rho_2$ has n vertices, then $\rho_3 \cup \rho_4$ must have at least 2^n vertices. The group G has Fitting height 3.

1 Introduction

Let G be a finite solvable group, $\mathrm{Irr}(G)$ the set of irreducible characters and $\mathrm{cd}(G)$ the set of character degrees. We study solvable groups and their character degrees by studying $\Delta(G)$, the character degree graph. The set of vertices for $\Delta(G)$ is the set of all primes p where p divides a character degree. There is an edge between two distinct primes p and q if their product divides a character degree.

Many of the known results on character degree graphs can be found in the expository paper, [11]. One of the most important properties for $\Delta(G)$ when G is solvable is Pálfy's Condition.

Theorem 1.1 ([15]). *A graph is said to satisfy Pálfy's Condition if given any three vertices, two of them are adjacent. If G is a solvable group, then $\Delta(G)$ satisfies Pálfy's Condition.*

Two corollaries follow immediately. If $\Delta(G)$ is disconnected, there are at most two components, [12] and both components are complete. If $\Delta(G)$ is connected, the diameter is at most three [13].

There are two main questions that arise when studying character degree graphs. What can we say about the group or family of groups that have a particular character degree graph? Which graphs can occur as a character degree graph? The following three theorems answer the second question for the graphs with five vertices or fewer, and give a partial answer for the graphs with six vertices.

Theorem 1.2 ([17]). *The graph in Figure* 1 *is not the character degree graph of any solvable group.*

Figure 1. Graph with four vertices and diameter three.

Theorem 1.3 ([9]). *The graphs in Figure* 2 *are not the character degree graphs of any solvable group.*

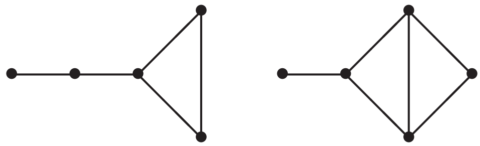

Figure 2. Graphs with five vertices and diameter three.

A consequence of Theorems 1.2 and 1.3 is that when $\Delta(G)$ has diameter three, it must have at least six vertices. In fact, there is a family of solvable groups that have a character degree graph with diameter three and six vertices. Lewis gave an example in [8] and Dugan generalized in [2] that result to show that a family of groups have a character degree graph with diameter three.

Theorem 1.4 ([8]). *There exists a solvable group that has a character degree graph as shown in Figure* 3.

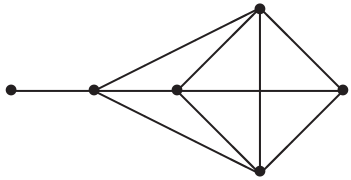

Figure 3. Graph with six vertices and diameter three.

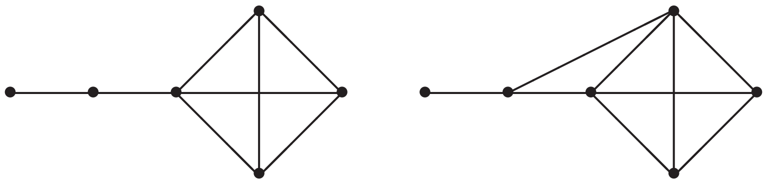

Figure 4. Graphs with six vertices and diameter three.

In [11], the question was posed, do the subgraphs shown in Figure 4, of the graph in Figure 3, occur as $\Delta(G)$ where G is a solvable group? In this paper we show that in fact, the graphs that are isomorphic to those in Figure 4 do not occur as $\Delta(G)$ for any solvable group G.

Let G be a solvable group and $\Delta(G)$ the character degree graph that has diameter three. To easily describe the graph $\Delta(G)$, we define a partition on the set of vertices $\rho(G)$ as Lewis did in [10]. Because $\Delta(G)$ has diameter three, we can find two vertices distance three from each other. Label them p_1 and p_4. Because p_1 and p_4 are necessarily not adjacent, if we consider any other prime $q \in \rho(G)$, the prime q must be adjacent to either p_1 or p_4. We define the following four sets:

Definition 1.5. (i) Define $\rho_4(G)$ to be the set of all vertices that are distance three from the vertex p_1. As p_4 is distance three from p_1, the vertex $p_4 \in \rho_4$.

(ii) Define $\rho_3(G)$ to be the set of all vertices that are distance two from the vertex p_1.

(iii) Define $\rho_2(G)$ to be the set of all vertices that are adjacent to p_1 and adjacent to some prime in $\rho_3(G)$.

(iv) Define $\rho_1(G)$ to be the vertices consisting of p_1 and those that are adjacent to p_1 and not adjacent to anything in $\rho_3(G)$.

(v) We relabel if necessary so that $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$

With the above partition in mind, we state our main theorems:

Theorem 1. *The graphs in Figure* 4 *do not occur as* $\Delta(G)$ *for any solvable group* G. *In particular, if* G *is a solvable group where* $\Delta(G)$ *has diameter three and* $|\rho(G)| = 6$, *then* $\Delta(G)$ *is isomorphic to the graph in Figure* 3.

We are able to give a lower bound on the number of vertices in the subset ρ_3.

Theorem 2. *Let* G *be a solvable group where* $\Delta(G)$ *has diameter three. Then* $|\rho_3| \geq 3$.

Theorem 3. *Let G be a solvable group where $\Delta(G)$ has diameter three. Then G has a normal non-abelian Sylow p-subgroup for exactly one prime p and $p \in \rho_3$.*

In order to prove Theorem 3, we use the previous results but we also need to show more generally that G would not have a normal non-abelian Sylow p-subgroup where p is in the subset ρ_4. Finally, we confirm [11, Conjecture 4.8] and give the Fitting height of all solvable groups that have a character degree graph with diameter three.

Theorem 4. *Let G be a solvable group with $\Delta(G)$ having diameter three. If $n = |\rho_1 \cup \rho_2|$, then $|\rho_3 \cup \rho_4| \geq 2^n$.*

Theorem 5. *Let G be a solvable group with $\Delta(G)$ having diameter three. Then G has Fitting height* 3.

2 Background

Let M be a normal subgroup of G. The graphs $\Delta(M)$ and $\Delta(G/M)$ must be subgraphs of $\Delta(G)$. If $\rho(M)$ or $\rho(G/M)$ contains primes from both $\rho_1(G)$ and $\rho_4(G)$ then the graph $\Delta(M)$ or $\Delta(G/M)$ must either have diameter three or be disconnected. Because we often know that $|G|$ is minimal with $\Delta(G)$ having diameter three, we rely heavily on the classification of disconnected character degree graphs in [7]. We state the theorem here for convenience.

Theorem 2.1 ([7]). *Let G be a solvable group where $\Delta(G)$ has two connected components. Then G is one of the following examples:*

(2.1) *G has a normal non-abelian Sylow p-subgroup P and an abelian p-complement K for some prime p. The subgroup $P' \subseteq \mathbb{C}_P(K)$ and every non-linear irreducible character of P is fully ramified with respect to $P/\mathbb{C}_P(K)$.*

(2.2) *G is the semi-direct product of a subgroup H acting on a subgroup P, where P is elementary abelian of order 9 and $\mathrm{cd}(H) = \{1, 2, 3\}$. Let $Z = \mathbb{C}_H(P)$. We have $Z \subseteq \mathbb{Z}(H)$, and $H/Z \cong \mathrm{SL}_2(3)$, where the action of H on P is the natural action of $\mathrm{SL}_2(3)$ on P.*

(2.3) *G is the semi-direct product of a subgroup H acting on a subgroup P, where P is elementary abelian of order 9 and $\mathrm{cd}(H) = \{1, 2, 3, 4\}$. Let $Z = \mathbb{C}_H(P)$. We have $Z \subseteq \mathbb{Z}(H)$, and $H/Z \cong \mathrm{GL}_2(3)$, where the action of H on P is the natural action of $\mathrm{GL}_2(3)$ on P.*

(2.4) *G is the semi-direct product of a subgroup H acting on an elementary abelian p-group V for some prime p. Let $Z = \mathbb{C}_H(V)$ and K the Fitting subgroup of H. Write $m = |H : K| > 1$, and $|V| = q^m$, where q is a p-power.*

We have $Z \subseteq \mathbb{Z}(H)$*,* K/Z *is abelian,* K *acts irreducibly on* V*,* m *and* $|K : Z|$ *are relatively prime, and* $(q^m - 1)/(q - 1)$ *divides* $|K : Z|$*.*

(2.5) *G has a normal non-abelian 2-subgroup Q, and an abelian 2-complement K with the property that $|G : KQ| = 2$ and the quotient G/Q is not abelian. Let $Z = \mathbb{C}_K(Q)$, and $C = \mathbb{C}_Q(K)$. The subgroup $Q' \subseteq C$, and Z is central in G. Every non-linear irreducible character of Q is fully ramified with respect to Q/C. Furthermore, Q/C is an elementary abelian 2-group of order 2^{2a} for some positive integer a, Q/C is irreducible under the action of K, and K/Z is abelian of order $2^a + 1$.*

(2.6) *G is the semi-direct product of an abelian group D acting coprimely on a group T so that $[T, D]$ is a Frobenius group. The Frobenius kernel is $A = T' = [T, D]'$, A is a non-abelian p-group for some prime p, and a Frobenius complement is B with $[B, D] \subseteq B$. Every character in* $\mathrm{Irr}(T \mid A')$ *is invariant under the action of D and A/A' is irreducible under the action of B. If $m = |D : \mathbb{C}_D(A)|$, then $|A : A'| = q^m$ where q is a p-power, and $(q^m - 1)/(q - 1)$ divides $|B|$.*

Two of these families never show up in our work. They are Examples (2.2) and (2.3). They both have two connected components each with one vertex, two and three. They have no normal non-abelian Sylow p-subgroups. When we are considering a graph that has two connected components each with a singleton vertex, one of those vertices, p, corresponds to a normal non-abelian Sylow p-subgroup.

Two of the remaining families have no normal non-abelian Sylow p-subgroups. They are Examples (2.4) and (2.5). Example (2.5) has a character degree graph where the smaller component is the singleton 2. Examples (2.1) and (2.6) both have a normal non-abelian Sylow p-subgroup.

We use the Zsigmondy Prime Theorem to count primes or vertices in a disconnected graph. Let q and n be positive integers. A prime p is called a Zsigmondy prime divisor for $q^n - 1$ if p divides $q^n - 1$ and p does not divide $q^j - 1$ for $1 \leq j < n$. The Zsigmondy Prime Theorem says there exists a Zsigmondy prime for $q^n - 1$ unless either $n = 2$ and $q = 2^k - 1$ for some integer k, or $n = 6$ and $q = 2$. Because of these exceptional cases, we use the following lemma from [7] which takes care of the exceptional cases.

Lemma 2.2 ([7, Lemma 5.1]). *Let m be a positive integer, and let q be a prime power such that $(q^m - 1)/(q - 1)$ is relatively prime to m. If r is the number of distinct prime divisors of m, then the quotient $(q^m - 1)/(q - 1)$ has at least $2^r - 1$ prime divisors.*

Another result by Pálfy is used to count vertices in the larger component when the graph is disconnected. We call this result *Pálfy's inequality*.

Theorem 2.3 ([16, Theorem 3]). *Let G be a solvable group with a disconnected character degree graph $\Delta(G)$. If the smaller component has n vertices, then the larger component has at least $2^n - 1$ vertices.*

In Examples (2.4) and (2.6) from Theorem 2.1, let $F = \mathbb{F}(G)$, $E/F = \mathbb{F}(G/F)$ and $m = |G : E|$. The prime divisors of m are precisely the primes in the smaller component of the character degree graph. Let r be the number of distinct primes that divide m. We know from Pálfy's inequality that we must have at least $2^r - 1$ primes in the larger component. By [7, Theorem 5.4], we know that if the larger component has exactly $2^r - 1$ primes, then F is abelian and in particular, we are in Example (2.4). When there are more than $2^r - 1$ primes in the larger component, we may have to distinguish between Examples (2.4) and (2.6). Of course, if the group has a normal non-abelian Sylow p-subgroup P, then we are in Example (2.6). In this case, G/P' will satisfy Example (2.4) by [7, Lemma 3], and so the larger component in $\Delta(G)$ must have at least 2^r primes including p.

We know from Ito's Theorem [3, Corollary 12.34] that G has a normal abelian Sylow p-subgroup if and only if $p \notin \rho(G)$. If the prime or vertex p is in $\rho(G)$, we know that if the Sylow subgroup is normal, it must be non-abelian. It is redundant to keep saying that the Sylow subgroup is non-abelian, but we do so anyway to avoid confusion. With that in mind, this lemma from [9] is used frequently.

Lemma 2.4 ([9, Lemma 3]). *Suppose that G has a non-abelian normal Sylow p-subgroup P for a prime p. Then $\rho(G/P') = \rho(G)\backslash\{p\}$.*

One of our goals is to show that if G is a solvable group with a character degree graph having diameter three, then G has exactly one normal non-abelian Sylow p-subgroup and $p \in \rho_3$. This result from [18] tells us that we can have at most one normal non-abelian Sylow p-subgroup when $\Delta(G)$ has diameter three.

Theorem 2.5 ([18]). *If G is a solvable group and $\Delta(G)$ has diameter three, then G has at most one normal non-abelian Sylow p-subgroup for some prime $p \in \rho(G)$.*

Finally, this theorem from [9] gives us our strategy for proving that the graphs we are considering are not character degree graphs for any solvable group G.

Theorem 2.6 ([9, Theorem 2]). *Suppose G is a solvable group with $\Phi(G) = 1$. Assume that for all nontrivial normal subgroups M, $\Delta(G/M)$ has two connected components or the diameter of $\Delta(G/M)$ is at most 2. Then either $\Delta(G)$ has two connected components or the diameter of $\Delta(G)$ is at most 2.*

3 Observations of the partition in Definition 1.5

Let G be a solvable group and $\Delta(G)$ the character degree graph with diameter three such that the vertices $\rho(G) = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$.

(1) Every prime in $\rho(G)$ is in exactly one of the four sets.

(2) The sets ρ_2 and ρ_3 are non-empty.

Proof. Because there is a shortest path between p_1 and p_4, there exists primes p_2 and p_3 such that the path p_1, p_2, p_3, p_4 is a shortest path between p_1 and p_4. The prime p_3 is necessarily in ρ_3 as p_3 is distance 2 from p_1 otherwise it contradicts that p_1, p_2, p_3, p_4 is a shortest path. The prime p_2 is necessarily in ρ_2 as p_2 is adjacent to something in ρ_3 and also adjacent to p_1. □

(3) Because no prime in $\rho_3 \cup \rho_4$ is adjacent to the prime p_1, the subset $\rho_3 \cup \rho_4$ determines a complete subgraph of $\Delta(G)$.

(4) Because no prime in $\rho_1 \cup \rho_2$ is adjacent to the prime p_4, the subset $\rho_1 \cup \rho_2$ determines a complete subgraph of $\Delta(G)$.

(5) Every prime in ρ_2 is adjacent to some prime in ρ_3 and every prime in ρ_3 is adjacent to some prime in ρ_2.

(6) Every prime in ρ_3 is distance 2 from any prime in ρ_1.

(7) The subset ρ_2 is the set of vertices that are distance 2 from any prime in ρ_4.

(8) The subset ρ_1 is the set of all vertices that are distance 3 from any prime in ρ_4.

Proof. Let r be a prime in ρ_1 and s be a prime in ρ_4. Because $\rho_1 \cup \rho_2$ determines a complete subgraph of $\Delta(G)$ and ρ_2 is non-empty, there is a prime $p_2 \in \rho_2$ that is adjacent to the prime r. Because p_2 is adjacent to some prime in ρ_3 and ρ_3 is non-empty, there exists a prime $p_3 \in \rho_3$ where p_2 and p_3 are adjacent. If p_2 were adjacent to the prime s, then in particular, s would be distance two from p_1 and s would be in ρ_3. This contradicts the fact that $s \in \rho_4$. As $\rho_3 \cup \rho_4$ determines a complete subgraph of $\Delta(G)$, the prime p_3 is adjacent to s and we have found a shortest path r, p_2, p_3, s. □

(9) If r and s are primes in $\rho(G)$ that are distance three from each other, than one must be in ρ_1 and the other must be in ρ_4.

Proof. Suppose that r is a prime that is not in ρ_1 or ρ_4. Without loss, we can assume that r is in ρ_2. Because $\rho_1 \cup \rho_2$ determines a complete subgraph, we see that s cannot be in ρ_1. Thus s must be in either $\rho_3 \cup \rho_4$. Because r is adjacent to some prime in ρ_3, there exists a $p_3 \in \rho_3$ and r and p_3 are adjacent. Because $\rho_3 \cup \rho_4$ determines a complete subgraph, we see that p_3 is adjacent to s and the distance between r and s must be less than or equal to two. □

(10) If the partition is defined using r and s instead of p_1 and p_4, where the distance between r and s is three, the partition will be the same.

(11) Further, the partition is unique up to symmetry in the case where $|\rho_1 \cup \rho_2| = |\rho_3 \cup \rho_4|$. We will show that it is the case that $|\rho_1 \cup \rho_2| < |\rho_3 \cup \rho_4|$, and the partition for character degree graphs will be unique.

The following theorem shows that the only possible graphs with six vertices that satisfy Pálfy's condition that can be character degree graphs are those in Figure 3 and Figure 4 and in particular, the partition defined in Definition 1.5 is unique when $|\rho(G)| = 6$.

Theorem 3.1 ([9, Theorem 4]). *If* $|\rho(G)| = 6$*, then*

$$|\rho_1 \cup \rho_2| = 2 \quad \textit{and} \quad |\rho_3 \cup \rho_4| = 4.$$

4 Preliminary results

In this section, we prove many of the lemmas necessary to show that the graphs in Figure 4 do not occur as character degree graphs for any solvable group G. The proof that the graphs in Figure 4 do not occur follows in the next section.

The following lemma shows that if $\Delta(O^{\rho_2}(G))$ is a disconnected graph, then $O^{\rho_2}(G)$ cannot be Example (2.4) or Example (2.5) from Theorem 2.1.

Lemma 4.1. *Let G be a solvable group with $\Delta(G)$ having diameter three. Assume that $O^{\rho_2}(G) < G$ and that the graph $\Delta(O^{\rho_2}(G))$ does not have diameter three. Then, $O^{\rho_2}(G)$ must have a normal non-abelian Sylow p-subgroup and the graph $\Delta(O^{\rho_2}(G))$ is disconnected. In particular, the group $O^{\rho_2}(G)$ is not Example* (2.4) *or Example* (2.5) *from Theorem* 2.1.

Proof. Suppose that G is a solvable group and $\Delta(G)$ has diameter three. Let $K = O^{\rho_2}(G)$ and suppose $K < G$ and so $\Delta(K)$ is not diameter three. We assume that K has no normal non-abelian Sylow p-subgroups and derive a contradiction.

Claim. *The subgroup K is Example* (2.4) *or Example* (2.5) *from Theorem* 2.1.

Proof. Because G/K is a ρ_2-group, all primes in $\rho(G) \backslash \rho_2$ are contained in $\rho(K)$. Because the primes in ρ_1 and ρ_4 are contained in $\rho(K)$ and $\Delta(K)$ is a subgraph of $\Delta(G)$, we have $\Delta(K)$ is either disconnected or has diameter three. By the hypothesis, $\Delta(K)$ does not have diameter three, and so it must be disconnected.

If $p \in \rho(K)$ for $p \in \rho_2$, then since $|\rho_1| \geq 1$, the smaller component has at least two primes. By [7, Theorem 5.3], $O^p(K) < K$. This contradicts the definition of K, and so ρ_2 intersects $\rho(K)$ trivially. Thus, the two components of $\Delta(K)$ are ρ_1 and $\rho_3 \cup \rho_4$.

If K has a normal Sylow p-subgroup for some prime $p \in \rho(K)$, so does G as ρ_2 intersects $\rho(K)$ trivially. As G has no normal non-abelian Sylow p-subgroups, K also has no normal Sylow p-subgroups. Because $\Delta(K)$ is disconnected, the subgroup K is either Example (2.4) or Example (2.5) from Theorem 2.1. □

Thus K is Example (2.4) or Example (2.5) from Theorem 2.1 with the larger component being $\rho_3 \cup \rho_4$. By [7, Theorem 5.2], $\mathrm{cd}(K)$ contains only one degree that is divisible by primes in $\rho_3 \cup \rho_4$. Pick primes $p_2 \in \rho_2$, $p_3 \in \rho_3$, and $p_4 \in \rho_4$ such that p_2 and p_3 are adjacent in $\Delta(G)$. Then there exists a character $\chi \in \mathrm{Irr}(G)$ such that $p_2 p_3$ divides $\chi(1)$. Let θ be an irreducible constituent of χ_K in $\mathrm{Irr}(K)$. By [3, Corollary 11.29], we know $\chi(1)/\theta(1)$ divides $|G : K|$, and hence, p_3 divides $\theta(1)$. Because K has only one degree divisible by primes in $\rho_3 \cup \rho_4$ and that degree is divisible by p_3 and p_4, we conclude that p_4 also divides $\theta(1)$. Thus, p_2 and p_4 are adjacent in $\Delta(G)$, which is a contradiction. □

The next two lemmas follow from the same arguments given in Claim 3 and Claim 5 in the proof of Theorem 1.3 in [9].

Lemma 4.2. *Let G be a solvable group with a normal Sylow p-subgroup P for some prime $p \in \rho(G)$. Suppose $\rho(G/M) = \rho(G)$ implies that $M = 1$ whenever M is a normal subgroup of G. Then P' is a minimal normal subgroup in G and in particular, P' is a central subgroup in P.*

Proof. Let G be a solvable group with a normal Sylow p-subgroup P for some prime $p \in \rho(G)$. Assume $\rho(G/M) = \rho(G)$ implies $M = 1$. By Lemma 2.4, we have $\rho(G/P') = \rho(G) \backslash \{p\}$. Let X be a normal subgroup of G that is contained in P' so that P'/X is a chief factor for G. Because $\rho(G/X)$ contains p and $\rho(G/P')$, we see that $\rho(G/X) = \rho(G)$. Thus $X = 1$ by our hypothesis and P' is a minimal normal subgroup of G. Because $\mathbb{Z}(P)$ is a characteristic subgroup of P, it is normal in G. It follows that $P' \cap \mathbb{Z}(P)$ is a normal subgroup of G. Because P is a nilpotent subgroup of G, $P' \cap \mathbb{Z}(P)$ is not a trivial subgroup, and so $P' \subseteq \mathbb{Z}(P)$. □

Lemma 4.3. *Let G be a solvable group and suppose G has a normal Sylow p-subgroup for some prime $p \in \rho(G)$. Suppose there is a normal subgroup N in G such that p does not divide $|N|$. Then $\rho(G/N)$ contains every prime in $\rho(G)$ that is not adjacent to p in $\Delta(G)$.*

Proof. Let G be a solvable group with a normal Sylow p-subgroup P for some prime $p \in \rho(G)$. Let N be a normal subgroup of G such that p does not divide $|N|$. The subgroup PN/N is a normal non-abelian Sylow p-subgroup of G/N and so $p \in \rho(G/N)$. Let q be a prime in $\rho(G)\backslash\rho(G/N)$, and let Q be a Sylow q-subgroup of G. Since q divides no degree in $\mathrm{cd}(G/N)$, we use Itô's Theorem [3, Corollary 12.34] to see that QN/N is abelian and QN is normal in G. Thus, the direct product $P \times QN$ is normal in G. Let $\chi \in \mathrm{Irr}(G)$ with q dividing $\chi(1)$ and let $\theta \in \mathrm{Irr}(QN)$ be an irreducible constituent of χ_{QN}. We know that $\chi(1)/\theta(1)$ divides $|G : QN|$ by [3, Corollary 11.29]. Since q does not divide $|G : QN|$, we see q must divide $\theta(1)$. Thus, $q \in \rho(QN)$, and so p and q will be adjacent in $\Delta(P \times QN)$. Because $\Delta(P \times QN)$ is a subgraph of $\Delta(G)$, the primes p and q are adjacent in $\Delta(G)$. In particular, $\rho(G/N)$ contains every prime in $\rho(G)$ that is not adjacent to p in $\Delta(G)$. $\square$

Because we know that a solvable group G has at most one normal non-abelian Sylow p-subgroup, if $\Delta(G)$ has diameter three, then all factor groups G/M have restrictions on the possible normal Sylow p-subgroups. This is the case even if $\Delta(G/M)$ does not have diameter three.

Lemma 4.4. *Let G be a solvable group where $\Delta(G)$ has diameter three. Let M be a minimal normal subgroup of G such that $\Delta(G/M)$ has diameter three. If G has a normal Sylow p-subgroup for $p \in \rho(G)$, then G/M does not have a normal Sylow q-subgroup for $q \in \rho(G)\backslash\{p\}$.*

Proof. Let G be a solvable group with $\Delta(G)$ having diameter three. Suppose G has a normal Sylow p-subgroup P for $p \in \rho(G)$, and let M be a minimal normal subgroup of G and suppose $\Delta(G/M)$ has diameter three. Suppose that the subgroup G/M has a normal non-abelian Sylow q-subgroup Q/M for a prime $q \in \rho(G)\backslash\{p\}$. Because the subgroup PM/M is a normal Sylow p-subgroup of G/M, the Sylow subgroup PM/M must be abelian or we have a contradiction as G/M can have at most one normal non-abelian Sylow p-subgroup. Thus, $P' \subseteq M$. Let R be a Sylow q-subgroup of G. We see that RM is normal in G and so RP is a normal subgroup in G. Further, $[R, P] \subseteq M$ and so R centralizes P/P' by [5, Corollary 3.28]. Hence, $P = \mathbb{C}_P(R)P' = \mathbb{C}_P(R)\Phi(P)$. As $P' \subseteq \Phi(P)$, by the Frattini Argument [4], $P = \mathbb{C}_P(R)$ and so RP is a direct product. Thus R is normal in RP, so it is characteristic, and R is normal in G. This contradicts the fact that G can have at most one normal non-abelian Sylow p-subgroup. $\square$

We will eventually show that if $\Delta(G)$ has diameter three, then G has a normal Sylow p-subgroup for some prime $p \in \rho_3$. However, if ρ_3 is not large enough, the solvable group G cannot have a normal Sylow p_3-subgroup and a diameter three character degree graph. This is necessary to show that the graphs in Figure 4 do not exist as character degree graphs.

Lemma 4.5. *Let G be a solvable group with $\Delta(G)$ having diameter three. Assume for all proper nontrivial normal subgroups M that $\Delta(G/M)$ and $\Delta(M)$ do not have diameter three, and assume $|\rho_3| < 2^n - 1$ where $n = |\rho_1 \cup \rho_2|$. Then G does not have a normal Sylow p_3-subgroup for any prime $p_3 \in \rho_3$.*

Proof. Suppose G has a normal non-abelian Sylow p_3-subgroup P for a prime $p_3 \in \rho_3$. Then, since $\rho(G/P') = \rho(G) \setminus \{p_3\}$ by Lemma 2.4 and the group G/P' is a nontrivial proper factor group of G, the graph $\Delta(G/P')$ cannot have diameter three by the hypothesis. Because $\rho(G/P')$ contains primes from ρ_1 and ρ_4, we have that the graph $\Delta(G/P')$ must be disconnected with components $\rho_1 \cup \rho_2$ and $(\rho_3 \cup \rho_4) \setminus \{p_3\}$. Further, G/P' cannot have any normal non-abelian Sylow subgroups or G would have more than one normal non-abelian Sylow subgroup, and that violates Theorem 2.5. So G/P' is Example (2.4) from Theorem 2.1 as $\rho_1 \cup \rho_2 \subseteq \rho(G/P')$ and $|\rho_1 \cup \rho_2| \geq 2$.

Theorem 5.3 in [7] gives us that $O^{p_2}(G/P') < G/P'$, and so $O^{p_2}(G) < G$ for every prime $p_2 \in \rho_2$. Fix the prime $p_2 \in \rho_2$ and let $K = O^{p_2}(G)$. Because $\rho(K)$ contains ρ_1 and ρ_4, and K is a proper subgroup of G, the graph $\Delta(K)$ is a subgraph of $\Delta(G)$. By the hypothesis, the graph $\Delta(K)$ must be disconnected. As K contains the subgroup P, the subgroup K has a normal non-abelian Sylow p_3-subgroup and p_3 is in the larger component. So K must satisfy the hypotheses of Example (2.6) from Theorem 2.1.

By [7, Lemma 3.6], K/P' satisfies the hypotheses of Example (2.4). Let

$$S/P' = \mathbb{F}(K/P'), \quad R/S = \mathbb{F}(K/S),$$
$$F/P' = \mathbb{F}(G/P'), \quad E/F = \mathbb{F}(G/F).$$

Because S is a characteristic subgroup of K and K is normal in G, S is normal in G, and $S \subseteq F \cap K$. Because $F \cap K$ is a normal subgroup of K, we have $F \cap K = S$. Further, R is characteristic in K and so R is normal in G. By the Diamond Lemma in [5], we obtain $R/S \cong RF/F$ is a normal subgroup of G/F. Because RF/F is normal in G/F, we have $RF/F \subseteq E/F$ and $RF \subseteq E$. Since E/F is a Hall $(\rho_3 \cup \rho_4) \setminus \{p\}$-subgroup, and R/S is a Hall $(\rho_3 \cup \rho_4) \setminus \{p_3\}$-subgroup by [7, Lemma 3.4], $RF = E$.

Because G/P' satisfies the hypotheses of Example (2.4), G/P' is the semidirect product of a subgroup HP'/P' acting on an elementary abelian p_3-group P/P', the positive integer m is defined to be the index $|H : \mathbb{F}(H)|$, the order of P/P' is q^m where q is a p_3-power, and the quotient $(q^m - 1)/(q - 1)$ divides $|E : F|$. Since there are $|\rho_1 \cup \rho_2| = n$ primes that divide m, there must be $2^n - 1$

distinct primes that divide the quotient $(q^m - 1)/(q - 1)$ by Lemma 2.2. Because p_3 does not divide $|E : F|$, and there are less than $2^n - 1$ primes in ρ_3, there must be at least one prime $p_4 \in \rho_4$ that divides $(q^m - 1)/(q - 1)$.

As the graph $\Delta(G)$ is connected, there exist primes $p_2 \in \rho_2$ and $p_3 \in \rho_3$ where p_2 and p_3 are adjacent. Then there is a character $\chi \in \mathrm{Irr}(G)$ such that $p_2 p_3$ divides $\chi(1)$. Let θ be an irreducible constituent of χ_K. By [3, Corollary 11.29], p_3 divides $\theta(1)$ and so P' is not contained in $\ker\theta$. The size of the smaller component of $\Delta(K)$ is either $|\rho_1 \cup \rho_2| = n$ or $|(\rho_1 \cup \rho_2)\backslash\{p_2\}| = n - 1 = a$.

Because K is Example (2.6), K contains a Frobenius group where P is the Frobenius kernel and we let B be a Frobenius complement. The positive integer m_1 is defined to be the index $|K : R|$. There is a p_3-power q_1 such that $|P : P'| = q_1^{m_1}$ and $(q_1^{m_1} - 1)/(q_1 - 1)$ divides $|B|$. By [7, Lemma 3.6], the character degrees $\mathrm{cd}(K \mid P')$ are all divisible by $p|B|$. Because $\theta \in \mathrm{Irr}(K \mid P')$, the quotient $(q_1^{m_1} - 1)/(q_1 - 1)$ divides $\theta(1)$.

If the smaller component of $\Delta(K)$ is $\rho_1 \cup \rho_2$, then

$$m = m_1 \quad \text{and} \quad (q_1^{m_1} - 1)/(q_1 - 1) = (q^m - 1)/(q - 1).$$

Since the prime p_4 divides $(q^m - 1)/(q - 1)$, the prime p_4 divides $\theta(1)$. If the smaller component of $\Delta(K)$ is $\rho_1 \cup \rho_2\backslash\{p_2\}$, then let $m_2 = |G : K|$. Then

$$m_1 = m/m_2 \quad \text{and} \quad q_1^{m_1} = q^m = |P : P'|,$$

so $q_1 = q^{m_2}$. Let r be a Zsigmondy prime divisor of q^m. Then r divides $q^m - 1$ and r does not divide $q^s - 1$ for any $s < m$. Because $q_1^{m_1} - 1 = q^m - 1$, and $m_2 < m$, the prime r divides the quotient $(q_1^{m_1} - 1)/(q_1 - 1)$. Thus, as p_4 is a Zsigmondy prime divisor of $q^m - 1$, the prime p_4 divides $(q_1^{m_1} - 1)/(q_1 - 1)$ and so divides $\theta(1)$.

As p_4 divides $\theta(1)$, the prime p_4 also divides $\chi(1)$. Then the primes p_2 and p_4 are adjacent, but this is impossible as $p_2 \in \rho_2$ and $p_4 \in \rho_4$. Thus, G has no normal Sylow p_3-subgroup. □

The following lemma shows that under suitable hypotheses, if we have a nontrivial factor group G/M, then $|\rho(G/M)| < |\rho(G)|$.

Lemma 4.6. *Let G be a solvable group such that $\Delta(G)$ has diameter three. Assume for all proper nontrivial normal subgroups N that $\Delta(G/N)$ and $\Delta(N)$ do not have diameter three. Further, assume G has no normal Sylow p_4-subgroup for $p_4 \in \rho_4$, $O^{\rho_4}(G) = G$, $|\rho_1 \cup \rho_2| = n$, and $|\rho_3| < 2^n - 1$. Then $\rho(G/M) = \rho(G)$ implies $M = 1$, whenever M is a normal subgroup of G.*

Proof. Suppose G is a solvable group where $\Delta(G)$ has diameter three, G has no normal Sylow p_4-subgroups for any prime $p_4 \in \rho_4$, the subgroup $O^{\rho_4}(G) = G$,

$|\rho_1 \cup \rho_2| = n$, and $|\rho_3 \cup \rho_4| < 2^n - 1$. Assume that if N is a proper nontrivial normal subgroup of G that the graphs $\Delta(N)$ and $\Delta(G/N)$ do not have diameter three. Let M be a nontrivial normal subgroup of G, and suppose that $\rho(G/M) = \rho(G)$. Then $\Delta(G/M)$ does not have diameter three, and since $\Delta(G/M)$ is a subgraph of $\Delta(G)$, the graph $\Delta(G/M)$ must be disconnected with components $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$. The group G/M must be one of the examples from Theorem 2.1. Because both components have more than one vertex, we must be in Example (2.4) or Example (2.6). If G/M is Example (2.4), then it has no normal Sylow p-subgroups for any prime $p \in \rho(G/M)$ and in particular, G has no normal non-abelian Sylow p-subgroups either as $\rho(G/M) = \rho(G)$. By [7, Theorem 5.3], we have $O^{p_2}(G/M) < G/M$. Let $H/M = O^{p_2}(G/M)$. Then since $|G/M : H/M|$ is a nontrivial power of p_2, the index $|G : H|$ is a nontrivial power of the prime p_2. So $O^{p_2}(G) < G$. This contradicts Lemma 4.1, hence G/M is Example (2.6). By [7, Lemma 3.6], the group G/M has a normal Sylow p-subgroup for some prime $p \in \rho_3 \cup \rho_4$. Let Q/M be that Sylow p-subgroup of G/M and P a Sylow p-subgroup of G that is contained in Q.

Consider $K = O^{p_2}(G)$ for a fixed prime $p_2 \in \rho_2$. Because $O^{p_2}(G/M) < G/M$ by [7, Theorem 5.3], we have K is a proper subgroup of G, and so $\Delta(K)$ must be disconnected. Also, notice P is contained in K.

Claim. *The group K is Example* (2.6) *from Theorem* 2.1.

Proof. Because the only prime that could be missing from $\rho(K)$ is the prime p_2, at least one of the components is larger than one as $\rho(G)\backslash\{p_2\} \subseteq \rho(K)$. If K is Example (2.1) or Example (2.5), then $|\rho_1| = |\rho_2| = 1$, the prime $p_2 \notin \rho(K)$, and K has an abelian p_1-complement. However, this implies $O^{p_4}(K)$ is a proper subgroup of K. Then $O^{p_4}(G)$ is a proper subgroup in G and this is a contradiction.

Suppose K is Example (2.4). There exists a prime $p_3 \in \rho_3$ that is adjacent to p_2 and a character $\chi \in \mathrm{Irr}(G)$ such that $p_2 p_3$ divides $\chi(1)$. Let θ be an irreducible constituent of χ_K. Then by [3, Corollary 11.29], the quotient $\chi(1)/\theta(1)$ divides the index $|G : K|$. Because the index $|G : K|$ is a p_2-power, p_3 divides $\theta(1)$. Because K is Example (2.4), there is only one character degree that is divisible by the primes in the larger component, $\rho_3 \cup \rho_4$, and so $\theta(1)$ must be that character degree. Hence, every prime $p_4 \in \rho_4$ divides $\theta(1)$. Thus, p_4 divides $\chi(1)$ and p_2 and p_4 are adjacent, which is a contradiction. Thus, K must be Example (2.6) from Theorem 2.1. □

Since K is Example (2.6), K has a normal non-abelian Sylow r-subgroup R for some prime $r \in \rho_3 \cup \rho_4$. Because R is a characteristic subgroup of K, R is normal in G. Further, as $|G : K|$ is a power of p_2, R is a Sylow subgroup of G. Because G/M has a normal non-abelian Sylow p-subgroup, and RM/M is a normal non-abelian Sylow r-subgroup of G/M, $p = r$ and $R = P$. As G has no normal non-abelian Sylow p-subgroups for $p \in \rho_4$, the prime p must be in ρ_3. But this contradicts Lemma 4.5. Thus $\rho(G/M)$ does not equal $\rho(G)$. □

5 Graphs with six vertices

In this section we show that if G is a solvable group then $\Delta(G)$ is not one of the two graphs from Figure 4. We start by showing that under suitable conditions, if $\Delta(G)$ has diameter three, then G does not have a normal non-abelian Sylow p-subgroup for any prime $p \in \rho_1 \cup \rho_2$.

Lemma 5.1. *Let G be a solvable group with $\Delta(G)$ having diameter three. Assume for all proper nontrivial normal subgroups M that $\Delta(G/M)$ and $\Delta(M)$ do not have diameter three. Further, assume $\rho(G/M) = \rho(G)$ implies $M = 1$. Then G does not have a normal Sylow p_2-subgroup for any prime $p_2 \in \rho_2$.*

Proof. Suppose G has a normal Sylow p_2-subgroup P for a fixed prime $p_2 \in \rho_2$ and let H be a p_2-complement of G. By [9, Lemma 3], $\rho(G/P') = \rho(G)\setminus\{p_2\}$. Because $\rho(G/P')$ contains the primes in ρ_1 and ρ_4, and $\Delta(G/P')$ is a subgraph of $\Delta(G)$, the graph $\Delta(G/P')$ is disconnected. Because P is not central in G and H acts on P nontrivially, we see that P/P' is not central in G/P'. From [7, Theorem 5.5], the Fitting subgroup of G/P' has at most one non-central Sylow subgroup, which is P/P'. Let $F = \mathbb{F}(G)$. Since all of the other Sylow subgroups of $\mathbb{F}(G/P') = F/P'$ are central, F/P' is abelian, and thus G/P' is as described in Example (2.4) from Theorem 2.1. This is because any solvable group having an abelian Fitting subgroup whose graph has two connected components, where at least one connected component has size larger than one, must satisfy the hypotheses of Example (2.4). Further, G/P' has Fitting height 3.

Let $E/F = \mathbb{F}(G/F)$ and $Z = \mathbb{C}_H(P/P')$. Then the quotient G/E is a cyclic $(\rho_1 \cup \rho_2)\setminus\{p_2\}$-group, E/F is a cyclic $\rho_3 \cup \rho_4$-group and $E \cap H$ is abelian. Because $H/(E \cap H)$ is isomorphic to G/E, the subgroup $E \cap H$ contains every Sylow p_4 subgroups of G as G/E is a $\rho_1 \cup \rho_2$-subgroup. So, the subgroup H has a normal Sylow p_4-subgroup Q for some $p_4 \in \rho_4$. By [7, Lemma 3.4] we have that $E \cap H$ acts irreducibly on $[E, F]/P'$. Also, $\rho(PQ) = \{p_2, p_4\}$ and $\Delta(PQ)$ has two connected components. Because the Fitting height of PQ is 2, by [7, Lemma 4.1] we know that PQ is Example (2.1) of Theorem 2.1. Let $C = \mathbb{C}_P(Q)$. Because PQ is Example (2.1) from Theorem 2.1, $P' \subseteq C$, every non-linear irreducible character is fully ramified with respect to P/C, and Q acts nontrivially on P fixing every non-linear irreducible character of P. By [14, Theorem 19.3], we have $P' = [P, Q]'$, and so $P' < [P, Q]$. Because $P \subseteq F$ and $Q \subseteq E$, we have $[P, Q] \subseteq [E, F]$. Since $[E, F]/P'$ is irreducible under the action of $E \cap H$, we have $[E, F] = [P, Q]$.

Fix a prime $p_1 \in \rho_1$. As p_1 and p_2 are adjacent in $\Delta(G)$, there exists a character $\chi \in \mathrm{Irr}(G)$ such that $p_1 p_2$ divides $\chi(1)$. Let $\theta \in \mathrm{Irr}(P)$ be an irreducible constituent of χ_P. Notice that θ is invariant in F as F is nilpotent, and in E because no prime in $\rho_3 \cup \rho_4$ divides $\chi(1)$. By Glauberman's Lemma [3, Lemma 13.8], there exists an $E \cap H$-invariant irreducible constituent of $\theta_{P'}$. Thus, $\mathbb{C}_{P'}(E \cap H) > 1$. By Lemma 4.2, P' is central in P and so $\mathbb{C}_{P'}(E \cap H)$ is normal in P. Also, H normalizes P' and $E \cap H$. So, H and $G = PH$ normalize $\mathbb{C}_{P'}(E \cap H)$. Furthermore, P' is minimal normal in G by Lemma 4.2, thus, $P' \subseteq \mathbb{C}_P(E \cap H)$. By Fitting's Lemma,

$$P/P' = [P, Q]/P' \times C/P' = [E, F]/P' \times \mathbb{C}_P(E \cap H)/P'.$$

Because $\mathbb{C}_P(E \cap H) \subseteq C$, we have $\mathbb{C}_P(E \cap H) = C$ and so E satisfies the hypotheses of Example (2.1) and has components $\{p_2\}$ and $\rho_3 \cup \rho_4$ by [7, Lemma 3.1]. There exists a character $\psi \in \mathrm{Irr}(G)$ such that $p_2 p_3$ divides $\psi(1)$ for some $p_3 \in \rho_3$. Let $\gamma \in \mathrm{Irr}(E)$ such that γ is an irreducible constituent of ψ_E. By [3, Corollary 11.29], the quotient $\psi(1)/\gamma(1)$ divides $|G : E|$, and since neither p_2 nor p_3 divides $|G : E|$, we have that p_2 and p_3 divide $\gamma(1)$. This is a contradiction to the fact that $\Delta(E)$ is disconnected. Thus, G has no normal Sylow p_2-subgroup. □

The hypotheses for the following lemma have the additional condition that $O^p(G) = G$ for all primes $p \in \rho_3 \cup \rho_4$.

Lemma 5.2. *Let G be a solvable group with $\Delta(G)$ having diameter three. Assume for all proper nontrivial normal subgroups M that $\Delta(G/M)$ and $\Delta(M)$ do not have diameter three. Further, suppose that $\rho(G/M) = \rho(G)$ implies $M = 1$ and $O^{\rho_3}(G) = G$ or $O^{\rho_4}(G) = G$. Then G does not have a normal Sylow p_1-subgroup for any prime $p_1 \in \rho_1$.*

Proof. Suppose G has a normal Sylow p_1-subgroup P for some $p_1 \in \rho_1$, and let $N = O_{p_1'}(G)$. By Lemma 4.3, we have $\{p_1\} \cup \rho_3 \cup \rho_4 \subseteq \rho(G/N)$. By the hypotheses, if $N > 1$, then $|\rho(G/N)| < |\rho(G)|$, and so, by Lemma 4.3, there exists a prime $p_2 \in (\rho_1 \cup \rho_2)\backslash\{p_1\}$ that is not in $\rho(G/N)$. Since $\rho(G/N)$ contains $\{p_1\}$ and ρ_4, it follows that $\Delta(G/N)$ has two connected components. By our assumption, G/N has a normal Sylow p_1-subgroup, and so G/N is either Example (2.1) or Example (2.6) from Theorem 2.1. Because p_1 is in the smaller component, the group G/N must be Example (2.1), and the components of $\Delta(G/N)$ are $\{p_1\}$ and $\rho_3 \cup \rho_4$. In Example (2.1), G has an abelian Hall p_1-complement. Thus for $p \in \rho_3 \cup \rho_4$, the subgroup $O^p(G)$ is a proper subgroup of G that contains P, which is a contradiction to the hypothesis that either $O^{\rho_3}(G) = G$ or $O^{\rho_4}(G) = G$. Thus, $N = 1$.

Because $O_{p_1'}(G) = 1$, the Fitting subgroup of G is P. Let H be a p_1-complement for G; H acts faithfully on P, and by [14, Lemma 18.1 and the discussion on p. 254], every prime divisor of $|H|$ occurs in $\rho(G)$. Pick a character $\gamma \in \mathrm{Irr}(P)$ with $\gamma(1) > 1$ and a character $\chi \in \mathrm{Irr}(G \mid \gamma)$. Then p_1 divides $\chi(1)$ and no prime in $\rho_3 \cup \rho_4$ divides $\chi(1)$. Thus, $G/P \cong H$ has an abelian Hall $\rho_3 \cup \rho_4$-subgroup by [14, Theorem 12.9]. Let $L = O_{(\rho_1 \cup \rho_2)\setminus\{p_1\}}(H)$ and $E/L = \mathbb{F}(H/L)$. Then H/L has an abelian Hall $\rho_3 \cup \rho_4$-subgroup, which must be E/L by the Hall–Higman Theorem [5, Theorem 3.21]. If $E = H$, then we have $O^{p_3}(H) < H$ for $p_3 \in \rho_3 \cup \rho_4$, and so $O^{p_3}(G) < G$. This is a contradiction to the hypothesis that either $O^{\rho_3}(G) = G$ or $O^{\rho_4}(G) = G$, thus, $E < H$.

Consider the normal subgroup PE. We know $\{p_1\} \cup \rho_3 \cup \rho_4 \subseteq \rho(PE)$. Since $PE < G$, we have that $\Delta(PE)$ must be disconnected and the components are $(\rho_1 \cup \rho_2) \cap \rho(PE)$ and $\rho_3 \cup \rho_4$. If $p_2 \in \rho(PE)$ for any prime $p_2 \in (\rho_1 \cup \rho_2)\setminus\{p_1\}$, then because PE has a normal Sylow p_1-subgroup and both components would have size larger than one, we must be in Example (2.6) of Theorem 2.1. However, p_1 must be in the larger component, which it is not, and so PE satisfies Example (2.1) of Theorem 2.1. Because PE is Example (2.1) from Theorem 2.1, the components of $\Delta(PE)$ are $\{p_1\}$ and $\rho_3 \cup \rho_4$, the subgroup E is an abelian Hall $\rho_3 \cup \rho_4$-subgroup, $L = 1$, and $P' \subseteq \mathbb{Z}(PE)$.

Let Q be the Sylow p_4-subgroup of E for some prime $p_4 \in \rho_4$. Because PE is Example (2.1) from Theorem 2.1, every non-linear irreducible character of P is fully ramified with respect to $P/\mathbb{C}_P(E)$, the subgroup Q acts faithfully on P, fixing every non-linear irreducible character of P. By [14, Theorem 19.3], we have $P' = [P, Q]'$. Let $\lambda \in \mathrm{Irr}([P, Q]/P')$ be non-principal. Because P' is central in P by Lemma 4.2, the stabilizer of λ in G is $P\,\mathbb{C}_H(\lambda)$.

Because Q acts faithfully, $\mathbb{C}_Q(\lambda) < Q$, which implies p_4 divides $|Q : \mathbb{C}_Q(\lambda)|$. Thus p_4 divides $|H : \mathbb{C}_H(\lambda)|$. Since p_2 and p_4 are not adjacent in $\Delta(G)$ for $p_2 \in (\rho_1 \cup \rho_2)\setminus\{p_1\}$, we have $\mathbb{C}_H(\lambda)$ contains a Hall $(\rho_1 \cup \rho_2)\setminus\{p_1\}$-subgroup of H. Further, λ extends to $P\,\mathbb{C}_H(\lambda)$ and

$$\mathrm{cd}(P\,\mathbb{C}_H(\lambda) \mid \lambda) = \mathrm{cd}(\mathbb{C}_H(\lambda)).$$

So by Clifford's theory, no degree in $\mathrm{cd}(\mathbb{C}_H(\lambda))$ is divisible by any prime in $(\rho_1 \cup \rho_2)\setminus\{p_1\}$. By Itô's Theorem [3, Theorem 12.34], $\mathbb{C}_H(\lambda)$ contains a unique Hall $(\rho_1 \cup \rho_2)\setminus\{p_1\}$-subgroup of H, which is abelian. Recall, $E = \mathbb{F}(H)$ is abelian and the index $|H : E|$ is only divisible by the primes in $(\rho_1 \cup \rho_2)\setminus\{p_1\}$. By [3, Theorem 6.15], $\mathrm{cd}(H)$ contains only products of primes in $(\rho_1 \cup \rho_2)\setminus\{p_1\}$ and so $\rho(H) = (\rho_1 \cup \rho_2)\setminus\{p_1\}$. Thus, $\mathbb{C}_H(\lambda)$ is abelian. The stabilizer of λ in $[P, Q]H$ is $[P, Q]\,\mathbb{C}_H(\lambda)$ and λ extends to this stabilizer.

Consider the group $[P, Q]H$. We have

$$\begin{aligned}\operatorname{cd}([P, Q]H) &= \operatorname{cd}\left(\frac{[P, Q]H}{P'}\right) \cup \operatorname{cd}([P, Q]H \mid P') \\ &= \operatorname{cd}\left(\frac{[P, Q]H}{[P, Q]}\right) \cup \operatorname{cd}\left(\frac{[P, Q]H}{P'} \,\middle|\, \frac{[P, Q]}{P'}\right) \cup \operatorname{cd}([P, Q]H \mid P').\end{aligned}$$

Observe that

$$\operatorname{cd}([P, Q]H \mid \lambda) = \{|H : \mathbb{C}_H(\lambda)|a \mid a \in \operatorname{cd}(\mathbb{C}_H(\lambda))\} = \{|H : \mathbb{C}_H(\lambda)|\}.$$

Thus the primes in $\rho_3 \cup \rho_4$ are the only possible prime divisors of degrees in

$$\operatorname{cd}\left(\frac{[P, Q]H}{P'} \,\middle|\, \frac{[P, Q]}{P'}\right).$$

By [6, Lemma 1], $[P, Q]/P'$ is irreducible under the action of Q and so

$$P' = \mathbb{C}_{[P,Q]}(E).$$

Since $\Delta(PQ)$ has two connected components, every non-linear irreducible character of $[P, Q]$ is fully ramified with respect to $[P, Q]/P'$. Then $[P, Q]E$ is Example (2.1) from Theorem 2.1 and the two connected components are $\{p_1\}$ and $\rho_3 \cup \rho_4$.

By [7, Lemma 3.1], $\operatorname{cd}([P, Q]E \mid P')$ consists of powers of the prime p_1. Since $|[P, Q]H : [P, Q]E| = |H : E|$ is only divisible by primes in $(\rho_1 \cup \rho_2)\backslash\{p_1\}$, the only primes that divide degrees in $\operatorname{cd}([P, Q]H \mid P')$ are the primes in $\rho_1 \cup \rho_2$. Because Q acts faithfully on $[P, Q]$, the prime $p_4 \in \rho([P, Q]H)$. As p_4 was chosen arbitrarily from $\rho_3 \cup \rho_4$, the graph $\Delta([P, Q]H)$ has two connected components: $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$. Because $[P, Q]H$ has a normal non-abelian Sylow p_1-subgroup and both components have more than one vertex, $[P, Q]H$ must satisfy Example (2.6) from Theorem 2.1. However, p_1 is in the component $\rho_1 \cup \rho_2$ and $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$. This is a contradiction to [7, Lemma 3.6] and Pálfy's inequality, Theorem 2.3, and so, G has no normal Sylow p_1-subgroups. □

As the following argument is used frequently, we have made a lemma. This lemma shows how Theorem 2.6 can be applied once the group G has been shown to have no normal non-abelian Sylow p-subgroups and satisfies the condition that $|\rho(G/M)| < |\rho(G)|$ for all proper nontrivial normal subgroups M.

Lemma 5.3. *Let G be a solvable group and suppose G has no normal Sylow p-subgroups for $p \in \rho(G)$. Suppose for all minimal normal subgroups M that $\rho(G/M) < \rho(G)$ and $\Delta(G/M)$ does not have diameter three. Then $\Delta(G)$ does not have diameter three.*

Proof. We write F for the Fitting subgroup of G and $\Phi(G)$ for the Frattini subgroup of G. By [14, Lemma 18.1 and the discussion on p. 254] we know that $\rho(G) = \pi(|G : F|)$. Because $F(G)/\Phi(G) = F(G/\Phi(G))$, we see that

$$\rho(G/\Phi(G)) = \rho(G)$$

and so $\Phi(G) = 1$. Let M be a proper nontrivial normal subgroup of G and consider $\Delta(G/M)$. Since $\Delta(G/M)$ is a subgraph of $\Delta(G)$, we see that it cannot have diameter three by hypothesis. Thus, $\Delta(G/M)$ must either have two connected components or have diameter two. By Theorem 2.6, we see that $\Delta(G)$ must also have two connected components or diameter two. □

We now have all the tools to show that the graphs in Figure 4 are not the character degree graphs for any solvable group G.

Proof of Theorem 1. Let $\mathcal{F}$ be the family of graphs that are isomorphic to the graphs in Figure 4. In particular, the graphs satisfy Pálfy's Condition, have six vertices, and have diameter three. Let G be a minimal counter-example with respect to $|G|$, such that $\Delta(G) \in \mathcal{F}$. The partition of vertices is as follows: $|\rho_1 \cup \rho_2| = 2$, $|\rho_3| = 1$ or 2, $|\rho_4| \geq 2$.

Working by induction, if M is a proper normal subgroup of G, then $\Delta(M)$ is a subgraph of $\Delta(G)$. If $\Delta(M)$ has diameter three, then either $\Delta(M) \in \mathcal{F}$, or $\Delta(M)$ is a graph that does not exist as a character degree graph by Theorem 1.2, Theorem 1.3, or by Pálfy's Condition. As G is minimal, $\Delta(M)$ cannot have diameter three and so is either disconnected or has diameter at most two.

If G has a normal Sylow p_4-subgroup P for some prime $p_4 \in \rho_4$, then

$$\rho(G/P') = \rho(G)\backslash\{p_4\}$$

by Lemma 2.4. The graph $\Delta(G/P')$ has five vertices. Because ρ_4 has more than one vertex, $\Delta(G/P')$ either has diameter three or is disconnected. Let $p_2 \in \rho_2$ and $p_3 \in \rho_3$ be adjacent in $\Delta(G)$ such that the product $p_2 p_3$ divides $\chi(1)$ for a character $\chi \in \mathrm{Irr}(G)$. Because p_2 and p_4 are not adjacent, we have $P' \leq \ker \chi$. Thus, $\chi \in \mathrm{Irr}(G/P')$ and so p_2 and p_3 are adjacent in $\Delta(G/P')$. Because a graph with five vertices and diameter three does not occur as a character degree graph by Theorem 1.3, G does not have a normal Sylow p_4-subgroup. Notice that deleting one or more edges incident to the vertex p_4 produces a graph that does not satisfy Pálfy's Condition, and so $O^{p_4}(G) = G$ by [10, Lemma 3.1].

By Lemma 4.5, G does not have a normal Sylow p_3-subgroup for any prime $p_3 \in \rho_3$. Because $|\rho_1 \cup \rho_2| = 2$ and $|\rho_3| < 3$, Lemma 4.6 says that

$$|\rho(G/M)| < |\rho(G)|$$

for all proper nontrivial normal subgroups M. We see by Lemmas 5.1 and 5.2

that G does not have a normal Sylow p-subgroup for any prime $p \in \rho_1 \cup \rho_2$. Thus G has no normal non-abelian Sylow subgroups and so by Lemma 5.3, the graph $\Delta(G)$ does not have diameter three, a contradiction. □

An immediate corollary is the following:

Corollary 5.4. *Let G be a solvable group where $\Delta(G)$ has diameter three and $|\rho(G)| = 6$. Then $|\rho_3| = 3$.*

Proof. We know from Theorem 3.1 that if G is a solvable group where $|\rho(G)| = 6$, then $\Delta(G)$ is either the graph from Figure 3 or the graphs from Figure 4. From Theorem 1 we see that $\Delta(G)$ must be the graph in Figure 3. We observe that ρ_3 must have three vertices. □

Using similar methods to show that the two graphs with six vertices do not occur as diameter three character degree graphs of solvable groups, we can also show that ρ_3 must have at least three vertices.

Proof of Theorem 2. We prove this using induction on the size of $\rho(G)$. Let G be a minimal counter-example such that $|\rho(G)| > 6$, the graph $\Delta(G)$ has diameter three, and $|\rho_3| = 1$ or 2. Because $|\rho_1 \cup \rho_2| \leq |\rho_3 \cup \rho_4|$, the subset ρ_4 has at least two vertices. Suppose G has a normal Sylow p_4-subgroup P for some prime $p_4 \in \rho_4$. Then by Lemma 2.4, the set of vertices for $\rho(G/P')$ is $\rho(G)\setminus\{p_4\}$. The graph $\Delta(G/P')$ is not disconnected. Let $p_2 \in \rho_2$ and $p_3 \in \rho_3$ be adjacent in $\Delta(G)$ such that the product $p_2 p_3$ divides $\chi(1)$ for a character $\chi \in \mathrm{Irr}(G)$. Because p_2 and p_4 are not adjacent, we have $P' \leq \ker \chi$. Thus, $\chi \in \mathrm{Irr}(G/P')$ and so p_2 and p_3 are adjacent in $\Delta(G/P')$. Thus, the graph $\Delta(G/P')$ is connected and has diameter three. This contradicts the induction hypothesis and so G has no normal Sylow p_4-subgroups for any prime $p_4 \in \rho_4$. Furthermore, deleting one or more edges incident to the vertex p_4 produces a graph that does not satisfy Pálfy's Condition, and so, $O^{p_4}(G) = G$ by [10, Lemma 3.1].

Let M be a proper, nontrivial, normal subgroup of G. The graphs $\Delta(M)$ and $\Delta(G/M)$ are subgraphs of $\Delta(G)$. In particular, $\rho(G/M) \cap \rho_3$ and $\rho(M) \cap \rho_3$ have at most two vertices. Therefore, the graphs $\Delta(M)$ and $\Delta(G/M)$ cannot have diameter three.

Since $|\rho_3| \leq 2$ and $|\rho_1 \cup \rho_2| = a$ is at least 2, $|\rho_3| < 2^a - 1$. Thus by Lemma 4.5, G does not have any normal Sylow p_3-subgroups for any prime $p_3 \in \rho_3$. By Lemma 4.6, if M is a normal subgroup of G and $\rho(G/M) = \rho(G)$, then $M = 1$. We see by Lemmas 5.1 and 5.2 that G does not have a normal Sylow p-subgroup for any prime in $\rho_1 \cup \rho_2$. Hence, G has no normal non-abelian Sylow subgroups. Thus, by Lemma 5.3, $\Delta(G)$ does not have diameter three and $|\rho_3| \geq 3$. □

6 No normal non-abelian Sylow p_4-subgroup

When showing that the graphs in Figure 4 were not the character degree graphs for any solvable group G, we were able to use the fact that the graphs in Figure 2 are not character degree graphs for any solvable group G. Is it possible that there is a group G that has a character degree graph $\Delta(G)$ as in Figure 3 and a normal Sylow p_4-subgroup P for the prime $p_4 \in \rho_4$? The graph $\Delta(G/P')$ will have diameter two in this case, and so none of our current tools answer this question. The following lemma shows that it is not possible for a solvable group G to have a character degree graph with diameter three and a normal Sylow p_4-subgroup for prime $p_4 \in \rho_4$.

Lemma 6.1. *Let G be a solvable group with $\Delta(G)$ having diameter three. Then G does not have a normal Sylow p_4-subgroup for any prime $p_4 \in \rho_4$.*

Proof. Let G be a minimal counter-example where $\Delta(G)$ has diameter three and G has a normal Sylow p_4-subgroup P for a prime $p_4 \in \rho_4$. We know that $|\rho_3| \geq 3$ by Theorem 2.

Claim 1. *We have $O^{p_3}(G) = G$ for all primes $p_3 \in \rho_3 \cup \rho_4 \backslash \{p_4\}$.*

Proof. Suppose $O^{p_3}(G) < G$. Since $\Delta(O^{p_3}(G))$ contains all edges not incident to $\{p_3\}$ and the primes except possibly the prime p_3, we can find a prime $q \in \rho_3$ other than p_3 that is adjacent to some prime $p_2 \in \rho_2$. Thus $\Delta(O^{p_3}(G))$ is connected and has diameter three. This contradicts our assumption that G is a minimal counter-example, as $P \subseteq O^{p_3}(G)$. □

Claim 2. *The subgroup $O_{p_4'}(G) = 1$ and the Fitting subgroup of G is P.*

Proof. Suppose that M is a minimal normal subgroup contained in $O_{p_4'}(G)$. Then $\rho(G/M)$ contains $\rho_1 \cup \rho_2 \cup \{p_4\}$ by Lemma 4.3, and $\Delta(G/M)$ is disconnected or has diameter three. Since G/M has a normal Sylow p_4-subgroup PM/M, we know that $\Delta(G/M)$ must be disconnected by our assumption, and further G/M is either Example (2.6) or Example (2.1) from Theorem 2.1. The components are either $\rho_1 \cup \rho_2$ and $\{p_4\}$, or $\rho_1 \cup \rho_2$ and $(\rho_3 \cup \rho_4) \cap \rho(G/M)$, where $|\rho_1 \cup \rho_2| = n$ and $|(\rho_3 \cup \rho_4) \cap \rho(G/M)| \geq 2^n - 1$, by Pálfy's inequality.

We assume first that both of the connected components of $\Delta(G/M)$ have size larger than one and so the group G/M is Example (2.6) in Theorem 2.1. Let $F/M = \mathbb{F}(G/M)$ and

$$(E/M)/(F/M) = \mathbb{F}((G/M)/(F/M)) = E/F = \mathbb{F}(G/F).$$

Then by [7, Lemma 3.6], $F/M = PM/M \times Z/M$ where Z/M is a central subgroup of G/M. Let ϕ be a nonlinear irreducible character of the subgroup P so that p_4 divides $\phi(1)$. Then $\phi \times 1_M \in \mathrm{Irr}(P \times M)$ and M is contained in the kernel of $\phi \times 1_M$, ie $(\phi \times 1_M)(a) = 1$ for all $a \in M$. By [7, Lemma 3.6], there exists a subgroup B of G such that $PM/M \cdot B/M$ is a Frobenius group, PM/M is the Frobenius kernel and B/M is a Frobenius complement. By [3, Theorem 6.34], $(\phi \times 1_M)^{PB/M}$ is an irreducible character of PB/M, and so $(\phi \times 1_M)^{PB}$ is an irreducible character of PB.

Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1)$ divisible by $p_2 p_3$ for primes $p_2 \in \rho_2$ and $p_3 \in \rho_3$. Let $\psi \in \mathrm{Irr}(PZ)$ and $\gamma \in \mathrm{Irr}(Z)$ where ψ is an irreducible constituent of χ_{PZ} and γ is an irreducible constituent of ψ_Z. Notice that if $\gamma = 1$, then $Z \subseteq \ker(\chi)$ and so M is also in $\ker(\chi)$. But then p_2 and p_3 are adjacent in $\Delta(G/M)$, which is not the case, and so γ is not the principal character.

Consider $\phi \times \gamma$, an irreducible character of PZ, and its stabilizer $G_{\phi\times\gamma}$, which is the intersection of the stabilizers, G_ϕ and G_γ. Notice, that by [3, Corollary 6.28], ϕ extends to G_ϕ. If $\theta \in \mathrm{Irr}(G)$ lies over ϕ, then since p_4 divides $\theta(1)$, we know that no prime in $\rho_1 \cup \rho_2$ divides $\theta(1)$. By [14, Theorem 12.9], we see that G/P contains an abelian Hall $\rho_1 \cup \rho_2$-subgroup of G/P. As ϕ extends to G_ϕ, we can apply Gallagher's Theorem [3, Corollary 6.17]. Any character degree $a \in \mathrm{cd}(G_\phi/P)$ can be multiplied by $\phi(1)$ to get a character degree in $\mathrm{cd}(G)$, and so no prime in $\rho_1 \cup \rho_2$ divides any character degree of G_ϕ/P. Thus, by Itô's Theorem [3, Corollary 12.34], G_ϕ/P contains a unique Hall $\rho_1 \cup \rho_2$-subgroup AP/P. By Clifford's theory, we get that no prime in $\rho_1 \cup \rho_2$ divides $|G : G_\phi|$.

Further, no prime in $\rho_1 \cup \rho_2$ divides $|G : G_{\gamma\times\phi}|$. If a prime $p_1 \in \rho_1 \cup \rho_2$ did divide $|G : G_{\gamma\times\phi}|$, then p_1 would divide $|G : G_\phi|$ and p_1 and p_4 would be adjacent, which is not possible. So. since $G_{\phi\times\gamma}/P$ also contains a Hall $\rho_1 \cup \rho_2$-subgroup of G, it must contain AP/P. Thus, G_γ contains A.

Consider the character ϕ^g and its stabilizer G_{ϕ^g} for some $g \in G$. The character ϕ^g also extends to its stabilizer by [3, Corollary 6.28]. By Gallagher's theorem [3, Corollary 6.17], any character degree in $\mathrm{cd}(G_{\phi^g}/P)$ is a multiple of $\phi^g(1)$, and so no prime in $\rho_1 \cup \rho_2$ divides any character degree of G_{ϕ^G}/P. So, this factor group contains a unique Hall $\rho_1 \cup \rho_2$-subgroup $A^g P/P$. Because $G_{\phi^g\times\gamma}/P$ contains A^g/P, G_γ also contains A^g, and so G_γ contains all conjugates of A. Hence, G_γ contains $O^{(\rho_1\cup\rho_2)'}(G)$. However, G_γ contains P and $O^{p_3}(G) = G$ for all $p_3 \in \rho_3 \cup \rho_4 \backslash \{p_4\}$, and so $G_\gamma = G$.

Because $1_P \times \gamma \in \mathrm{Irr}(P \times Z)$, and E/F is cyclic, we get that $1_P \times \gamma$ extends to E by [3, Corollary 11.22]. Then, as G/E is cyclic, $1_P \times \gamma$ extends to all of the Sylow subgroups of G/E. By [3, Corollary 11.31], we get that $1_P \times \gamma$ extends to G, and hence γ extends to G. Call the extension of γ to G $\hat{\gamma}$. By Gallagher's Theorem, $\eta\hat{\gamma}$ are all of the irreducible constituents of γ^G, where $\eta \in \mathrm{Irr}(G/Z)$. In particular, $\chi = \eta\hat{\gamma}$ for some $\eta \in \mathrm{Irr}(G/Z)$. Since M is contained in Z, we know that $\Delta(G/Z)$ is disconnected and any $p_2 \in \rho_2$ and $p_3 \in \rho_3$ are not adjacent. Since $p_2 p_3$ divides $\chi(1)$ and p_2 does not divide $\gamma(1)$, we know that p_2 must divide $\eta(1)$. Thus, p_3 cannot divide $\eta(1)$ as p_2 and p_3 are not adjacent in $\Delta(G/Z)$, and so p_3 divides $\gamma(1)$. However, there exists an $\alpha \in \mathrm{Irr}(G/Z)$ such that p_1 divides $\alpha(1)$ for some prime $p_1 \in \rho_1$ and $\alpha\gamma \in \mathrm{Irr}(G)$. This contradicts the fact that p_1 and p_3 are not adjacent in $\Delta(G)$.

Thus, $(\rho_3 \cup \rho_4) \cap \rho(G/M) = \{p_4\}$ and so G/M is Example (2.1) from Theorem 2.1. Then, G/M has an abelian Hall p_4-complement, and hence $O^{p_3}(G) < G$ for some prime $p_3 \in \rho_3 \cup \rho_4 \backslash \{p_4\}$, which we showed cannot happen. Thus we have $O_{p_4'}(G) = 1$ and $\mathbb{F}(G) = P$. □

Claim 3. *The subgroup P' is a minimal normal subgroup of G and P' is central in P.*

Proof. Suppose there exists $1 \neq X \subset P'$ where P'/X is a chief factor. Then since $\rho(G/P') = \rho(G)\backslash\{p_4\}$ and $\Delta(G/P')$ is connected, we have that $\rho(G/X) = \rho(G)$ and in particular $\Delta(G/X)$ has diameter three. However, this contradicts our assumption, and P' is minimal normal. Because $\mathbb{Z}(P)$ is characteristic in P and so normal in G, it follows that $P' \cap \mathbb{Z}(P)$ is normal in G. Because P is nilpotent, $1 < P' \cap \mathbb{Z}(P)$, and so $P' \subseteq \mathbb{Z}(P)$ as P' is a minimal normal subgroup of G. □

Let $\theta \in \mathrm{Irr}(P)$ be nonlinear and H a p_4-complement in G. Notice that H acts faithfully on P and every prime divisor of $|H|$ occurs in $\rho(G)$. Let $\chi \in \mathrm{Irr}(G)$ be an irreducible constituent of θ^G. Then, since no prime in $\rho_1 \cup \rho_2$ divides $\chi(1)$, by [14, Theorem 12.9], $G/P \cong H$ has an abelian Hall $\rho_1 \cup \rho_2$-subgroup. Let $L = O_{\rho_3 \cup \rho_4 \backslash \{p_4\}}(H)$ and $E/L = \mathbb{F}(H/L)$. Note that E/L is a $\rho_1 \cup \rho_2$-subgroup. Since H/L has an abelian Hall $\rho_1 \cup \rho_2$-subgroup, it follows that E/L is the Hall $\rho_1 \cup \rho_2$-subgroup by the Hall–Higman Theorem in [5, Theorem 3.21] and since $O^{\rho_3 \cup \rho_4 \backslash \{p_4\}}(G) = G$, $E = H$. In particular, L is a Hall $\rho_3 \cup \rho_4 \backslash \{p_4\}$-subgroup of H.

Let K be a p_2-complement in H for a fixed prime $p_2 \in \rho_2$. Note that $L \subseteq K$. Since $E/L = H/L$ is abelian, K is normal in H. Because G is a minimal counterexample and PK has a normal non-abelian Sylow p_4-subgroup, the graph $\Delta(PK)$ is disconnected. The subgroup PK is a Hall p_2-complement of G and so we have $\rho(PK) = \rho(G) \setminus \{p_2\}$. Also, the only edges in $\Delta(PK)$ that are not retained from $\Delta(G)$ are those that are incident to the prime p_2. The subsets $\rho_1 \cup \rho_2 \backslash \{p_2\}$ and $\rho_3 \cup \rho_4$ induce complete subgraphs of $\Delta(PK)$ and so the two connected components of $\Delta(PK)$ are $(\rho_1 \cup \rho_2)\backslash\{p_2\}$ and $\rho_3 \cup \rho_4$. As the prime p_4 is in a component with size larger than one, the group PK must be Example (2.6) in Theorem 2.1. Thus, $PK = TD$ where D is an abelian group acting coprimely on the group T. By [7, Lemma 3.6], we have that $P \subseteq T$, and T has a p_4-complement Q. So, $PK = PQD$ and the primes that divide Q are precisely the primes in $\rho_3 \cup \rho_4 \setminus \{p_4\}$. Thus, $Q = L$ and, as Q is abelian, L is abelian.

Because the group $PK = PLD$ is Example (2.6), the subgroup $[PL, D]$ is a Frobenius group where $[P, L]$ is the Frobenius kernel by [7, Lemma 3.6]. A p_4-complement in $[PL, D]$ is $[L, D]$ and so we call a Frobenius complement $B = [L, D]$, which is contained in L. From [7, Lemma 3.6], we have

$$P' = [P, L]' \subseteq [P, L].$$

We see that $[P, L]K$ satisfies the hypotheses of Example (2.6) of Theorem 2.1. So $\Delta([P, L]K)$ is disconnected with components $(\rho_1 \cup \rho_2)\setminus\{p_2\}$ and $\rho_3 \cup \rho_4$. The action of B on P' is a Frobenius action, so $\mathbb{C}_{P'}(B) = 1$ and $\mathbb{C}_P(L) \subseteq \mathbb{C}_P(B)$. Also, $\mathbb{C}_P(L)' \subseteq \mathbb{C}_P(L) \cap P' = 1$. Thus, $\mathbb{C}_P(L)$ is abelian.

Let the character $\lambda \in \mathrm{Irr}(P')$ be non-principal. Because P' is central in P, the stabilizer of λ is $P\,\mathbb{C}_H(\lambda)$. By [3, Theorem 13.28], we can find a $\mathbb{C}_H(\lambda)$-invariant irreducible constituent θ of λ^P. Note that the stabilizer of θ in G is $P\,\mathbb{C}_H(\theta)$ and $\mathbb{C}_H(\lambda) \subseteq \mathbb{C}_H(\theta)$. Because P' is central and $\theta_{P'}$ has a unique constituent λ, we have that $\mathbb{C}_H(\lambda) = \mathbb{C}_H(\theta)$.

As p_4 divides every degree in $\mathrm{cd}(G \mid \lambda)$, we have that $\mathbb{C}_H(\lambda)$ contains an abelian Hall $\rho_1 \cup \rho_2$-subgroup of H. Further, θ must extend to $P\,\mathbb{C}_H(\lambda)$ by [3, Corollary 6.28] and so, by Gallagher's Theorem, no prime in $\rho_1 \cup \rho_2$ is in $\rho(\mathbb{C}_H(\lambda))$. Since λ extends to $P'\mathbb{C}_H(\lambda)$, and p_4 divides every degree in $\mathrm{cd}(G \mid \lambda)$, we see that no prime in $\rho_1 \cup \rho_2$ divides any degree in $\mathrm{cd}(P'H \mid P')$. On the other hand, H has a normal abelian Hall $\rho_3 \cup \rho_4\setminus\{p_4\}$-subgroup L, and so no prime in $\rho_3 \cup \rho_4$ divides a degree in $\mathrm{cd}(H)$. If H is nilpotent, then by [14, discussion on p. 254], there would be a character degree that equals $|H|$. Since all of the primes in $\rho(G)\setminus\{p_4\}$ divide $|H|$, this is not the case. Because $L \subseteq \mathbb{F}(H)$, we deduce that $p_1 \in \rho(H)$ for some $p_1 \in \rho_1 \cup \rho_2\setminus\{p_2\}$. Recall that L contains B, so that $P'B$ is a Frobenius group. It follows that at least one prime in $\rho_3 \cup \rho_4\setminus\{p_4\}$ is in $\rho(P'H)$ and $\Delta(P'H)$ is disconnected. The components are a nonempty subset of $\rho_1 \cup \rho_2$ and a nonempty subset of $\rho_3 \cup \rho_4\setminus\{p_4\}$. Because $P'H$ has no normal non-abelian Sylow subgroups, $P'H$ is Example (2.4) from Theorem 2.1.

First, we suppose that $\rho_1 \cup \rho_2 \subseteq \rho(P'H)$. Recall that PK is Example (2.6) from Theorem 2.1 and so by [7, Lemma 3.6], PK/P' is Example (2.4). Because L is a normal Hall subgroup of H, we know $L \subseteq \mathbb{F}(K)$. By [7, Lemma 3.6], only the primes in $\rho_3 \cup \rho_4$ can divide $\mathbb{F}(K)$. Thus, $\mathbb{F}(K) \subseteq L$ and $L = \mathbb{F}(K)$. Further, both L and K/L are cyclic groups.

Consider the subgroup HP'/P' acting coprimely on the group $[P, L]/P'$. Define $E/P' = \mathbb{F}(HP'/P') \cong \mathbb{F}(H)$. Certainly, $L \subseteq \mathbb{F}(H)$. Because p_2 divides $|H|$ and K is a normal p_2-complement of H, the Fitting subgroup of K is contained in the Fitting subgroup of H. Hence, $\mathbb{F}(H) = L \times O_{p_2}(H)$. Because the

Sylow p_2-subgroup of H is not a normal subgroup of H, the subgroup $O_{p_2}(H)$ is a proper subgroup of H. Further, the subgroup $PKO_{p_2}(H)$ is a proper normal subgroup of G and $\rho(PKO_{p_2}(H)) = \rho(G)$. As the subgroup $PKO_{p_2}(H)$ has a normal Sylow p_4-subgroup, and by the minimality of G, it follows that the graph $\Delta(PKO_{p_2}(H))$ must be disconnected. Both components have size larger than one and so $PKO_{p_2}(H)$ is Example (2.6) from Theorem 2.1. By [7, Lemma 3.6], the primes that divide $|\mathbb{F}(H) : \mathbb{F}(PKO_{p_2}(H))|$ are precisely the primes in $\rho_3 \cup \rho_4$, and so p_2 cannot divide that index, which is a contradiction. Hence, $O_{\rho_2}(H) = 1$ and $\mathbb{F}(H) = L$ and $E/P' = LP'/P'$.

Recall that the group $[P, L]K$ is Example (2.6) from Theorem 2.1. By [7, Lemma 3.6], the factor group $[P, L]K/P'$ is Example (2.4) from Theorem 2.1. First, we define

$$Z_1/P' = \mathbb{C}_{KP'/P'}([P, L]/P') \quad \text{and} \quad Z_2/P' = \mathbb{C}_{HP'/P'}([P, L]/P').$$

As $\mathbb{F}(G/P') = P/P'$ and $\mathbb{C}_{G/P'}(P/P') \subseteq P/P'$, we see that

$$Z_1/P' = Z_2/P' = P'/P'.$$

The Fitting subgroup of K is L and so the Fitting subgroup of KP'/P' is LP'/P' and it is abelian. As $[P, L]K$ is Example (2.6) from Theorem 2.1, L acts irreducibly on $[P, L]$, we have LP'/P' acts irreducibly on $[P, L]/P'$. Define

$$m = |HP'/P' : LP'/P'|,$$

which is equal to $|H : L|$, and so $(m, |L|) = 1$. Let $m_2 = |H : K|$ be the power of p_2 that divides $|H|$.

Let q_1 be a power of p_4 such that $q_1^{m/m_2} = |P/P'|$. Because $[P, L]K/P'$ is Example (2.4), there is a p_4-power q such that $q^{m/m_2} = |[P, L] : P'|$. Clearly, we have $q \leq q_1$. Let s be a Zsigmondy prime divisor of $q_1^{m/m_2} - 1$. We recall that Zsigmondy prime divisors exist except if $m = 2$ and $p_4 = 2$ or $m = 6$ and $q = 2$. These exceptions do no occur as $[P, L]K/P'$ is Example (2.4). Then s divides $q_1^{m/m_2} - 1$ and does not divide $q_1^i - 1$ for any $i < m/m_2$. Further, as $q_1 - 1$ is a factor of $q_1^{m/m_2} - 1$, the prime s divides $(q_1^{m/m_2} - 1)/(q_1 - 1)$, and as the quotient $(q_1^{m/m_2} - 1)/(q_1 - 1)$ divides $|L|$, so does s. Now the subgroup L acts Frobeniusly on $[P, L]/P'$. So L divides $|[P, L]/P'| - 1 = q^{m/m_2} - 1$. Hence, s divides $q^{m/m_2} - 1$, and since s is a Zsigmondy prime divisor, we have $q^{m/m_2} \geq q_1^{m/m_2}$. Thus $q = q_1$.

Because $P'H$ is also Example (2.4), we know that there is a p_4-power q_2 such that $q_2^m = |P'|$. Because L acts Frobeniusly on P', we know that $|L|$ divides $q_2^m - 1$. Because s divides $|L|$, we know that s divides $q_2^m - 1$ and so $q^{m/m_2} \leq q_2^m$. Let r be a Zsigmondy prime divisor of $q_2^m - 1$. The prime r exists because the exceptions do not occur as $P'H$ is Example (2.4). Then r divides $q_2^m - 1$ and r

also divides $(q_2^m - 1)/(q_2 - 1)$. Hence, r divides $|L|$. But as $|L|$ divides $q^{m/m_2} - 1$, so does r. But then $q_2^m \le q^{m/m_2}$ and so $q_2^m = q^{m/m_2}$. Hence, $q_2 = q^{1/m_2}$. Since $(q_2^m - 1)/(q_2 - 1)$ divides $|L|$, we know that $(q^{m/m_2} - 1)/(q^{1/m_2} - 1)$ divides $|L|$. Thus $[P, L]H/P'$ satisfies Example (2.4) and so the graph $\Delta([P, L]H/P')$ is disconnected.

By Fitting's Lemma, $P/P' = \mathbb{C}_{P/P'}(L) \times [P, L]/P'$. So

$$PH/P' = P/P' \cdot HP'/P' = (\mathbb{C}_{P/P'}(L) \times [P, L]/P') \cdot HP'/P'.$$

If $\mathbb{C}_P(L)/P'$ is not central in HP'/P', then there is a character degree divisible by $|H : \mathbb{C}_H(\lambda)|\theta(1)$. Because K is a p_2-complement, p_2 divides $|H : \mathbb{C}_H(\lambda)|$, and so p_2 and p_4 are adjacent. This is a contradiction, and so ρ_2 is not contained in $\rho(P'H)$. In particular, the prime p_2 is not in $\rho(H)$. So H has an abelian Sylow p_2-subgroup Q by [3, Corollary 12.34]. Further, PQ is a normal subgroup of G.

Now, the graph $\Delta(PQ)$ has two connected components, $\{p_2\}$ and $\{p_4\}$. Observe that Q acts coprimely on P, fixing all non-linear characters. From [14, Theorem 19.3], we have $[P, Q]' = P'$ and $[P, Q]$ is not abelian. Consider $\Delta([P, L]H)$ determined by

$$\mathrm{cd}([P, L]H) = \mathrm{cd}\left(\frac{[P, L]H}{[P, L]}\right) \cup \mathrm{cd}([P, L]H \mid [P, L]).$$

Because $\rho(H) = (\rho_1 \cup \rho_2)\backslash\{p_2\}$, the prime p_2 does not divide a character degree in $\mathrm{cd}(H) = \mathrm{cd}([P, L]H/[P, L])$. Recall that $[P, L]K$ is Example (2.6) from Theorem 2.1 and B is a Frobenius complement. Thus by [7, Lemma 3.6], every degree in $\mathrm{cd}([P, L]K \mid [P, L])$ is divisible by $p_4|B|$ and so p_2 does not divide any of those character degrees. We see that p_2 is not in $\mathrm{cd}([P, L]H)$, and so the subgroup $[P, L]H$ is proper in G. The graph $\Delta([P, L]H)$ must be disconnected. Thus, Q is a normal subgroup of $[P, L]H$. In particular, Q centralizes $[P, L]$ as Q is normal in H. Because $[[P, L], Q] = 1$ and $[L, Q] = 1$, we have $[[L, Q], P] = 1$. Thus $[[Q, P], L] = 1$ by the Three Subgroup Lemma [5, Lemma 4.9], and so $[P, Q] \subseteq \mathbb{C}_P(L)$. This is a contradiction because $\mathbb{C}_P(L)$ is abelian and $[P, Q]$ is not. □

7 Main theorems

We now prove that when G is a solvable group with $\Delta(G)$ having diameter three, then G must have exactly one normal non-abelian Sylow p-subgroup and $p \in \rho_3$. Because Theorem 2.5 tells us that G has at most one normal non-abelian Sylow p-subgroup when G is solvable and $\Delta(G)$ has diameter three, it is possible that a solvable group G could have no normal non-abelian Sylow p-subgroups and have a character degree graph $\Delta(G)$ that has diameter three.

In our proof, we assume that G has no normal non-abelian Sylow p-subgroups for $p \in \rho_3$ and our goal is to use Lemma 5.3 to show that $\Delta(G)$ could not have had diameter three. To do this, we must show that G has no normal non-abelian Sylow p-subgroups for any prime p. We will apply Lemma 6.1 to show that G has no normal non-abelian Sylow p-subgroups for any prime $p \in \rho_4$. To show that there are no normal non-abelian Sylow p-subgroups for any prime $p \in \rho_1 \cup \rho_2$ we must apply Lemma 5.2 and Lemma 5.1. Most of the work of this proof is to show that the hypotheses of these lemmas are met.

Theorem 3. *Let G be a solvable group with character degree graph $\Delta(G)$ with diameter three. Then G has a normal Sylow p-subgroup for exactly one prime p and $p \in \rho_3$.*

Proof. Let G be a counter-example with $|G|$ minimal such that $\Delta(G)$ has diameter three and G has no normal Sylow p_3-subgroups for $p_3 \in \rho_3$. Because $\Delta(G)$ has diameter three, we know $|\rho_3| \geq 3$ by Theorem 2. By Lemma 6.1, we see that G does not have a normal Sylow p_4-subgroup for any prime $p_4 \in \rho_4$.

Claim 1. *Let p_2 be a prime in ρ_2. If $O^{p_2}(G)$ is a proper subgroup of G, then $\Delta(O^{p_2}(G))$ is disconnected.*

Proof. Let $K = O^{p_2}(G)$ for some prime $p_2 \in \rho_2$. Since $\rho(K)$ contains $\rho_1 \cup \rho_4$, the graph $\Delta(K)$ either has diameter three or is disconnected. Suppose $\Delta(O^{p_2}(G))$ has diameter three. Then by the hypothesis, $O^{p_2}(G)$ has a normal Sylow p_3-subgroup P for some prime $p_3 \in \rho_3$. The subgroup P is characteristic in $O^{p_2}(G)$ and so is normal in G, which is a contradiction. Hence, $\Delta(O^{p_2}(G))$ cannot have diameter three. Because $\rho(O^{p_2}(G))$ contains all the primes of $\rho(G)$ with the possible exception of the prime p_2, we see $\Delta(O^{p_2}(G))$ must be disconnected. $\square$

Claim 2. *The graph $\Delta(G/N)$ cannot have diameter three for any proper, nontrivial, normal subgroup N.*

Proof. Suppose there exists a normal subgroup N of G such that $\Delta(G/N)$ has diameter three. Then we can find a minimal normal subgroup M contained in N, where $\Delta(G/M)$ has diameter three. By the minimality of G, G/M has a normal Sylow p_3-subgroup P/M for $p_3 \in \rho_3$. Because M is an elementary abelian p-group for some prime p, if $p = p_3$, then G has a normal Sylow p_3-subgroup and so, $p \neq p_3$.

Because $P' \neq 1$, either $M \subseteq P'$ or $M \cap P' = 1$. Suppose that $M \cap P' = 1$. Then because $[P, M] \subseteq P'$ and $[P, M] \subseteq M$, we see $[P, M] = 1$ and so M is central in P. Let P_3 be a Sylow p_3-subgroup of G such that $P_3 \subseteq P$. Since M

normalizes P_3, P_3 is characteristic in P and so is normal in G. This is a contradiction, and so $M \subseteq P'$. Hence $G/M/(P/M)' = G/M/P'/M \cong G/P'$.

Suppose that G has a normal Sylow q-subgroup Q for $q \in \rho_1 \cup \rho_2 \cup \rho_4$. Then we have a contradiction of Lemma 4.4 because G/M has a normal Sylow p_3-subgroup. Hence, G has no normal non-abelian Sylow subgroups.

Consider the graph $\Delta(G/M/(P/M)') = \Delta(G/P')$. By our hypothesis, if the graph $\Delta(G/P')$ has diameter three, then it has a normal Sylow q-subgroup for $q \in \rho_3$. Since $q \neq p_3$, this contradicts the fact that G/M can have at most one normal non-abelian Sylow p-subgroup. Hence, G/P' does not have diameter three and further, does not have any normal Sylow p-subgroups for $p \in \rho(G/P')$. Because $\rho(G/P')$ contains primes from ρ_1 and ρ_4, we know that $\Delta(G/P')$ must be disconnected and is Example (2.4) from Theorem 2.1.

Let p_2 be a prime in ρ_2. By [7, Theorem 5.3], $O^{p_2}(G/P') < G/P'$ and so $O^{p_2}(G) < G$. Recall that G has no normal non-abelian Sylow p-subgroups for $p \in \rho(G)$. Suppose $O^{p_2}(G)$ has a normal non-abelian Sylow t-subgroup T. Then because T is normal in $O^{p_2}(G)$ and $O^{p_2}(G)$ is characteristic in G, the subgroup T is normal in G. This is a contradiction, and so $O^{p_2}(G)$ has no normal non-abelian Sylow p-subgroups. However, $\Delta(O^{p_2}(G))$ is disconnected and so is either Example (2.4) or Example (2.5) from Theorem 2.1 which contradicts Lemma 4.1. Thus, $\Delta(G/N)$ cannot have diameter three for any proper, nontrivial, normal subgroup N. □

Claim 3. *The graph $\Delta(M)$ does not have diameter three for any proper normal subgroup M of G.*

Proof. Suppose that M is a proper normal subgroup of G such that $\Delta(M)$ has diameter three. Then M has a normal Sylow p_3-subgroup P for a prime $p_3 \in \rho_3$. Since P is characteristic in M, and M is normal in G, we see P is normal in G. Further, P' is normal in G, and $P' \neq 1$. Recall that $\Delta(G/P')$ cannot have diameter three. Because $\rho(G/P')$ contains $\rho(G)\backslash\{p_3\}$, the graph has components $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4\backslash\{p_3\}$, with the possibility of containing p_3 as well. It must be disconnected with both components at least size 2. By [7, Theorem 5.3], we have $O^{p_2}(G/P') < G/P'$ for some prime $p_2 \in \rho_2$, and $O^{p_2}(G) < G$. Further, $P \subseteq O^{p_2}(G)$. We have shown that $\Delta(O^{p_2}(G))$ is disconnected and Lemma 4.1 tells us that $O^{p_2}(G)$ cannot be Example (2.4) or Example (2.5) from Theorem 2.1.

Suppose $|\rho_1 \cup \rho_2| = 2$ and $p_2 \notin \rho(O^{p_2}(G))$. Then it is possible that $O^{p_2}(G)$ is Example (2.1) from Theorem 2.1 and $O^{p_2}(G)$ has a normal Sylow p_1-subgroup R for $p_1 \in \rho_1$. Since $\Delta(M)$ has diameter three, $p_1 \in \rho(M)$ and so M has a normal Sylow p_1-subgroup, which contradicts the fact that M can have at most one normal non-abelian Sylow p-subgroup. Thus $O^{p_2}(G)$ cannot be Example (2.1) from Theorem 2.1. Since at least one component, if not both, has size at least 2, $O^{p_2}(G)$ is Example (2.6) from Theorem 2.1. Hence $O^{p_2}(G)$ has a normal Sylow p-subgroup Q for a prime $p \in \rho_3 \cup \rho_4$. But this is a contradiction because Q is characteristic in $O^{p_2}(G)$ and so Q is a normal Sylow p-subgroup of G and G does not have a normal Sylow p-subgroup for any prime in $\rho_3 \cup \rho_4$. Thus, $\Delta(M)$ cannot have diameter three whenever M is a proper normal subgroup of G. □

Claim 4. *The subgroup $O^{p_3}(G) = G$ for all primes $p_3 \in \rho_3$.*

Proof. Suppose there exists a prime $p_3 \in \rho_3$ such that $O^{p_3}(G)$ is proper in G. Since $O^{p_3}(G)$ is proper in G we know that $\Delta(O^{p_3}(G))$ cannot have diameter three. Because $\rho(O^{p_3}(G))$ contains all of $\rho(G)$ except perhaps the prime p_3, we know that $\Delta(O^{p_3}(G))$ is disconnected. Because $|\rho_3| \geq 3$, there is a prime $q \in \rho_3$, not equal to p_3, and a prime $p_2 \in \rho_2$, such that q and p_2 are adjacent in $\Delta(G)$. However, as $\Delta(O^{p_3}(G))$ contains all edges not incident to the prime p_3, we see that p_2 and q are adjacent in $\Delta(O^{p_3}(G))$ and $\Delta(O^{p_3}(G))$ must have diameter three. This is a contradiction. □

Claim 5. *If M is a normal subgroup of G, then $\rho(G/M) = \rho(G)$ implies that $M = 1$.*

Proof. Suppose there exists a nontrivial normal subgroup M of G such that $\rho(G/M) = \rho(G)$. Without loss of generality, M is a minimal normal subgroup of G. We know that the graph $\Delta(G/M)$ cannot have diameter three, and because $\rho(G/M)$ contains primes from ρ_1 and ρ_4, the graph $\Delta(G/M)$ must be disconnected. The components are $\rho_1 \cup \rho_2$ and $\rho_3 \cup \rho_4$. Because $\rho_1 \cup \rho_2$ is the smaller component and has size at least 2, G/M is either Example (2.4) or Example (2.6) from Theorem 2.1. Suppose G/M is Example (2.6). Then G/M has a normal Sylow p-subgroup P/M for a prime $p \in \rho_3 \cup \rho_4$. Let Q be a Sylow p-subgroup of G such that Q is contained in P. If M is not contained in P', then since $[P, M] \subseteq P'$ and $[P, M] \subseteq M$, we have $[P, M] = 1$. Thus M is central in P and, as M normalizes Q, we have Q is characteristic in P and so Q is normal in G, which is a contradiction and so $M \subseteq P'$. Since $\rho(G/M)/(P/M)') = \rho(G)\backslash\{p\}$ and $G/M/(P/M)' = G/P'$, the graph $\Delta(G/P')$ is disconnected. The factor group G/P' is Example (2.4) from Theorem 2.1. By [7, Lemma 5.3], $O^{p_2}(G/P')$ is proper in G/P' and so $O^{p_2}(G) < G$. But this contradicts Lemma 4.1, thus, G/M is not Example (2.6), and so is Example (2.4). If G has a normal non-abelian Sylow subgroup, then so does G/M. Therefore, G has no normal non-abelian Sylow subgroups. By [7, Theorem 5.3], $O^{p_2}(G/M) < G/M$. But then $O^{p_2}(G) < G$, which contradicts Lemma 4.1. Thus, $\rho(G/M) = \rho(G)$ implies that $M = 1$. □

By Claim 2, Claim 3, Claim 4, and Claim 5, the hypotheses for Lemma 5.1 and Lemma 5.2 are satisfied. Thus, G does not have a normal non-abelian Sylow p-subgroup for any prime $p \in \rho_1 \cup \rho_2$. Recall that G has no normal non-abelian Sylow p-subgroup for $p \in \rho_4$ and so G has no normal non-abelian Sylow p-subgroups. Thus, by Lemma 5.3, G does not have diameter three, which is our final contradiction. □

Finally, because G has a normal Sylow p_3-subgroup P when $\Delta(G)$ has diameter three, we can observe that $\Delta(G/P')$ must be disconnected, and so, G/P' is in one of the families of groups from Theorem 2.1.

Theorem 4. *Let G be a solvable group with $\Delta(G)$ having diameter three. If $|\rho_1 \cup \rho_2| = n$, then $|\rho_3 \cup \rho_4| \geq 2^n$*

Proof. By Theorem 3, G has a normal non-abelian Sylow p_3-subgroup P for some prime $p_3 \in \rho_3$. By Lemma 2.4, $\rho(G/P') = \rho(G)\backslash\{p_3\}$. Because ρ_3 has more than three vertices by Theorem 2, either $\Delta(G/P')$ is disconnected or it has diameter three. If $\Delta(G/P')$ has diameter three, then by Theorem 3, $\Delta(G/P')$ has a normal non-abelian Sylow q-subgroup Q/P' for some prime $q \in \rho(G)\backslash\{p_3\}$. Let R be a Sylow q-subgroup of G contained in Q. As RP'/P' is a normal subgroup of G/P', the Sylow subgroup R is a normal subgroup of G, which contradicts the fact that G can have at most one normal non-abelian Sylow p-subgroup. Hence, $\Delta(G/P')$ is disconnected. By Pálfy's inequality, if $|\rho_1 \cup \rho_2| = n$, then $|\rho_3 \cup \rho_4\backslash\{p_3\}| \geq 2^n - 1$. Hence, $|\rho_3 \cup \rho_3| \geq 2^n$. □

Theorem 5. *Let G be a solvable group with $\Delta(G)$ having diameter three. Then G has Fitting height* 3.

Proof. By Theorem 3, G has a normal Sylow p_3-subgroup P for $p_3 \in \rho_3$. So G/P' has a normal abelian Sylow p_3-subgroup. Notice that $\Delta(G/P')$ is disconnected, both components have size larger than 2, and G/P' has no normal Sylow p-subgroups for $p \in \rho(G/P')$. So G/P' is Example (2.4) from Theorem 2.1 and so has Fitting height 3. Let H be a p_3-complement of G. Anything in H that centralizes P/P' also centralizes P. Let $F = \mathbb{F}(G)$ and $E/P' = \mathbb{F}(G/P')$. We have F/P' is a nilpotent normal subgroup of G/P' and $F \subseteq E$. Conversely, $E = P(E \cap H)$, and $E \cap H$ is a nilpotent subgroup that centralizes P/P'. So $E = P \times (E \cap H)$ is nilpotent and $E \subseteq F$. Thus the Fitting height of G is the same as G/P', and so G has Fitting height three. □

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