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A generalization of Burnside's p-nilpotency criterion

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Abstract. Let p be a prime number, let G be a finite group, and let P be a Sylow p-subgroup of G. Under the assumption that $N_G(P)$ is p-nilpotent, we give an equivalent condition of the p-nilpotency of G. Our result is a generalization of the famous Burnside p-nilpotency criterion and some recent results.

1 Introduction and statements of results

All groups considered in this paper are finite.

Let P be a Sylow p-subgroup of a finite group G for some prime p. If G is p-nilpotent, then the normalizer $N_G(P)$ is p-nilpotent. But the converse is not true in general, i.e., the p-nilpotency of $N_G(P)$ does not imply the p-nilpotency of G in general (see [10, Example 1.1]). Hence we must add some extra condition or impose stronger hypotheses. For example, for an odd prime p, the well-known Glauberman–Thompson theorem [3, Theorem 8.3.1] states that G is p-nilpotent if, and only if $N_G(Z(J(P)))$ is p-nilpotent, where $J(P)$ is the Thompson subgroup of P (ref. [3]). Note that $N_G(P) \le N_G(Z(J(P))))$.

The main purpose of this paper is to generalize the following result which is known as Burnside's p-nilpotency criterion.

Theorem 1.1 ([8, Theorem 7.2.1]). *Let p be a prime number, let G be a finite group, and let P be a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if P is abelian.*

As P being abelian is equivalent to P being contained in $Z(P)$, the center of P, one may interpret the following result of P. Hall as a generalization of Burnside's criterion.

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Theorem 1.2 ([4]). *Let p be a prime number, let G be a finite group, and let P be a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if the nilpotency class of P is less than p, i.e., $P \leq Z_{p-1}(P)$, where $Z_{p-1}(P)$ denotes the $(p-1)$-th term of the ascending central series of P.*

Remark 1.3. Theorem 1.2 can be deduced from Yoshida's transfer theorem (see [7, Theorem 10.1]).

Suppose that P is a p-group for some prime p and that i is an integer. Denote the subgroups $\Omega_i(P)$ and $\Omega(P)$ of P as follows:

$$\Omega_i(P) = \langle x \in P | x^{p^i} = 1 \rangle, \quad \Omega(P) = \begin{cases} \Omega_1(P) & \text{if } p \text{ is odd,} \\ \Omega_2(P) & \text{if } p = 2. \end{cases}$$

Recently the authors of [2] improved Hall's result in the case that p is odd.

Theorem 1.4 ([2, Theorem D]). *Let p be an odd prime number, let G be a finite group, and let P be a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if $\Omega(P) \leq Z_{p-1}(P)$.*

The authors of [1] obtained the following nice result which extends Burnside's result in a different way:

Theorem 1.5 ([1, Theorem 1 and Theorem 2]). *Let p be a prime number, let G be a finite group, and let P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if either of the following conditions holds:*

(1) $\Omega(P \cap G') \leq Z(P)$.

(2) *When $p = 2$, $\Omega_1(P \cap G') \leq Z(P)$ and P is quaternion-free.*

where G' is the commutator subgroup of G.

As in Huppert ([6]), denote the subgroup of G generated by all p'-elements of G by $O^p(G)$. Note that $P \cap O^p(G) \leq P \cap G'$ if P is a Sylow p-subgroup of G, and that G is p-nilpotent if and only if $P \cap O^p(G) = 1$. The main theorem of this paper is a generalization of all the results mentioned above.

Before the statement of the main theorem, we introduce the notion of the n-th Engel word $E_n(x, y)$:

$$E_n(x, y) = \underbrace{[x, [x, \ldots, [x}_{n \text{ copies of } x}, y]] \ldots].$$

Suppose that H, K are subgroups of G, we say that G satisfies the n-th Engel condition for (H, K) if $E_n(h, k) = 1$ for all $h \in H$ and $h \in K$.

Main Theorem. *Let p be a prime number, let G be a finite group, and let P be a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if either of the following holds:*

(1) *G satisfied the $(p-1)$-th Engel condition for $(P, \Omega(P \cap O^p(G)))$.*

(2) *$p = 2$, $\Omega_1(P) \leq Z(P \cap O^2(G))$ and P is quaternion-free.*

As an immediate consequence one has the following.

Corollary. *Suppose that p is a prime. Let G be a group and P a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if either of the following holds:*

(1) *$\Omega(P \cap O^p(G)) \leq Z_{p-1}(P)$.*

(2) *$p = 2$, $\Omega_1(P \cap O^p(G)) \leq Z(P)$ and P is quaternion-free.*

2 The proof of the Main Theorem

In the proof of the Main Theorem we will frequently use the fact that

$$O^p(H) \leq O^p(G)$$

for every subgroup H of G.

We also need the following lemmata.

Lemma 2.1 ([9, Lemma 2.3]). *Let P be a 2-group and A an automorphism group of P of odd order. If A acts trivially on $\Omega_1(P)$ and P is quaternion-free, then $A = 1$.*

Lemma 2.2 ([5, Theorem B]). *Let H be a p-soluble linear group over a field of characteristic p satisfying $O_p(H) = 1$. If g is an element of order p^m in H, then the minimal polynomial of g is $(x-1)^r$, where $r = p^m$, unless there is an integer m_0, not greater than m, such that $p^{m_0} - 1$ is a power of a prime q for which $c_q(H) > 1$ (where $c_q(H)$ denotes the class of a Sylow q-subgroup of H; that is, the length of the upper or lower central series of H). In the latter case, if m_0 is the smallest such integer, one has*

$$p^{m-m_0}(p^{m_0} - 1) \leq r \leq p^m.$$

Proof of the Main Theorem. Assume that either condition (1) or (2) holds. Suppose that the assertion is false, and let G be a counterexample with minimal order.

Step 1: $O_{p'}(G) = 1$. Clearly, the hypotheses hold for $G/O_{p'}(G)$. Hence the minimal choice of G implies that $O_{p'}(G) = 1$.

Step 2: For any subgroup S of G with $P \leq S \lneq G$, S is p-nilpotent. Since we have $N_S(P) \leq N_G(P)$, $N_S(P)$ is p-nilpotent. Note that $O^p(S) \leq O^p(G)$, the hypotheses hold for S as well, and thus S is p-nilpotent by the minimal choice of G.

Step 3: We have $P \cap O^p(G) \trianglelefteq G$ and thus $O^p(G)$ is p-soluble. We first prove that $P \cap O^p(G) \lneq P$ and thus $O^p(G) \lneq G$. Assume that $P \cap O^p(G) = P$.

Suppose that $p = 2$. If condition (1) holds, then G satisfies the 1-st Engel condition for $(P, \Omega(P))$. This means that $\Omega(P) \leq Z(P)$. Hence G is p-nilpotent by Theorem 1.5 (1), a contradiction. If condition (2) holds, applying Theorem 1.5 (2) directly, we have G is p-nilpotent, a contradiction. Hence p is odd. Write

$$\overline{G} = G/O_p(G) \quad \text{and} \quad \overline{P} = P/O_p(G).$$

Let H be a subgroup of G such that $H/O_p(G) = N_{\overline{G}}(Z(J(\overline{P})))$, where $J(\overline{P})$ is the Thompson subgroup of $\overline{P}$. Since $O_p(\overline{G}) = 1$, $H \lneq G$. Noticing that $P \leq H$, we have that H is p-nilpotent by Step 2. Thus $H/O_p(G) = N_{\overline{G}}(Z(J(\overline{P})))$ is p-nilpotent as well. It then follows from the Glauberman–Thompson theorem (see [3, Theorem 8.3.1]) that $\overline{G}$ is p-nilpotent. If $\overline{G}$ is a p'-group, then $P = O_p(G)$ and $G = N_G(P)$ is p-nilpotent by hypothesis, a contradiction. Hence $\overline{G}$ is not a p'-group. Let $K/O_p(G)$ be the normal p-complement of $\overline{G}$. Then we have that $G/K \cong (G/O_{p'}(G))/(K/O_{p'}(G))$ is a p-group, and thus $O^p(G) \leq K$. Moreover, $K \lneq G$ as $\overline{G}$ is not a p'-group. Therefore $P \nleq K$. But $P \leq O^p(G) \leq K$, a contradiction. Hence $P \cap O^p(G) \lneq P$, as desired.

Remark. For the proof that $P \cap O^p(G) \lneq P$ in the case that p is odd, the referee gave us a short argument as follows.

Suppose that $P \cap O^p(G) = P$ and p is odd. As G is not p-nilpotent, by [10, Main Theorem], P contains a subgroup $Y = Y_m(P)$ for some integer $m \geq 1$ (see [10]). But G does not satisfy the $(p-1)$-th Engel condition for $(Y, \Omega_1(Y))$, a contradiction. Therefore $P \cap O^p(G) \lneq P$, as desired.

Suppose that $N_G(P \cap O^p(G)) \lneq G$. Noticing that $P \leq N_G(P \cap O^p(G))$, we have that $N_G(P \cap O^p(G))$ is p-nilpotent by Step 2. Then $N_{O^p(G)}(P \cap O^p(G))$ is p-nilpotent. It follows that the hypotheses of the theorem hold for $O^p(G)$. Since $O^p(G) \lneq G$, $O^p(G)$ is p-nilpotent by the minimal choice of G. This implies that G is p-nilpotent, a contradiction. Therefore $N_G(P \cap O^p(G)) = G$. It follows that $P \cap O^p(G)$ is a normal Sylow p-subgroup of $O^p(G)$ and thus $O^p(G)$ is p-soluble.

Step 4: The group G is p-soluble and $C_G(O_p(G)) \leq O_p(G) \neq 1$. By Step 3, G is p-soluble. Hence $C_G(O_p(G)) \leq O_p(G)$ by Step 1 and [3, Theorem 6.3.2].

Step 5: $N_G(P) = P$ is a maximal subgroup of G. Let T be the normal p-complement of $N_G(P)$. Since $T \leq C_G(O_p(G)) \leq O_p(G)$ by Step 4, $T = 1$. Hence $N_G(P) = P$.

Suppose that M is a maximal subgroup of G containing $N_G(P)$. Then M is p-nilpotent by Step 2. Hence $[O_{p'}(M), O_p(G)] = 1$ and thus $O_{p'}(M) = 1$. Therefore $M = P$ is a maximal subgroup of G.

Step 6: $G = PQ$, where Q is a Sylow q-subgroup of G for some $q \neq p$. Moreover, $P = (P \cap O^p(G))\langle a \rangle$ for some $a \in N_G(Q)$ and Q has no proper subgroups that are $\langle a \rangle$-invariant. Moreover, $O_p(G)Q/O_p(G)$ is a chief factor of G and Q is elementary abelian. Clearly, G is not a p-group. Let q be a prime such that $q \in \pi(G)$ and $q \neq p$. Since G is p-soluble, there exists a Sylow q-subgroup Q of G such that PQ is a subgroup of G (see [3, Theorem 6.3.5]). By Step 5, we have $G = PQ$.

Now let $P_1/(P \cap O^p(G))$ be a maximal subgroup of $P/(P \cap O^p(G))$. Then $P \leq N_G(P_1)$. The maximality of P implies that $N_G(P_1) = P$ or $N_G(P_1) = G$. Let $H = P_1 O^p(G)$. By construction, $H \neq G$. Suppose that $N_G(P_1) = P$. Then $N_H(P_1) = P_1$ is p-nilpotent. It follows that H satisfies the hypotheses of the Main Theorem. By the minimality of G, we have that $O^p(G) \leq H$ is p-nilpotent and thus G is p-nilpotent, a contradiction. Therefore, $N_G(P_1) = G$ and P_1 is normal in G. As G is not p-nilpotent, this yields $O_p(G) = P_1$. In particular, $P/P \cap O^p(G)$ has a unique maximal subgroup and hence is cyclic. On the other hand, by the Frattini argument, we have that

$$G = O^p(G)N_G(Q).$$

As $O^p(G) = P \cap O^p(G)Q$, this yields

$$G = (P \cap O^p(G))N_G(Q).$$

Clearly, $P = P \cap [(P \cap O^p(G))N_G(Q)] = (P \cap O^p(G))(P \cap N_G(Q))$. Since $P/(P \cap O^p(G))$ is a cyclic group, we can find a cyclic group $\langle a \rangle \leq P \cap N_G(Q)$ such that $P = (P \cap O^p(G))\langle a \rangle$.

If X is a proper subgroup of Q that is $\langle a \rangle$-invariant, then PX will be a subgroup strictly between P and G, which is contrary to Step 5.

Let $T/O_p(G)$ be a chief factor of G. Then $T/O_p(G)$ is an elementary abelian q-group and there exists a Sylow q-subgroup Q_1 of T such that $T = Q_1 O_p(G)$. It is clear that $PT = PQ_1$. By Step 5, we have $PT = PQ = G$. Hence $Q \cong Q_1$ is elementary abelian.

Step 7: Let $P_2 = \Omega(P \cap O^p(G))\langle a\rangle$ *when condition* (1) *of the Main Theorem holds, and let* $P_2 = \Omega_1(P \cap O^p(G))\langle a\rangle$ *when condition* (2) *of the Main Theorem holds. Then* $P = P_2$. Suppose that $P > P_2$. Since $P \cap O^p(G) \trianglelefteq G$, we have $\Omega(P \cap O^p(G)) \trianglelefteq G$ and $\Omega_1(P \cap O^p(G)) \trianglelefteq G$. Clearly, $\langle a\rangle Q$ is a subgroup of G. It follows that $G_1 = P_1 Q$ is a subgroup of G. Since $P > P_2$, $G_1 \subsetneq G$.

Assume that K is a subgroup of G_1 containing P_2. Then

$$K = K \cap G_1 = P_2(K \cap Q).$$

Since $K \cap Q$ is a subgroup of Q that is $\langle a\rangle$-invariant, it follows that $K \cap Q = 1$ or Q. Hence $K = P_2$ or G_1. This means that P_2 is maximal in G_1.

Hence $N_{G_1}(P_2) = P_2$ or G_1. Suppose that $N_{G_1}(P_2) = G_1$. Then we have $Q \leq N_G(P_2)$. It follows that

$$Q \leq N_G(P_2(P \cap O^p(G))) = N_G(P)$$

and thus $N_G(P) = G$ is p-nilpotent, a contradiction.

Suppose that $N_{G_1}(P_2) = P_2$. Then $N_{G_1}(P_2)$ is p-nilpotent. It follows that G_1 satisfies the hypotheses of the theorem and thus G_1 is p-nilpotent by the minimal choice of G. If condition (1) of the Main Theorem holds, then Q centralizes $\Omega(P \cap O^p(G))$. It follows that Q centralizes $P \cap O^p(G)$ by [3, Theorem 5.3.2] and thus $O^p(G)$ is p-nilpotent, a contradiction. If condition (2) of the Main Theorem holds, then Q centralizes $\Omega_1(P \cap O^p(G))$ and $P \cap O^p(G)$ is quaternion-free. In this case we also have that Q centralizes $P \cap O^p(G)$ by Lemma 2.2 and $O^p(G)$ is p-nilpotent, a contradiction.

Therefore we have $P = P_2$.

Step 8: The final contradiction. We want to apply Lemma 2.2 to obtain the final contradiction. First of all we shall construct the groups in Lemma 2.2. Put $\overline{G} = G/\Phi(O_p(G))$.

Step 8.1: $O_{p'}(\overline{G}) = 1$. Suppose that $O_{p'}(\overline{G}) \neq 1$. Let K be a normal subgroup of G such that $O_{p'}(\overline{G}) = K/\Phi(O_p(G))$. We can write $K = K_{p'}\Phi(O_p(G))$, where $K_{p'}$ is a Hall p'-subgroup of K. Since $\Phi(O_p(G))$ is a normal Sylow p-subgroup of K, it follows that any two Hall p'-subgroups of K are conjugated in K by the Schur–Zassenhaus Theorem. Then, by Frattini's argument, we have

$$G = KN_G(K_{p'}) = K_{p'}\Phi(O_p(G))N_G(K_{p'}) = \Phi(O_p(G))N_G(K_{p'}).$$

As $\Phi(O_p(G)) \leq \Phi(G)$, this yields that $G = N_G(K_{p'})$ and thus $K_{p'} \trianglelefteq G$. But then $1 \neq K_{p'} \leq O_{p'}(G)$, contrary to Step 1.

Step 8.2: $C_{\overline{G}}(O_p(\overline{G})) = O_p(\overline{G})$. By Step 8.1 and [3, Theorem 6.3.2].

Step 8.3: Put $H = \overline{G}/O_p(\overline{G})$. Then H is a p-soluble linear group over a field of characteristic p satisfying $O_p(H) = 1$. It is easy to see that

$$H = \overline{G}/O_p(\overline{G}) = \overline{G}/C_{\overline{G}}(O_p(\overline{G}))$$

acts faithfully on $O_p(\overline{G})$, where $O_p(\overline{G}) = O_p(G)/\Phi(O_p(G))$ is an elementary abelian p-group. So H is a p-soluble linear group over a field of characteristic p. Clearly, $O_p(H) = O_p(\overline{G}/O_p(\overline{G})) = 1$. Hence Step 8.3 holds.

Step 8.4: Let g denote the image of a in $H = \overline{G}/O_p(\overline{G})$. Then $g \neq 1$. Suppose $g = 1$. Then $a\Phi(O_p(G)) \in O_p(\overline{G}) = O_p(G)/\Phi(O_p(G))$ and thus $a \in O_p(G)$. From Step 3 one concludes that $P \cap O^p(G) \leq O_p(G)$. But $P = (P \cap O^p(G))\langle a \rangle$ by Step 7. So $P = O_p(G)$ and $G = N_G(P)$ is p-nilpotent by the hypothesis, a contradiction.

Step 8.5: There exists an element $y \in O_p(\overline{G})$ such that $E_{p-1}(g, y) \neq 1$. Since Q is abelian by Step 6, Lemma 2.1 shows that the minimal polynomial of g is $(x-1)^r$, where r is the order of g. From Step 8.4, one deduces that $r \geq p$. This means that $(g-1)^{p-1} \neq 0$ in H. Hence Step 8.5 holds.

Step 8.6: Completing Step 8. For any $u \in P$, by Step 7, we can write $u = st$, where $t \in \langle a \rangle$ and $s \in \Omega(P \cap O^p(G))$ when condition (1) of the Main Theorem holds, or $s \in \Omega_1(P \cap O^p(G))$ when condition (2) of the Main Theorem holds. Hence, by hypotheses, $E_{p-1}(a, u) = E_{p-1}(a, s) = 1$. The above equation implies that, for any $z \in O_p(\overline{G})$, we have $E_{p-1}(g, z) = 1$. This is contrary to Step 8.5.

These completes the proof of Main Theorem. □

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