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On the Hurwitz action in finite Coxeter groups

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Abstract. We provide a necessary and sufficient condition on an element of a finite Coxeter group to ensure the transitivity of the Hurwitz action on its set of reduced decompositions into products of reflections. We show that this action is transitive if and only if the element is a parabolic quasi-Coxeter element. We call an element of the Coxeter group parabolic quasi-Coxeter element if it has a factorization into a product of reflections that generate a parabolic subgroup. We give an unusual definition of a parabolic subgroup that we show to be equivalent to the classical one for finite Coxeter groups.

1 Introduction

This paper is concerned with the so-called dual approach to Coxeter groups. A *dual Coxeter system* is a Coxeter group together with a generating set consisting of all the reflections in the group, that is, the set of conjugates of all the elements of a simple system.

There are several motivations for studying dual Coxeter systems. They were introduced by Bessis [3] and independently by Brady and Watt [8, 10]. The dual Coxeter systems are crucial in the theory of dual braid monoids, which are alternative braid monoids embedding in the Artin–Tits group attached to a finite Coxeter group (that is, a spherical Artin–Tits group) and providing an alternative Garside structure of it. The latter allows a new presentation of the group and thereby for instance to get better solutions to the word problem in the spherical Artin–Tits groups (see [3, 5]). Each dual braid monoid depends on a choice of a Coxeter element and its poset of simple elements ordered by left-divisibility is isomorphic to the generalized noncrossing partition lattice with respect to that Coxeter element (see [3]).

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Bessis [4] later generalized these notions to complex reflections groups and their braid groups where the dual approach is natural. Indeed, there is in general no canonical choice of a simple system for a complex reflection group.

Having replaced the set of simple generators of a Coxeter group by the whole set of reflections in the Coxeter group (and in the Artin–Tits group), one needs to find new sets of relations between these new generators that define the respective groups. The idea is to take the so-called *dual braid relations* [3]. Unlike the classical braid relations, a dual braid relation can involve three generators and has the form $ab = ca$ (or $ba = ac$), where a, b and c are reflections.

In the classical case Matsumoto's Lemma [30] allows one to pass from any reduced decomposition of an element to any other one by successive applications of braid relations. The same question can be asked for reduced decompositions with respect to the new set of generators, and can be studied using the so-called *Hurwitz action* on reduced decompositions.

Let us say a bit more on this action. Let G be an arbitrary group, $n \geq 2$. There is an action of the braid group $\mathcal{B}_n$ on n strands on G^n where the standard generator $\sigma_i \in \mathcal{B}_n$ which exchanges the i-th and $(i+1)$-th strands acts as

$$\sigma_i \cdot (g_1, \ldots, g_n) := (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n).$$

Notice that the product of the entries stays unchanged and that all the tuples in a given orbit generate the same subgroup of the Coxeter group. This action is called the *Hurwitz action* since it was first studied by Hurwitz in 1891 ([23]) in the case where $G = \mathfrak{S}_n$.

Two elements $g, h \in G^n$ are called *Hurwitz equivalent* if there is a braid $\beta \in \mathcal{B}_n$ such that $\beta \cdot g = h$. It has been shown by Liberman and Teicher (see [29]) that the question of whether two elements in G^n are Hurwitz equivalent or not is undecidable in general. Nevertheless, there are results in many cases (see for instance [19] and [35]). The Hurwitz action also plays a role in algebraic geometry, more precisely in the braid monodromy of a projective curve (e.g., see [11, 28] or [25]).

In the case of finite Coxeter groups, the Hurwitz action can be restricted to the set of minimal length decompositions of a given fixed element w into products of reflections. Given a reduced decomposition $(t_1, \ldots, t_k)$ of w where $t_1, \ldots, t_k$ are reflections (that is, $w = t_1 \cdots t_k$ with k minimal), the generator $\sigma_i \in \mathcal{B}_n$ then acts as

$$\sigma_i \cdot (t_1, \ldots, t_k) := (t_1, \ldots, t_{i-1}, t_i t_{i+1} t_i, t_i, t_{i+2}, \ldots, t_k).$$

The right-hand side is again a reduced decomposition of w. In fact, we see that the braid group generator σ_i acts on the i-th and $(i+1)$-th entries by replacing (t_i, t_{i+1}) by $(t_i t_{i+1} t_i, t_i)$ which corresponds exactly to a dual braid relation. Hence determining whether one can pass from any reduced decomposition of an element to any other just by applying a sequence of dual braid relations is equivalent to determining whether the Hurwitz action on the set of reduced decompositions of the element is transitive.

The transitivity of the Hurwitz action on the set of reduced decompositions has long been known to be true for a family of elements commonly called *parabolic Coxeter elements* (note that there are several inequivalent definitions of these in the literature). For more on the topic we refer to [2], and the references therein, where a simple proof of the transitivity of the Hurwitz action was shown for (suitably defined) parabolic Coxeter elements in a (not necessarily finite) Coxeter group. See also [24]. The Hurwitz action in Coxeter groups has also been studied outside the context of parabolic Coxeter elements (see [22, 31, 37] – be aware that there are mistakes in [22]).

The aim of this paper is to provide a necessary and sufficient condition on an element of a finite Coxeter group to ensure the transitivity of the Hurwitz action on its set of reduced decompositions. We call an element of a Coxeter group a *parabolic quasi-Coxeter element* if it admits a reduced decomposition which generates a parabolic subgroup. A *quasi-Coxeter element* is an element admitting a reduced decomposition which generates the whole Coxeter group.

The main result of this paper is the following characterization.

Theorem 1.1. *Let (W, T) be a finite, dual Coxeter system of rank n and let $w \in W$. The Hurwitz action on* $\mathrm{Red}_T(w)$ *is transitive if and only if w is a parabolic quasi-Coxeter element for (W, T).*

The following theorem is an immediate consequence.

Theorem 1.2. *Let (W, T) be a finite, dual Coxeter system of rank n and let $w \in W$. If w is a quasi-Coxeter element for (W, T), then for* each $(t_1, \dots, t_n) \in \mathrm{Red}_T(w)$ *one has $W = \langle t_1, \dots, t_n \rangle$.*

The proof that being a parabolic quasi-Coxeter element is a necessary condition to ensure the transitivity of the Hurwitz action is uniform. The other direction is case-by-case. In the simply laced types we first prove Theorem 1.2 (see Theorem 6.1) and use it to prove the second direction of Theorem 1.1.

Another consequence of Theorem 1.1 and of work of Dyer [16, Theorem 3.3] is the following.

Corollary 1.3. *Let (W, T) be a dual Coxeter system and let $w \in W$. Further, let $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$, set $W' := \langle t_1, \dots, t_m \rangle$ and $T' := W' \cap T$. If W' is finite, then the Hurwitz action on* $\mathrm{Red}_{T'}(w)$ *is transitive.*

It is an open question whether this statement remains true if there is no finiteness assumption on W'.

The structure of the paper is as follows. We adopt the approach and terminology introduced in [2] (see Section 2). In particular, we use an unusual definition of parabolic subgroup, which we show in Section 4 (after recalling some well-known facts on root systems in Section 3) to be equivalent to the classical definition for finite Coxeter groups and (as a consequence of results in [18]) for a large family of irreducible infinite Coxeter groups, the so-called irreducible 2-spherical Coxeter groups. As a byproduct, we obtain some results on parabolic subgroups of finite Coxeter groups. In particular, we show the following:

Proposition 1.4. *Let (W, S) be a finite Coxeter system and $S' \subseteq T$ such that (W, S') is a simple system. Then the parabolic subgroups with respect to S coincide with those with respect to S'.*

Given a root subsystem Φ' of a given root system Φ, we discuss in Section 5 the relationship between the corresponding Coxeter groups and root lattices, especially in the simply laced types. These results are needed later in Section 6. It is known for the Coxeter groups of type A_n that all the elements are parabolic Coxeter elements in the sense of [2]. For types B_n and $I_2(m)$, the sets of parabolic Coxeter elements and parabolic quasi-Coxeter elements coincide as it is shown in Section 6. In particular, Theorem 1.1 is true for A_n, B_n and $I_2(m)$ as a consequence of [2]. For the other types it is in general false that parabolic quasi-Coxeter elements coincide with parabolic Coxeter elements. In Section 6 we also show:

Theorem 1.5. *Let w be a quasi-Coxeter element in a finite dual Coxeter system (W, T) of rank n and $(t_1, \ldots, t_n) \in \mathrm{Red}_T(w)$ such that $W = \langle t_1, \ldots, t_n \rangle$. Then the reflection subgroup $W' := \langle t_1, \ldots, t_{n-1} \rangle$ is parabolic.*

The proof of this theorem is uniform for the simply laced types, but case-by-case for the other ones. As a corollary we obtain a new characterization of the maximal parabolic subgroups of a finite dual Coxeter system (see Corollary 6.10). Moreover, it follows from Theorem 1.5 that an element is a parabolic quasi-Coxeter element if and only if it is a prefix of a quasi-Coxeter element (see Corollary 6.11).

Theorem 1.5 allows us to argue by induction to prove the main theorem. For type D_n we need to show that every two maximal parabolic subgroups intersect non-trivially provided that $n \geq 6$ (see Section 7) which then allows us to conclude Theorem 1.1 by induction. For the types E_6, E_7 and E_8 we first check by computer that every reflection occurs in the Hurwitz orbit of every reduced decomposition of a quasi-Coxeter element, and then argue by induction. Theorem 1.1 is verified for types F_4, H_3 and H_4 by computer. All this is done in Section 8.

2 Dual Coxeter systems and Hurwitz action

2.1 Dual Coxeter systems

Let (W, T) be a *dual Coxeter system* of finite rank n in the sense of [3]. This is to say that there is a subset $S \subseteq T$ with $|S| = n$ such that (W, S) is a (not necessarily finite) Coxeter system, and $T = \{wsw^{-1} \mid w \in W,\ s \in S\}$ is the set of *reflections* for the Coxeter system (W, S) (unlike Bessis, we specify no Coxeter element). We call (W, S) a *simple system* for (W, T) and S a set of *simple reflections*. If $S' \subseteq T$ is such that (W, S') is a Coxeter system, then $\{wsw^{-1} \mid w \in W,\ s \in S'\} = T$ (see [9, Lemma 3.7]). Hence a set $S' \subseteq T$ is a simple system for (W, T) if and only if (W, S') is a Coxeter system. The *rank* of (W, T) is defined as $|S|$ for a simple system S. This is well-defined by [9, Theorem 3.8]. It is equal to the rank of the corresponding root system.

Simple systems for (W, T) have been studied by several authors (see [18]). Clearly, if S is a simple system for (W, T), then so is wSw^{-1} for any $w \in W$. Moreover, it is shown in [18] that for an important class of infinite Coxeter groups including the irreducible affine Coxeter groups, all simple systems for (W, T) are conjugate to one another in this sense.

The following result is well known and follows from [16].

Proposition 2.1. *Let (W, S) be a (not necessarily finite) Coxeter system of rank n. Then W cannot be generated by less than n reflections.*

Proof. Assume that $W = \langle t_1, \dots, t_k \rangle$, with $k \leq n$. Following [16], write

$$\chi(W) = \{t \in T \mid \{u \in T \mid \ell(ut) < \ell(t)\} = \{t\}\}$$

for the set of canonical Coxeter generators of W where ℓ is the length function in (W, S). It follows from [16, Corollary 3.1 (i)] that $|\chi(W)| \leq k$. But since W is of rank n the set $\chi(W)$ contains S hence we have $|\chi(W)| \geq n$, which concludes the proof. □

A *reflection subgroup* W' is a subgroup of W generated by reflections. It is well known that $(W', W' \cap T)$ is again a dual Coxeter system (see [16]). For $w \in W$, a reduced T-decomposition of w is a shortest length decomposition of w into reflections, and we denote by $\mathrm{Red}_T(w)$ the set of all such reduced T-decompositions. When the context is clear, we drop the T and elements of $\mathrm{Red}_T(w)$ are referred to as *reduced decompositions* of w. The length of any element in $\mathrm{Red}_T(w)$ is called the *reflection length* (or *absolute length*) of w and we denote it by $\ell_T(w)$. The absolute length function $\ell_T : W \to \mathbb{Z}_{\geq 0}$ can be used to define the *absolute* *order* $\leq_T$ on W as follows. For $u, v \in W$:

$$u \leq_T v \text{ if and only if } \ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v).$$

The reflection subgroup generated by $\{s_1, \dots, s_m\}$ is called a *parabolic subgroup* for (W, T) if there is a simple system $S = \{s_1, \dots, s_n\}$ for (W, T) with $m \leq n$. Notice that this differs from the usual notion of a parabolic subgroup generated by a conjugate of a subset of a fixed simple system S (see [21, Section 1.10]). However we prove in Section 4 the equivalence of the definitions for finite Coxeter groups.

2.2 Coxeter and quasi-Coxeter elements

We now define (parabolic) Coxeter elements and (parabolic) quasi-Coxeter elements. The second item of the definition below is borrowed from [2]. The third one is a generalization of Voigt's original definition in [37], see also Remark 2.8.

Definition 2.2. Let (W, T) be a dual Coxeter system and $S = \{s_1, \dots, s_n\}$ be a simple system for (W, T).

(a) We say that $c \in W$ is a *classical Coxeter element* if c is conjugate to some $s_{\pi(1)} \cdots s_{\pi(n)}$ for π a permutation of the symmetric group $\mathfrak{S}_n$. An element $w \in W$ is a *classical parabolic Coxeter element* if $w \leq_T c$ for some classical Coxeter element c.

(b) An element $c \in W$ is called a *Coxeter element* if there exists a simple system $S' = \{s'_1, \dots, s'_n\}$ for (W, T) such that $c = s'_1 \cdots s'_n$. An element $w \in W$ is a *parabolic Coxeter element* if there exists a simple system $S' = \{s'_1, \dots, s'_n\}$ for (W, T) such that $w = s'_1 \cdots s'_m$ for some $m \leq n$.

(c) An element $w \in W$ is called a *quasi-Coxeter element* for (W, T) if there exists $(t_1, \dots, t_n) \in \mathrm{Red}_T(w)$ such that $W = \langle t_1, \dots, t_n \rangle$. An element $w \in W$ is a *parabolic quasi-Coxeter element* for (W, T) if there is a simple system $S' = \{s'_1, \dots, s'_n\}$ and $(t_1, \dots, t_m) \in \mathrm{Red}_T(w)$ such that

$$\langle t_1, \dots, t_m \rangle = \langle s'_1, \dots, s'_m \rangle$$

for some $m \leq n$.

Remark 2.3. Let us point out a few facts about these definitions.

(a) An element of the form $s_{\pi(1)} \cdots s_{\pi(n)}$ obtained as product of elements of a fixed simple system in some order as in Definition 2.2 (a) is usually called a *standard Coxeter element*. In the case where the Dynkin diagram of the Coxeter system is a tree, every two standard Coxeter elements are conjugate to each other by a sequence of cyclic conjugations (see [7, V, 6.1, Lemme 1]). Hence the set of classical Coxeter elements forms in that case a single conjugacy class. In particular, this holds for all the finite Coxeter groups.

(b) If the Coxeter group is finite, then an element $w \in W$ is a parabolic Coxeter element if and only if $w \leq_T c$ for some Coxeter element c (see [14, Corollary 3.6]).

(c) It is clear that a classical Coxeter element is a Coxeter element and hence it follows from Remark 2.3 (b) that a classical parabolic Coxeter element is a parabolic Coxeter element. The difference between classical Coxeter elements and Coxeter elements is somewhat subtle. For finite Weyl groups (see Section 3 for the definition), the two definitions are equivalent, as a consequence of [34, Theorem 1.8 (ii) and Remark 1.10]. It seems however that no case-free proof of this fact is known. An example where these two definitions differ is the dihedral group $I_2(5)$ (see [2, Remark 1.1]).

(d) We will show in Corollary 6.11 the same statement as the one given in Remark 2.3 (b) but for parabolic quasi-Coxeter elements, namely that $w \in W$ is a parabolic quasi-Coxeter element if and only if there exists a quasi-Coxeter element $w' \in W$ such that $w \leq_T w'$.

Example 2.4. In type D_4 with simple system $\{s_0, s_1, s_2, s_3\}$ where s_2 does not commute with any other simple reflection, the element

$$c := s_1(s_2s_1s_2)(s_2s_0s_2)s_3$$

is a quasi-Coxeter element. It has a reduced decomposition generating the whole group since if we write

$$(t_1, \dots, t_4) = (s_1, s_2s_1s_2, s_2s_0s_2, s_3)$$

we have that $t_1t_2t_1 = s_2$ and $s_2t_3s_2 = s_0$. Using the permutation model for a group of type D_4 (see Section 7), it can be shown that there is no reduced decomposition of this element yielding a simple system for the group. By computer we checked that the poset $\{w \in W \mid w \leq_T c\}$ has 54 elements and is not a lattice. There is a single conjugacy class of quasi-Coxeter elements which are not Coxeter elements in that case. Therefore we can not define a new Garside structure on the Artin–Tits group of type D_4 by replacing the Coxeter element by a quasi-Coxeter element.

2.3 Rigidity

Finite Coxeter groups are *reflection rigid*, that is, if (W, S) and (W, S') are simple systems for (W, T), then both systems determine the same diagram (see [9, Theorem 3.10]). We define (W, T) to be *irreducible* if (W, S) is irreducible and its type to be the type of (W, S) for some (equivalently each) simple system $S \subseteq T$. In most cases, the type is determined by the group itself. There are only two exceptions, namely

$$W_{B_{2k+1}} \cong W_{A_1} \times W_{D_{2k+1}} \qquad (k \geq 1),$$
$$W_{I_2(4k+2)} \cong W_{A_1} \times W_{I_2(2k+1)} \quad (k \geq 1),$$

which follows from the classification of the finite irreducible Coxeter groups and [32, Theorem 2.17, Lemma 2.18 and Theorem 3.3].

We say that a reflection group is *strongly reflection rigid* if whenever (W, S) and (W, S') are simple systems for (W, T), then S and S' are conjugate in W. Notice that (W, T) is strongly reflection rigid if and only if every Coxeter element is a classical Coxeter element. Hence in particular let us point out that Remark 2.3 (c) implies:

Remark 2.5. The finite Weyl groups are strongly reflection rigid.

2.4 Hurwitz action on reduced decompositions

The *braid group* on n strands denoted $\mathcal{B}_n$ is the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for } |i - j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \ldots, n-2.$$

It acts on the set T^n of n-tuples of reflections as

$$\sigma_i \cdot (t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, t_i t_{i+1} t_i, t_i, t_{i+2}, \ldots, t_n),$$
$$\sigma_i^{-1} \cdot (t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, t_{i+1}, t_{i+1} t_i t_{i+1}, t_{i+2}, \ldots, t_n).$$

We call this action of $\mathcal{B}_n$ on T^n the *Hurwitz action* and an orbit of this action an *Hurwitz orbit*.

Lemma 2.6 ([2, Lemma 1.2]). *Let W' be a reflection subgroup of W and let $T' = T \cap W'$ be the set of reflections in W'. For an element $w \in W'$ such that $\ell_{T'}(w) = n$, the braid group on n strands acts on* $\mathrm{Red}_{T'}(w)$.

We now give the main result of [2].

Theorem 2.7. *Let (W, T) be a dual Coxeter system of finite rank n and let $c = s_1 \cdots s_m$ be a parabolic Coxeter element in W. The Hurwitz action on* $\mathrm{Red}_T(c)$ *is transitive. In symbols, for each $(t_1, \ldots, t_m) \in T^m$ such that $c = t_1 \cdots t_m$, there is a braid $\beta \in \mathcal{B}_m$ such that*

$$\beta \cdot (t_1, \ldots, t_m) = (s_1, \ldots, s_m).$$

Hence in the case where W is finite, Theorem 1.1 generalizes Theorem 2.7.

Remark 2.8. For the case where (W, T) is simply laced and w is a quasi-Coxeter element in W, Voigt observed in his thesis [37] that the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive. His definition of a quasi-Coxeter element is slightly different. Let Φ be the root system associated to W (see Sections 3 and 5 for definitions and notations on root systems and lattices), then Voigt defined $w = s_{\alpha_1} \cdots s_{\alpha_n} \in W$ to be quasi-Coxeter if $\mathrm{span}_{\mathbb{Z}}(\alpha_1, \ldots, \alpha_n)$ is equal to the root lattice of Φ. The connection with our definition will be explained in Section 5.

3 Root systems and geometric representation

In this section we fix some notation, and for the convenience of the reader we recall some facts on root systems and the geometric representation of a Coxeter group as can be found for example in [21] or [7]. Let V be a finite-dimensional Euclidean vector space with positive definite symmetric bilinear form $(- \mid -)$. For $0 \neq \alpha \in V$, let $s_\alpha : V \to V$ be the reflection in the hyperplane orthogonal to α, that is, the map defined by

$$v \mapsto v - \frac{2(v \mid \alpha)}{(\alpha \mid \alpha)}\alpha.$$

Then s_α is an involution and $s_\alpha \in \mathrm{O}(V)$, the orthogonal group of V with respect to $(- \mid -)$.

Definition 3.1. A finite subset $\Phi \subseteq V$ of nonzero vectors is called *root system* in V if

(1) $\mathrm{span}_{\mathbb{R}}(\Phi) = V$,

(2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$,

(3) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in \Phi$.

The root system is called *crystallographic* if in addition

(4) $\langle \beta, \alpha \rangle := \frac{2(\beta|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The *rank* $\mathrm{rk}(\Phi)$ of Φ is the dimension of V. The group $W_\Phi = \langle s_\alpha \mid \alpha \in \Phi \rangle$ associated to the root system Φ is a Coxeter group. In the case where the root system is crystallographic, W_Φ is a *(finite) Weyl group*. A subset $\Phi' \subseteq \Phi$ is called a *root subsystem* if Φ' is a root system in $\mathrm{span}_{\mathbb{R}}(\Phi')$.

Conversely, to any finite Coxeter group one can associate a root system. For an infinite Coxeter system (W, S), one can still associate a set of vectors (again called a root system) with slightly relaxed conditions (see for instance [21, Section 5.3–5.7]). In the following, when dealing with root systems it will always be in the sense of the above definition unless otherwise specified since this paper primarily concerns finite Coxeter groups. When results generalize to arbitrary Coxeter groups, we will mention that we work with the generalized root systems.

Let $\Phi \neq \varnothing$ be a root system. Then Φ is *reducible* if $\Phi = \Phi_1 \dot{\cup} \Phi_2$ where Φ_1, Φ_2 are nonempty root systems such that $(\alpha \mid \beta) = 0$ whenever $\alpha \in \Phi_1$, $\beta \in \Phi_2$. Otherwise Φ is *irreducible*. For an irreducible crystallographic root system Φ, the set $\{(\alpha \mid \alpha) \mid \alpha \in \Phi\}$ has at most two elements (see [21, Section 2.9]). If this set has only one element, up to rescaling we can assume it to be 2 and call Φ *simply laced*. It follows from the classification of irreducible root systems that simply laced root systems are crystallographic. Simply laced root systems have types A_n ($n \geq 1$), D_n ($n \geq 4$) or E_n ($n \in \{6, 7, 8\}$) in the classification. We will sometimes use the notation W_{X_n} for the Coxeter group with corresponding root system of type X_n for convenience. For more on the topic we refer the reader to [21].

4 Equivalent definitions of parabolic subgroups

In this section, we show one direction of Theorem 1.1 with a case-free argument. To this end, we show Proposition 1.4, that is, we show that for finite Coxeter groups, the definition of parabolic subgroups given in Section 2 coincides with the usual one. We also mention that they coincide for a large class of infinite Coxeter groups.

Let (W, S) be a finite Coxeter system with root system Φ and $V := \mathrm{span}_{\mathbb{R}}(\Phi)$. We say that a subgroup generated by a conjugate of a subset of S is *parabolic in the classical sense*. It is well known that these are exactly the subgroups of the form

$$C_W(E) := \{w \in W \mid w(v) = v \text{ for all } v \in E\}$$

where $E \subseteq V$ is any set of vectors (see for instance [26, Section 5-2]).

Definition 4.1. Given a subset $\mathcal{A} \subseteq W$, the *parabolic closure* $P_{\mathcal{A}}$ of $\mathcal{A}$ is the intersection of all the parabolic subgroups in the classical sense containing $\mathcal{A}$. It is again a parabolic subgroup in the classical sense (see [36, 12.2–12.5] or [33]).

We denote by $\mathrm{Fix}(\mathcal{A})$ the subspace of vectors in V which are fixed by every element of $\mathcal{A}$. If $\mathcal{A} = \{w\}$, then we simply write $\mathrm{Fix}(w)$ for $\mathrm{Fix}(\mathcal{A}) = \ker(w - 1)$ and P_w for $P_{\mathcal{A}}$. For convenience we also set $\mathrm{Mov}(w) := \mathrm{im}(w - 1)$. Note that $V = \mathrm{Fix}(w) \oplus \mathrm{Mov}(w)$ (see [1, Definition 2.4.6]). It follows from the above description that $P_{\mathcal{A}} = C_W(\mathrm{Fix}(\mathcal{A}))$.

In this section we give a case-free proof that for finite Coxeter groups, the parabolic subgroups as defined in Section 2.1 coincide with the parabolic subgroups in the classical sense. As a consequence we are able to show one direction of Theorem 1.1. We first recall the following result

Lemma 4.2 ([3, Lemma 1.2.1 (i)], [1, Theorem 2.4.7] after [12]). *Let $w \in W$, $t \in T$. Then*

$$\mathrm{Fix}(w) \subseteq \mathrm{Fix}(t) \text{ if and only if } t \leq_T w.$$

Notice that Lemma 4.2 implies [2, Theorem 1.4] if W is finite.

Proposition 4.3. *Let (W, S) be a finite Coxeter system, $T = \bigcup_{w \in W} wSw^{-1}$ and $w \in W$. If the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive, then the subgroup generated by the reflections in any reduced decomposition of w is equal to P_w.*

Proof. We prove the contrapositive of the statement. Let $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$ and assume that $W' := \langle t_1, \ldots, t_m \rangle$ is not equal to P_w. Since $t_i \leq_T w$ for each i, we have $t_i \in P_w$ for all $i = 1, \ldots, m$ by Lemma 4.2. It follows that $W' \subsetneq P_w$. Since both W' and P_w are reflection subgroups of W, there exists a reflection $t \in P_w$ with $t \notin W'$. It follows that $\mathrm{Fix}(w) \subseteq \mathrm{Fix}(t)$, hence also that $t \leq_T w$ by Lemma 4.2. In particular, there exists $(q_1, \ldots, q_m) \in \mathrm{Red}_T(w)$ with $q_1 = t$. Since the Hurwitz orbit of $(t_1, \ldots, t_m)$ remains in W' and $t \notin W'$, the Hurwitz action on $\mathrm{Red}_T(w)$ can therefore not be transitive. □

Corollary 4.4. *Let (W, S) be a finite Coxeter system and $T = \bigcup_{w \in W} wSw^{-1}$. A subgroup $P \subseteq W$ is parabolic if and only if it is parabolic in the classical sense. In particular, if $S' \subseteq T$ is such that (W, S') is a simple system, then the parabolic subgroups in the classical sense defined by S coincide with those defined by S'.*

Proof. If P is parabolic, then $P = \langle s_1, \ldots, s_m \rangle$ where $\{s_1, \ldots, s_n\} = S' \subseteq T$ is a simple system for W and $m \leq n$. By [2, Theorem 1.3], the Hurwitz action on $\mathrm{Red}_T(w)$ where $w = s_1 s_2 \cdots s_m$ is transitive. By Proposition 4.3, it follows that P is parabolic in the classical sense.

Conversely, if P is parabolic in the classical sense, then P is generated by a conjugate of a subset of S, and a conjugate of S is again a simple system for W. Hence P is parabolic. □

As a corollary we get a proof of one direction of Theorem 1.1:

Corollary 4.5. *Let (W, T) be a finite, dual Coxeter system of rank n and let $w \in W$. If the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive, then w is a parabolic quasi-Coxeter element for (W, T).*

Proof. Let $w = t_1 \cdots t_m \in \mathrm{Red}_T(w)$. By Proposition 4.3, $W' := \langle t_1, \dots, t_m \rangle$ is parabolic in the classical sense. By Corollary 4.4, it follows that W' is parabolic and hence w is a parabolic quasi-Coxeter element. □

If (W, S) is of type $\tilde{A}_1$, i.e. if W is a dihedral group of infinite order, then the non-trivial parabolic subgroups are precisely those subgroups that are generated by a reflection. Therefore parabolic subgroups in the classical sense coincide with parabolic subgroups in this case. As an immediate consequence of a theorem of Franzsen, Howlett and Mühlherr [18] we also get the equivalence of the definitions for a large family of infinite Coxeter groups (including the irreducible affine Coxeter groups):

Proposition 4.6. *Let (W, S) be an infinite irreducible 2-spherical Coxeter system, that is, S is finite, and ss' has finite order for every $s, s' \in S$. Then a subgroup of W is parabolic if and only if it is parabolic in the classical sense.*

Proof. Under these assumptions, if (W, S') is a Coxeter system (without assuming $S' \subset T$), it follows from [18, Theorem 1 b)] that there exists $w \in W$ such that $S' = wSw^{-1}$, hence any parabolic subgroup is a parabolic subgroup in the classical sense. □

Question 4.7. Do parabolic subgroups always coincide with parabolic subgroups in the classical sense?

5 Root lattices

In this section, we study root lattices and their sublattices. The results will be needed for the better understanding of quasi-Coxeter elements in the simply laced types. Parts of this section have been inspired by [37].

Definition 5.1. Let V be a Euclidean vector space with symmetric bilinear form $(- \mid -)$. A *lattice* L in V is the integral span of a basis of V. The lattice L is called *integral* if $(\alpha \mid \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in L$ and *even* if $(\alpha \mid \alpha) = 2$ for all basis elements α.

For a set of vectors $\Phi \subseteq V$ we set $L(\Phi) := \mathrm{span}_{\mathbb{Z}}(\Phi)$.

Remark 5.2. If Φ is a root system, then $L(\Phi)$ is a lattice, called the *root lattice*. If Φ is a crystallographic root system, then $L(\Phi)$ is an integral lattice.

Proposition 5.3. *For an even lattice L the set $\Phi(L) := \{\alpha \in L \mid (\alpha \mid \alpha) = 2\}$ is a simply laced, crystallographic root system in* $\mathrm{span}_{\mathbb{R}}(L)$.

Proof. The set $\Phi(L)$ is contained in the ball around 0 with radius 2, therefore bounded, thus finite. The rest of the proof is straightforward. □

Definition 5.4. Let Φ be a crystallographic root system in V. The *weight lattice* $P(\Phi)$ of Φ is defined by $P(\Phi) := \{x \in V \mid \langle x, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. By [7, VI, 1.9] it is again a lattice containing $L(\Phi)$ and the group $P(\Phi)/L(\Phi)$ is finite. We call its order the *connection index* of Φ and denote it by $i(\Phi)$.

Note that if Φ is simply laced, the weight lattice is equal to the dual root lattice, namely,

$$P(\Phi) = L^*(\Phi) := \{x \in V \mid (x \mid y) \in \mathbb{Z} \text{ for all } y \in L(\Phi)\}.$$

Proposition 5.5. *Let Φ be a simply laced root system and let C be the Cartan matrix of Φ. Then*

$$i(\Phi) = \det(C).$$

Proof. Let $\Delta = \{\alpha_1, \dots, \alpha_m\} \subseteq \Phi$ be a basis of the root system Φ. Then Δ is a basis of $L(\Phi)$. Denote by M the Gram matrix of $L(\Phi)$ with respect to Δ. By general lattice theory (see for instance [17, Section 1.1]) one has

$$|L^*(\Phi) : L(\Phi)| = \det(M).$$

Since Φ is simply laced, we have $L^*(\Phi) = P(\Phi)$ and hence

$$i(\Phi) = |L^*(\Phi) : L(\Phi)|.$$

Again since Φ is simply laced, we have $C = M$, which concludes the proof. □

We list $i(\Phi)$ for the irreducible, simply laced root systems. These can be found in [7, Planches I, IV , V, VI, VII].

Type of Φ	A_n	D_n	E_6	E_7	E_8
$i(\Phi)$	$n+1$	4	3	2	1

As a consequence we obtain the following result.

Proposition 5.6. *Let Φ be an irreducible, simply laced root system. Then Φ is determined by the pair* $(\operatorname{rk}(\Phi), i(\Phi))$.

The following lemma seems to be folklore, but we could not find a proof in the literature, hence we state it here.

Lemma 5.7. *Let Φ be a simply laced root system. Then the root lattice determines the root system, that is,*

$$\Phi(L(\Phi)) = \Phi.$$

Proof. By the previous proposition, we have to show that the rank and connection indices of Φ and $\Phi(L(\Phi))$ coincide. Since $\Phi \subseteq \Phi(L(\Phi))$, we have

$$\mathrm{rk}(\Phi) \leq \mathrm{rk}(\Phi(L(\Phi))).$$

On the other hand, the rank of $\Phi(L(\Phi))$ is bounded above by the dimension of the ambient vector space which equals $\mathrm{rk}(\Phi)$.

In the proof of Proposition 5.5, we used the fact that the Cartan matrix of a root system and the Gram matrix of the corresponding root lattice coincide. Denote by C_Φ the Cartan matrix with respect to Φ. Then

$$\det(C_\Phi) = \mathrm{vol}(L(\Phi)) = \mathrm{vol}(L(\Phi(L(\Phi)))) = \det(C_{\Phi(L(\Phi))}),$$

which yields $i(\Phi) = i(\Phi(L(\Phi)))$ by Proposition 5.5. □

Lemma 5.8. *Let Φ be as in Lemma* 5.7 *and let $\Phi' \subseteq \Phi$ be a root subsystem. Then $L(\Phi') \cap \Phi = \Phi'$.*

Proof. We have the equalities

$$\begin{aligned} L(\Phi') \cap \Phi &= L(\Phi') \cap \{\alpha \in L(\Phi) \mid (\alpha \mid \alpha) = 2\} && \text{by Lemma 5.7} \\ &= \{\alpha \in L(\Phi') \mid (\alpha \mid \alpha) = 2\} \\ &= \Phi' && \text{by Lemma 5.7.} \end{aligned}$$ □

Notation 5.9. Let Φ be a finite root system and $R \subseteq \Phi$ a set of roots. We denote by W_R the group $\langle s_r \mid r \in R\rangle$.

Note that this is consistent with the notation introduced in Definition 3.1 in the case where $R = \Phi$.

Proposition 5.10. *Let Φ be a crystallographic root system, let $\Phi' \subseteq \Phi$ be a root subsystem, and let $R := \{\beta_1, \dots, \beta_k\} \subseteq \Phi'$ be a set of roots. The following statements are equivalent:*

(a) *The root subsystem Φ' is the smallest root subsystem of Φ containing R (i.e., the intersection of all root subsystems containing R).*

(b) $\Phi' = W_R(R)$.

(c) $W_{\Phi'} = W_R$.

Moreover, if any of the above equivalent conditions is satisfied, then

(d) $L(\Phi') = L(R)$.

Proof. Obviously, $W_R(R)$ is a root system with $R \subseteq W_R(R)$. Thus if (a) holds, then $\Phi' \subseteq W_R(R)$. As $W_R(R) \subseteq W_{\Phi'}(\Phi') = \Phi'$, it follows that (a) implies (b). The converse direction follows from the definition of a root subsystem.

Statement (b) implies that $R \subseteq \Phi'$ and therefore we have $W_R \subseteq W_{\Phi'}$. We show that in this case $W_{\Phi'} \subseteq W_R$. To this end, let $\alpha \in \Phi' = W_R(R)$. Then $\alpha = w(\beta_i)$ for some $w \in W_R$ and $1 \leq i \leq k$. Then

$$s_\alpha = s_{w(\beta_i)} = w s_{\beta_i} w^{-1} \in W_R,$$

which shows the claim. Thus (b) implies (c).

Assume (c) and let Φ'' be the smallest root subsystem of Φ containing R. Then $W_{\Phi'} = W_R \subseteq W_{\Phi''}$. By the definition of Φ'' we have $\Phi'' \subseteq \Phi'$. If $\Phi'' \subsetneq \Phi'$, then $W_{\Phi''} \subsetneq W_{\Phi}$, a contradiction. Hence $\Phi'' = \Phi'$, which shows (a).

It remains to show that (c) implies (d). So assume (c) and let $t_i := s_{\beta_i}$, $1 \leq i \leq k$. Let $T_{\Phi'}$ be the set of reflections in $W_{\Phi'}$. By [16, Corollary 3.11 (ii)], we have $T_{\Phi'} = \{w t_i w^{-1} \mid 1 \leq i \leq k,\ w \in W_{\Phi'}\}$. In particular, any root in Φ' has the form $w(\beta_i)$ for some $w \in W_{\Phi'}$, $1 \leq i \leq k$. Since $W_{\Phi'} = \langle t_1, \ldots, t_k \rangle$, we can write $w = t_{i_1} \cdots t_{i_m}$ with $1 \leq i_j \leq k$ for each $1 \leq j \leq m$. Since Φ' is crystallographic, it follows that $w(\beta_i) = t_{i_1} \cdots t_{i_m}(\beta_i)$ is an integral linear combination of the roots β_j, hence that $\Phi' \subseteq L(R)$. Since $R \subseteq \Phi'$, we get $L(R) = L(\Phi')$, which proves (d). □

Remark 5.11. Notice that condition (d) in Proposition 5.10 is in general not equivalent to conditions (a)–(c). For example, if Φ is of type B_2, then one can choose two orthogonal roots α and β generating a proper root subsystem of type $A_1 \times A_1$ while one has $L(\{\alpha, \beta\}) = L(\Phi)$. Nevertheless, one has the equivalence for simply laced root systems:

Lemma 5.12. *Let* Φ, Φ', R *be as in Proposition* 5.10 *and assume in addition that* Φ *is simply laced. Then condition* (d) *in Proposition* 5.10 *is equivalent to any of conditions* (a)–(c).

Proof. Since conditions (a)–(c) are equivalent, it suffices to show that condition (d) implies (a). Assume that there exists a root subsystem Φ'' of Φ with $\Phi'' \subseteq \Phi'$ and $R \subseteq \Phi''$. It follows that $L(\Phi') = L(R) \subseteq L(\Phi'')$. One then has

$$\Phi' = L(\Phi') \cap \Phi \subseteq L(\Phi'') \cap \Phi = \Phi''$$

by Lemma 5.8. Hence $\Phi' = \Phi''$, which concludes the proof. □

The following two results are useful tools for the next section. The following proposition and its proof are borrowed from [37, Proposition 1.5.2].

Proposition 5.13. *Let Φ be a simply laced root system and let $\Phi' \subseteq \Phi$ be a root subsystem with* $\mathrm{rk}(\Phi') = m = \mathrm{rk}(\Phi)$. *Then:*

(a) $|L(\Phi) : L(\Phi')| = |P(\Phi') : P(\Phi)|$,

(b) $|L(\Phi) : L(\Phi')| = \sqrt{i(\Phi') \cdot i(\Phi)^{-1}}$.

Proof. Since Φ and Φ' have the same rank, there is a lattice isomorphism

$$\mathcal{L} : L(\Phi) \xrightarrow{\sim} L(\Phi'),$$

hence $|L(\Phi) : L(\Phi')| = |\det(\mathcal{L})|$ (see [17, Section 1.1]). Furthermore, we have

$$\begin{aligned} P(\Phi') &= \{x \in V \mid (x \mid y) \in \mathbb{Z} \text{ for all } y \in L(\Phi')\} \\ &= \{x \in V \mid (x \mid \mathcal{L}(v)) \in \mathbb{Z} \text{ for all } v \in L(\Phi)\} \\ &= \{x \in V \mid (\mathcal{L}^t(x) \mid v) \in \mathbb{Z} \text{ for all } v \in L(\Phi)\} \\ &= \{x \in V \mid \mathcal{L}^t(x) \in P(\Phi)\} \\ &= (\mathcal{L}^t)^{-1}(P(\Phi)). \end{aligned}$$

Thus $[P(\Phi') : P(\Phi)] = |\det(\mathcal{L}^t)| = |\det(\mathcal{L})|$, which shows (a). We have

$$L(\Phi') \subseteq L(\Phi) \subseteq P(\Phi) \subseteq P(\Phi').$$

It follows that

$$|L(\Phi) : L(\Phi')| \underbrace{|P(\Phi) : L(\Phi)|}_{= i(\Phi)} \underbrace{|P(\Phi') : P(\Phi)|}_{= |L(\Phi):L(\Phi')|} = \underbrace{|P(\Phi') : L(\Phi')|}_{= i(\Phi')},$$

which concludes the proof. □

For a simply laced root system Φ we extend the definition of connection index to subsets of Φ. For an arbitrary subset $R \subseteq \Phi$ we define $i(R) := |L^*(R) : L(R)|$. Note that $i(R)$ is well-defined by Proposition 5.10, because $i(R) = i(\Phi')$, where Φ' is the smallest root subsystem of Φ in $\mathrm{span}_{\mathbb{R}}(R)$ containing R.

The following theorem is part of the Diploma thesis of Kluitmann [27].

Theorem 5.14. *Let Φ be a simply laced root system. Further, let $w \in W_\Phi$ and let $(s_{\alpha_1}, \ldots, s_{\alpha_k})$, $(s_{\beta_1}, \ldots, s_{\beta_k}) \in \mathrm{Red}_T(w)$, where $\alpha_i, \beta_i \in \Phi$, for $1 \leq i \leq k$. Then for $R := \{\alpha_1, \ldots, \alpha_k\}$ and $Q := \{\beta_1, \ldots, \beta_k\}$ we have*

$$i(R) = i(Q).$$

Proof. Consider $L(R)$ and $L^*(R)$ (respectively $L(Q)$ and $L^*(Q)$) as lattices in $V' := \mathrm{span}_{\mathbb{R}}(R)$ (respectively in $V'' := \mathrm{span}_{\mathbb{R}}(Q)$). By [12, Lemma 3] the set R

is a basis for $L(R)$. Let $\{\alpha_1^*, \dots, \alpha_k^*\}$ be the basis of $L^*(R)$ dual to R. In particular, $s_{\alpha_j}(\alpha_i^*) = \alpha_i^*$ for $i \neq j$ and $s_{\alpha_i}(\alpha_i^*) = \alpha_i^* - \alpha_i$. Therefore

$$\theta_i := (w-1)(\alpha_i^*) = -s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i) \in L(R) \cap \Phi.$$

Note that $\theta_1 = -\alpha_1$, $\theta_2 \in -\alpha_2 + \operatorname{span}_{\mathbb{Z}}(\alpha_1)$, and more generally

$$\theta_i \in -\alpha_i + \operatorname{span}_{\mathbb{Z}}(\alpha_1, \dots, \alpha_{i-1}).$$

It follows that $\{\theta_1, \dots, \theta_k\}$ is a basis of V' and that the map

$$(w-1)_{|V'} : L^*(R) \to L(R)$$

is bijective.

Thus we have $i(R) = |\det(w-1)_{|V'}|$. The same argument with Q instead of R yields that $i(Q) = |\det(w-1)_{|V''}|$. By the proof of [1, Theorem 2.4.7] we have that R and Q are both bases for $\operatorname{Mov}(w)$, hence $V' = V'' = \operatorname{Mov}(w)$ and hence $i(R) = i(Q)$. □

6 Reflection subgroups related to prefixes of quasi-Coxeter elements

In this section, we prove Theorem 1.2 for simply laced dual Coxeter systems, see Theorem 6.1. Further we show that the reflections in a reduced T-decomposition of an element $w \in W$ generate a parabolic subgroup whenever $w \leq_T w'$ for some quasi-Coxeter element w'. Last but not least we demonstrate that parabolic quasi-Coxeter elements coincide with parabolic Coxeter elements in types A_n, B_n and $I_2(m)$.

Recall that for $w \in W$, we denote by P_w the parabolic closure of w (see Definition 4.1) and that $P_w = C_W(\operatorname{Fix}(w))$.

Theorem 6.1. *Let (W, T) be a simply laced dual Coxeter system of rank n. If $w \in W$ is a parabolic quasi-Coxeter element, then the reflections in any reduced T-decomposition of w generate the parabolic subgroup P_w. That is, for each $(t_1, \dots, t_m) \in \operatorname{Red}_T(w)$ we have $P_w = \langle t_1, \dots, t_m \rangle$.*

Proof. By the definition of a parabolic quasi-Coxeter element, there exists an element $(t_1, \dots, t_m) \in \operatorname{Red}_T(w)$ such that $P := \langle t_1, \dots, t_m \rangle$ is a parabolic subgroup. By Lemma 4.2, we have $P \subseteq C_W(\operatorname{Fix}(w)) = P_w$. Since $w \in P$, we have by definition of the parabolic closure that $P = P_w$. Let $(q_1, \dots, q_m) \in \operatorname{Red}_T(w)$. Then for all $1 \leq i \leq m$ we have $q_i \leq_T w$, which yields that q_i is in $C_W(\operatorname{Fix}(w)) = P_w$. Thus $W' := \langle q_1, \dots, q_m \rangle$ is a subgroup of P_w. Let Φ be the root system of P_w and $\beta_i \in \Phi$ be such that $q_i = s_{\beta_i}$, for $1 \leq i \leq m$. Then $L(\beta_1, \dots, \beta_m)$ is a sublattice of $L(\Phi)$ and $\Phi' = L(\beta_1, \dots, \beta_m) \cap \Phi$ is the smallest root subsystem of Φ

that contains $\beta_1, \ldots, \beta_m$. Therefore Theorem 5.14 yields that $i(\Phi') = i(\Phi)$. This implies $L(\Phi) = L(\Phi')$ by Proposition 5.13. Thus $W' = P_w$ by Lemma 5.12. □

We will show in Corollary 6.11 that the following property of parabolic quasi-Coxeter elements does in fact characterize them.

Proposition 6.2. *Let (W, T) be a finite dual Coxeter system. If $w \in W$ is a parabolic quasi-Coxeter element, then there exists a quasi-Coxeter element $w' \in W$ such that $w \leq_T w'$.*

Proof. Let $w \in W$ be a parabolic quasi-Coxeter element. By definition, there exists a simple system $S = \{s_1, \ldots, s_n\}$ for W and a T-reduced decomposition $w = t_1 \cdots t_m$ such that

$$\langle t_1, \ldots, t_m \rangle = \langle s_1, \ldots, s_m \rangle,$$

with $m \leq n$. Set $w' := t_1 \cdots t_m s_{m+1} \cdots s_n$. It is clear that

$$\langle t_1, \ldots, t_m, s_{m+1}, \ldots, s_n \rangle = W.$$

Moreover, we have that $\ell_T(w') = n$, hence w' is a quasi-Coxeter element with $w \leq_T w'$. □

Lemma 6.3. *Let (W, T) be a dual Coxeter system of type A_n. Then each $w \in W$ is a classical parabolic Coxeter element.*

Proof. In type A_n, the set of $(n+1)$-cycles forms a single conjugacy class. Hence the set of classical Coxeter elements is exactly the set of $(n+1)$-cycles (see Remark 2.3 (a)). The assertion follows with Remark 2.3 (b) as for every element $w \in W$, we have $w \leq_T w'$ for at least one $(n+1)$-cycle w'. □

Lemma 6.4. *Let (W, T) be a dual Coxeter system of type B_n. Then every parabolic quasi-Coxeter element $w \in W$ for (W, T) is a classical parabolic Coxeter element.*

Proof. For the proof we use the combinatorial description of W_{B_n} as given in [6, Section 8.1]. Therefore let $S_{-n,n}$ be the group of permutations of

$$[-n, n] = \{-n, -n+1, \ldots, -1, 1, \ldots, n\}$$

and define

$$W = W_{B_n} := \{w \in S_{-n,n} \mid w(-i) = -w(i) \text{ for all } i \in [-n, n]\},$$

also known as the *hyperoctahedral group*. Then (W, S) is a Coxeter system of type B_n with

$$S = \{(1, -1), (1, 2)(-1, -2), \ldots, (n-1, n)(-n+1, -n)\}.$$

The set of reflections T for this choice of S is given by

$$T = \{(i, -i) \mid i \in [n]\} \cup \{(i, j)(-i, -j) \mid 1 \leq i < |j| \leq n\}.$$

We show that every quasi-Coxeter element for (W, T) is a classical Coxeter element. If w is a parabolic quasi-Coxeter element, then by Proposition 6.2, there exists a quasi-Coxeter element $w' \in W$ such that $w \leq_T w'$. Hence if w' is a classical Coxeter element, then $w \leq_T w'$ implies that w is a classical parabolic Coxeter element.

Let $R = \{r_1, \dots, r_n\} \subseteq T$ be such that $\langle R \rangle = W$. It suffices to show that $r_1 r_2 \cdots r_n$ is in fact a classical Coxeter element.

The group W cannot be generated only by reflections of type $(i, j)(-i, -j)$, $i \neq \pm j$. Therefore there exists $i \in [n]$ with $(i, -i) \in R$. If there exists $j \in [n]$, $j \neq i$, with $(j, -j) \in R$, then R cannot generate the whole group W. Since classical Coxeter elements are closed under conjugation, we can conjugate the set R with $(1, i)(-1, -i)$ (if necessary) and assume $(1, -1) \in R$.

Since R generates the whole group W, there does not exist $j \in [n]$ which is fixed by each $r \in R$. Thus for each $k \in [n], k \neq 1$, we can find $i_k \in [\pm n]$ with $k \neq \pm i_k$ such that $(k, i_k)(-k, -i_k) \in R$. Therefore

$$R = \{(1, -1), (2, i_2)(-2, -i_2), \dots, (n, i_n)(-n, -i_n)\}.$$

Note that some i_j has to equal ± 1, because otherwise $(1, -1)$ would commute with any element of W. By conjugating R with $(j, 2)(-j, -2)$ respectively $(j, -2)(-j, 2)$ (if necessary) we can assume that $i_2 = 1$. Hence after rearrangement we can assume that R is of the form

$$R = \{(1, -1), (2, 1)(-2, -1), (3, i_3)(-3, -i_3), \dots, (n, i_n)(-n, -i_n)\}.$$

Similarly to what we did above, there exists $j \geq 3$ with $i_j \in \{\pm 1, \pm 2\}$. By conjugating R with $(j, 3)(-j, -3)$ respectively $(j, -3)(-j, 3)$ (if necessary) we can assume that $i_3 \in \{1, 2\}$. Continuing in this manner we obtain

$$R = \{(1, -1), (2, i_2)(-2, -i_2), \dots, (n, i_n)(-n, -i_n)\}$$

with $i_j \in \{1, \dots, j-1\}$ for each $j \in \{2, \dots, n\}$. A direct computation shows that $c := r_1 r_2 \cdots r_n$ is a $2n$-cycle and thus a classical Coxeter element. Indeed, there is a single conjugacy class of $2n$-cycles in W. □

Remark 6.5. Notice that by Remark 2.3 (c), it is already known that classical Coxeter elements and Coxeter elements must coincide in type B_n. Moreover, it follows from [12, Lemma 8, Theorem A] that every quasi-Coxeter element is actually a Coxeter element. Hence one can derive Lemma 6.4 from these two observations. However since both of them rely on sophisticated methods, we preferred to give here a direct proof using the combinatorics of the hyperoctahedral group.

Remark 6.6. In type A_n, we even have that every element w such that $\ell_T(w) = n$ is a classical Coxeter element (thus quasi-Coxeter), because such an element is necessarily an $(n+1)$-cycle. Notice that this fails in type B_n. For instance, the product $(1,-1)(2,-2)\cdots(n,-n)$ in W_{B_n} has reflection length equal to n, but it is not a quasi-Coxeter element.

The following is well known (see [7, IV, 1.2, Proposition 2]):

Proposition 6.7. *A group W is a dihedral group if and only if it is generated by two elements s, t of order 2, in which case $\{s,t\}$ is a simple system for W.*

Corollary 6.8. *Let (W,T) be a dual Coxeter system of type $I_2(m)$, $m \geq 3$. Then w is a quasi-Coxeter element in W if and only if w is a Coxeter element in W. It follows that $w \in W$ is a parabolic quasi-Coxeter element if and only if w is a parabolic Coxeter element.*

Note that Coxeter elements and classical Coxeter elements do not coincide in general in dihedral type (see Remark 2.3 (c)).

Theorem 1.5. *Let w be a quasi-Coxeter element in a finite dual Coxeter system (W,T) of rank n and $(t_1,\ldots,t_n) \in \mathrm{Red}_T(w)$ such that $W = \langle t_1,\ldots,t_n\rangle$. Then the reflection subgroup $W' := \langle t_1,\ldots,t_{n-1}\rangle$ is parabolic.*

Proof. The reduction to the case where W is irreducible is immediate. The proof is uniform for the simply laced types and case-by-case for the non-simply laced types.

Dihedral type. The claim is obvious in that case.

Simply laced types. Let Φ be a root system for (W,T) with ambient vector space V. Let $P_{W'}$ be the parabolic closure of W'. For $1 \leq i \leq n$, let $\beta_i \in \Phi$ be a root corresponding to t_i and let Φ' be the smallest root subsystem of Φ containing $R := \{\beta_1,\ldots,\beta_{n-1}\}$ so that $W_R = W_{\Phi'} = W'$ (see Proposition 5.10). Let $\Psi \subseteq \Phi$ be the root subsystem of Φ associated to $P_{W'}$. By [12, Lemma 3] the set $R \cup \{\beta_n\}$ is a basis of V.

Let U be the ambient vector space for Ψ. As the linear independent set R is a subset of Ψ, the dimension of U is at least $n-1$. Since $P_{W'}$ is the parabolic closure of $t_1\cdots t_{n-1}$ it has to be the centralizer of a line in V and therefore $\dim U = n-1$. It follows that $U = \mathrm{span}_{\mathbb{R}}(\beta_1,\ldots,\beta_{n-1})$.

By Proposition 5.10 we have that

$$L(\Phi') = L(\{\beta_1,\ldots,\beta_{n-1}\}) \quad \text{and} \quad L(\Phi) = L(\{\beta_1,\ldots,\beta_n\}).$$

Since $V = U \oplus \mathbb{R}\beta_n$, we have that $L(\Phi) \cap U = L(\{\beta_1, \ldots, \beta_{n-1}\}) = L(\Phi')$. As $L(\Psi) \subseteq U$, it follows that $L(\Psi) \subseteq L(\Phi')$. But since $\Phi' \subseteq \Psi$, we get that $L(\Phi') \subseteq L(\Psi)$ and therefore $L(\Phi') = L(\Psi)$. Thus $W' = P_{W'}$ by Lemma 5.12.

Type B_n. By Lemma 6.4 the element w is a classical Coxeter element. It follows that wt_n is a classical parabolic Coxeter element, hence a parabolic Coxeter element (see Remark 2.3 (c)). It follows that W' is parabolic.

Type F_4. By [15, Table 6], the group W_{F_4} cannot be generated by just adding one reflection to one of its non-parabolic rank 3 reflection subgroups.

Types H_3 and H_4. We refer to [15, Tables 8 and 9], where the reflection subgroups of W_{H_3} and W_{H_4} and their parabolic closures are determined.

(i) Each rank 2 reflection subgroup of the group W_{H_3} is already parabolic.

(ii) The only rank 3 reflection subgroup of W_{H_4} that is not parabolic has type $A_1 \times A_1 \times A_1$. Taking a set of three reflections generating such a reflection subgroup, we checked using [38] that this set cannot be completed to obtain a generating set for W_{H_4} by adding a single reflection. □

Remark 6.9. Theorem 1.5 is not true in general. It can even fail if w is a Coxeter element, as the following example borrowed from [20, Example 5.7] shows: Let $W = \langle s, t, u \rangle$ be of affine type $\widetilde{A}_2$, and let $w = stu$. Then we have $s(tut) \leq_T c$, but $W' = \langle s, tut \rangle$ is an infinite dihedral group, hence it is not a parabolic subgroup in the classical sense since proper parabolic subgroups of W are finite. We mentioned in Section 4 that for affine Coxeter groups parabolic subgroups in the classical sense also coincide with parabolic subgroups as defined in Section 2.1.

Corollary 6.10. *Let (W, T) be a finite dual Coxeter system of rank n and let W' be a reflection subgroup of rank $n-1$. Then W' is a parabolic subgroup if and only if there exists $t \in T$ such that $\langle W', t \rangle = W$.*

Proof. The necessary condition is clear by the definition of parabolic subgroup. The sufficient condition is a direct consequence of Theorem 1.5. □

We also derive a characterization of parabolic quasi-Coxeter elements analogous to that of parabolic Coxeter elements (see Remark 2.3 (b)).

Corollary 6.11. *Let (W, T) be a finite dual Coxeter system and $w \in W$. Then w is a parabolic quasi-Coxeter element if and only if there exists a quasi-Coxeter element $w' \in W$ such that $w \leq_T w'$.*

Proof. The forward direction is given by Proposition 6.2. Now let $w \leq_T w'$ where $w' \in W$ is a quasi-Coxeter element. Using Theorem 1.5 inductively we get that w is a parabolic quasi-Coxeter element in W. □

Remark 6.12. Corollary 6.11 does not hold for infinite Coxeter groups as it fails for the Coxeter element given in Remark 6.9.

7 Intersection of maximal parabolic subgroups in type D_n

The aim of this section is to show the following result which will be needed in the next section in the proof of Theorem 1.1.

Proposition 7.1. *Let (W, S) be a Coxeter system of type D_n ($n \geq 6$). Then the intersection of two maximal parabolic subgroups is non-trivial.*

Remark 7.2. This statement is not true in general, not even in the simply laced case. In particular, it fails in types D_4, D_5, E_7 and E_8. For example:

(a) Let (W, S) be of type D_4 where $S = \{s_0, s_1, s_2, s_3\}$ with s_2 commuting with no other simple reflection, then both $P := \langle s_0, s_1, s_3\rangle$ and $s_2 P s_2$ are maximal parabolic subgroups of type $A_1 \times A_1 \times A_1$ and have trivial intersection.

(b) Let (W, S) be of type E_7 where $S = \{s_1, \ldots, s_7\}$ labelled as in [7, Planche VI]. Let $I = \{s_1, s_2, s_3, s_4, s_6, s_7\}$ and $J = \{s_1, s_2, s_3, s_5, s_6, s_7\}$. Then the non-conjugate parabolic subgroups W_J and $wW_I w^{-1}$ intersect trivially, where

$$w = s_6 s_2 s_4 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_6 s_7 s_4 s_5 s_6 s_4 s_5 s_3 s_4 s_1 s_3 s_2 s_4 s_5 s_4 s_3 s_2 s_4 s_1.$$

This was checked using [38].

For the rest of this section we work with the combinatorial realization of W as a subgroup (which we denote by W_{D_n}) of the hyperoctahedral group W_{B_n} (see Section 6). To this end, set

$$\begin{aligned} s_0 &= (1, -2)(-1, 2), \\ s_i &= (i, i+1)(-i, -(i+1)) \quad \text{for } i \in [n-1]. \end{aligned}$$

Then $\{s_0, s_1, \ldots, s_{n-1}\}$ is a simple system for a Coxeter group W_{D_n} of type D_n. The set of reflections is given by $T = \{(i, j)(-i, -j) \mid i, j \in [-n, n],\ i \neq \pm j\}$. Notice that W is a subgroup of the group W_{B_n} of type B_n; indeed, the above generators are clearly contained in W_{B_n}. Given $A \subset [-n, n]$, write $\mathrm{Stab}(A)$ for the subgroup of W_{D_n} of elements preserving the set A. Notice that since $W_{D_n} \subseteq W_{B_n}$, we have $\mathrm{Stab}(A) = \mathrm{Stab}(-A)$. The maximal standard parabolic subgroups of W_{D_n}

are then described as follows (see [6, Proposition 8.2.4]). Let $i \in \{0, 1, \ldots, n-1\}$ and $I = S \setminus \{s_i\}$. Then $W_I = \mathrm{Stab}(A_I)$, where

$$A_I := \begin{cases} [i+1, n] & \text{if } i \neq 1, \\ \{-1, 2, 3, \ldots, n\} & \text{if } i = 1. \end{cases}$$

Since W_I stabilizes both A_I and $-A_I$, it stabilizes also the complement A_I^0 of $A_I \cup (-A_I)$ in $[-n, n]$. Notice that $A_I^0 = -A_I^0$.

From this description we can easily achieve a description of maximal parabolic subgroups:

Lemma 7.3. *If $W_J = \mathrm{Stab}(A_J)$ is a maximal standard parabolic subgroup and $w \in W$, then $wW_Jw^{-1} = \mathrm{Stab}(w(A_J))$.*

Proof. Clear. □

Proof of Proposition 7.1. It is enough to show that $W_I \cap wW_Jw^{-1} \neq \{\mathrm{id}\}$ for two subsets $I, J \subseteq S$ with $|I| = |J| = n-1$ and $w \in W$. We therefore assume that $W_I \cap wW_Jw^{-1} = \{\mathrm{id}\}$ for some $I, J \subseteq S$ with $|I| = |J| = n-1$ and $w \in W$ and show that this implies that $n \leq 5$. Consider the intersections $A_I \cap w(A_J)$ and $A_I \cap (-w(A_J))$. If one of these intersections contains at least two elements, say k and l, then $(k, l)(-k, -l) \in W_I \cap wW_Jw^{-1}$ since

$$A_I \cap (-A_I) = \varnothing = w(A_J) \cap (-w(A_J)).$$

Therefore we can assume that $|A_I \cap w(A_J)| \leq 1$ and $|A_I \cap -w(A_J)| \leq 1$. Now if $|A_I| \geq 4$, then $|A_I \cap w(A_J)^0| \geq 2$, and since $A_I \cap (-A_I) = \varnothing$, it follows that there exist $k, \ell \in A_I \cap w(A_J)^0$ with $k \neq \pm\ell$, and we then have that

$$(k, \ell)(-k, -\ell) \in W_I \cap wW_Jw^{-1}.$$

Hence we can furthermore assume that $|A_I| < 4$. It follows that $|A_I^0| \geq 2n - 6$.

But arguing similarly we can also assume that $|A_I^0 \cap w(A_J)^0| < 4$, hence we have $|A_I^0 \cap w(A_J)^0| \leq 2$ since it has to be even and $|A_I^0 \cap w(A_J)| \leq 1$. It follows that $|A_I^0| \leq 4$. Together with the inequality above we get $2n - 6 \leq |A_I^0| \leq 4$, hence $n \leq 5$. □

8 The proof of Theorem 1.1

The aim of this section is to prove the sufficient condition for the transitive Hurwitz action as stated in Theorem 1.1. The proof is a case-by-case analysis.

Let (W, T) be a finite dual Coxeter system and let $w \in W$. For two elements $(t_1, \ldots, t_m), (r_1, \ldots, r_m) \in \mathrm{Red}_T(w)$ we write $(t_1, \ldots, t_m) \sim (r_1, \ldots, r_m)$ if both factorizations lie in the same Hurwitz orbit. Furthermore, note that if w is a parabolic quasi-Coxeter element in (W, T), then each conjugate of w is also a parabolic quasi-Coxeter element in (W, T). Since the Hurwitz operation commutes with conjugation, we can restrict ourselves to check transitivity for one representative of each conjugacy class of parabolic quasi-Coxeter elements of W. The proof of the following is easy:

Lemma 8.1. *Let (W_i, T_i), $i = 1, 2$, be dual Coxeter systems and let $w_i \in W_i$, $i = 1, 2$. Then $(W_1 \times W_2, T := (T_1 \times \{1\}) \cup (\{1\} \times T_2))$ is a dual Coxeter system. Furthermore, if the Hurwitz action is transitive on $\mathrm{Red}_{T_i}(w_i)$, $i = 1, 2$, then the Hurwitz action is transitive on $\mathrm{Red}_T((w_1, w_2))$.*

8.1 Types A_n, B_n and $I_2(m)$

In all these cases every parabolic quasi-Coxeter element is already a parabolic Coxeter element by Lemmas 6.3, 6.4 and Corollary 6.8. Therefore the assertion follows with [2, Theorem 1.3].

Remark 8.2. Notice that in types A_n and B_n, parabolic quasi-Coxeter elements, classical parabolic Coxeter elements and parabolic Coxeter elements coincide. In type $I_2(m)$, parabolic quasi-Coxeter elements and parabolic Coxeter elements coincide, but parabolic Coxeter elements and classical parabolic Coxeter elements do not (see Remark 2.3 (c)).

8.2 The simply laced types

We now treat the parabolic quasi-Coxeter elements in an irreducible, finite, simply laced dual Coxeter system (W, T) of rank n. As we already dealt with the type A_n, it remains to consider the types D_n, $n \geq 4$ and E_6, E_7, E_8.

We only need to show the assertion for quasi-Coxeter elements. Indeed, let w be a parabolic quasi-Coxeter element in (W, T) and let $(t_1, \ldots, t_m) \in \mathrm{Red}_T(w)$. Then $W' = \langle t_1, \ldots, t_m \rangle$ is by Theorem 6.1 a parabolic subgroup of (W, T), in fact $W' = P_w$.

Therefore, it follows from Lemma 4.2 that all the reflections in any reduced factorization of w are in W'. The latter group is a direct product of irreducible Coxeter groups of simply laced type. If we know that the Hurwitz action is transitive on $\mathrm{Red}_T(\tilde{w})$ for all the quasi-Coxeter elements $\tilde{w}$ in the irreducible Coxeter groups of simply laced type, then the Hurwitz action on $\mathrm{Red}_T(w)$ is transitive as well by Lemma 8.1.

The strategy to prove the theorem is as follows: we first show by induction on the rank n (with $n \geq 4$) that the Hurwitz action is transitive on the set of reduced decompositions of quasi-Coxeter elements of type D_n; for this we will need to use the result for parabolic subgroups, but since they are (products) of groups of type A with groups of type D of smaller rank, the result holds for groups of type A by Section 8.1 and they hold for groups of type D_k, $k < n$ by induction.

Using the fact that it holds for type D_n, $n \geq 4$, we then prove the result for the groups E_6, E_7 and E_8. Similarly as for type D_n, parabolic subgroups of type E are of type A, D or E. It was previously shown to hold for types A and D and holds for type E by induction.

We then prove by computer that the result holds for the remaining exceptional groups.

Let w be a quasi-Coxeter element and let $(t_1, \ldots, t_n) \in \mathrm{Red}_T(w)$.

8.2.1 Type D_n

For types D_4 and D_5 the assertion is checked directly using [38]. Therefore assume $n \geq 6$. Let $(r_1, \ldots, r_n) \in \mathrm{Red}_T(w)$ be a second reduced factorization of w. By Theorem 6.1 and Theorem 1.5 the groups $\langle t_1, \ldots, t_{n-1}\rangle$ and $\langle r_1, \ldots, r_{n-1}\rangle$ are maximal parabolic subgroups and since $\ell_T(wt_n) = n - 1 = \ell_T(wr_n)$, it follows that $C_W(V^{wt_n}) = P_{wt_n} = \langle t_1, \ldots, t_{n-1}\rangle$, $C_W(V^{wr_n}) = P_{wr_n} = \langle r_1, \ldots, r_{n-1}\rangle$. By Proposition 7.1 there exists a reflection t in their intersection. It follows by Lemma 4.2 that $t \leq_T wt_n, wr_n$. Hence there exists $t'_2, \ldots, t'_{n-1}, r'_2, \ldots, r'_{n-1} \in T$ such that $(t, t'_2, \ldots, t'_{n-1}) \in \mathrm{Red}_T(wt_n)$ and $(t, r'_2, \ldots, r'_{n-1}) \in \mathrm{Red}_T(wr_n)$. In particular, we get

$$(t'_2, \ldots, t'_{n-1}, t_n), (r'_2, \ldots, r'_{n-1}, r_n) \in \mathrm{Red}_T(tw).$$

By Theorem 1.5 the element tw is quasi-Coxeter in the parabolic subgroup

$$P_{tw} = \langle t'_2, \ldots, t'_{n-1}, t_n\rangle.$$

It follows from Lemma 4.2 that the reflections $r'_2, \ldots, r'_{n-1}, r_n$ are in P_{tw} since $r'_i \leq_T tw$ for each i. As P_{tw} is a direct product of irreducible Coxeter groups of type A and D of smaller rank, we have by induction together with Lemma 8.1 that

$$(t'_2, \ldots, t'_{n-1}, t_n) \sim (r'_2, \ldots, r'_{n-1}, r_n),$$

as well as

$$(t, t'_2, \ldots, t'_{n-1}) \sim (t_1, \ldots, t_{n-1}) \quad \text{and} \quad (t, r'_2, \ldots, r'_{n-1}) \sim (r_1, \ldots, r_{n-1}).$$

This implies

$$\begin{aligned}(t_1, \ldots, t_n) &\sim (t, t'_2, \ldots, t'_{n-1}, t_n) \\ &\sim (t, r'_2, \ldots, r'_{n-1}, r_n) \sim (r_1, \ldots, r_n) \in \mathrm{Red}_T(w),\end{aligned}$$

which concludes the proof.

8.2.2 Types E_6, E_7 and E_8

First we calculated representatives of the conjugacy classes of quasi-Coxeter elements using [38], see also Remark 8.3 (b) below. Then given a quasi-Coxeter element w we checked, again using [38], that there is a reduced factorization $(t_1, \dots, t_n)$ of w such that for every reflection t in T there exists an element $(t_1', \dots, t_{n-1}', t) \in \mathrm{Red}_T(w)$ with $(t_1, \dots, t_n) \sim (t_1', \dots, t_{n-1}', t)$.

Let $(r_1, \dots, r_n) \in \mathrm{Red}_T(w)$. By our computations in GAP there exists an element $(t_1', \dots, t_{n-1}', r_n) \in \mathrm{Red}_T(w)$ with $(t_1, \dots, t_n) \sim (t_1', \dots, t_{n-1}', r_n)$. Then

$$wr_n = t_1' \cdots t_{n-1}' = r_1 \cdots r_{n-1}$$

are reduced factorizations. Further, wr_n is a quasi-Coxeter element in (W', T') where $W' := \langle t_1', \dots, t_{n-1}' \rangle$ and $T' := T \cap W'$. By Theorem 1.5 we have that W' is a equal to the parabolic closure $P_{t_1' \cdots t_{n-1}'}$ of $t_1' \cdots t_{n-1}'$ and therefore $r_1, \dots, r_{n-1}$ are elements of W' by Lemma 4.2. Thus $(t_1', \dots, t_{n-1}')$ and $(r_1, \dots, r_{n-1})$ are reduced factorizations of a quasi-Coxeter element in a dual, simply laced Coxeter system of rank $n-1$. By induction and by Lemma 8.1 we get

$$(t_1', \dots, t_{n-1}') \sim (r_1, \dots, r_{n-1}),$$

thus

$$(t_1, \dots, t_n) \sim (t_1', \dots, t_{n-1}', r_n) \sim (r_1, \dots, r_n).$$

8.3 The types F_4, H_3 and H_4

For these cases we again calculated representatives of the conjugacy classes of quasi-Coxeter elements using [38] and then we checked Theorem 1.1 directly for each representative using [38].

Remark 8.3. (a) The computer programs that we used can be found at `www.math.uni-bielefeld.de/~baumeist/Dual-Coxeter/dual-Coxeter.html`.

(b) For the convenience of the reader we briefly describe Carter's classification of the conjugacy classes in finite Weyl groups by means of so-called admissible diagrams [12]. Due to [12, Lemma 8, Theorem A], we obtain the following description of conjugacy classes of quasi-Coxeter elements (in the notation of [12]):

- For the infinite families the conjugacy classes correspond to the admissible diagrams
$$A_n, B_n, D_n, D_n(a_1), D_n(a_2), \dots, D_n(a_{\lfloor \frac{1}{2}n-1 \rfloor}).$$
In particular, in types A_n and B_n the conjugacy class of the Coxeter element is the only quasi-Coxeter class (see Lemmata 6.3 and 6.4).

- For the exceptional types the conjugacy classes correspond to the admissible diagrams

$$E_6, E_6(a_1), E_6(a_2), E_7, E_7(a_1), \dots, E_7(a_4), E_8, E_8(a_1), \dots,$$
$$E_8(a_8), F_4, F_4(a_1), G_2.$$

For the remaining non-crystallographic types we found by computer:

- For the type H_3 respectively H_4 there are three respectively eleven conjugacy classes of quasi-Coxeter elements.

Note that there might be more than one admissible diagram for the same conjugacy class (e.g., the class $E_7(a_2)$ might also be parametrized by the diagram $E_7(b_2)$). For (W, T) irreducible and crystallographic, the conjugacy classes of quasi-Coxeter elements are precisely described by the connected admissible diagrams with number of nodes equal to the rank of (W, T). In [13] such a class is called *semi-Coxeter class*.

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Bibliography

[1] D. Armstrong, Generalized noncrossing partitions and combinatorics of Coxeter groups, *Mem. Amer. Math. Soc.* **202** (2009), no. 949, 1–159.

[2] B. Baumeister, M. Dyer, C. Stump and P. Wegener, A note on the transitive Hurwitz action on decompositions of parabolic Coxeter elements, *Proc. Amer. Math. Soc. Ser. B* **1** (2014), 149–154.

[3] D. Bessis, The dual braid monoid, *Ann. Sci. Éc. Norm. Supér. (4)* **36** (2003), 647–683.

[4] D. Bessis, Finite complex reflection arrangements are $K(\pi, 1)$, *Ann of Math. (2)* **181** (2015), no. 3, 809–904.

[5] D. Bessis, F. Digne and J. Michel, Springer theory in braid groups and the Birman–Ko–Lee monoid, *Pacific J. Math.* **205** (2002), 287–309.

[6] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Grad. Texts in Math. 231, Springer, New York, 2005.

[7] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4–6*, Hermann, Paris, 1968.

[8] T. Brady, A partial order on the symmetric group and new $K(\pi, 1)$'s for the braid groups, *Adv. Math.* **161** (2001), no. 1, 20–40.

[9] N. Brady, J. P. McCammond, B. Mühlherr and W. D. Neumann, Rigidity of Coxeter groups and Artin groups, *Geom. Dedicata* **94** (2002), no. 1, 91–109.

[10] T. Brady and C. Watt, $K(\pi, 1)$'s for Artin groups of finite type, *Geom. Dedicata* **94** (2002), 225–250.

[11] E. Brieskorn, Autmomorphic sets and braids and singularities, in: *Braids* (Santa Cruz 1986), Contemp. Math. 78, American Mathematical Society, Providence (1988), 45–115.

[12] R. W. Carter, Conjugacy classes in the Weyl group, *Compos. Math.* **25** (1972), 1–59.

[13] R. W. Carter and G. B. Elkington, A note on the parametrization of conjugacy classes, *J. Algebra* **20** (1972), 350–354.

[14] F. Digne and T. Gobet, Dual braid monoids, Mikado braids and positivity in Hecke algebras, *Math. Z.*, to appear.

[15] J. M. Douglass, G. Pfeiffer and G. Röhrle, On reflection subgroups of finite Coxeter groups, *Comm. Algebra* **41** (2013), no. 7, 2574–2592.

[16] M. J. Dyer, Reflection subgroups of Coxeter systems, *J. Algebra* **135** (1990), no. 1, 57–73.

[17] W. Ebeling, *Lattices and Codes*, Vieweg, Wiesbaden, 2002.

[18] W. N. Franzsen, R. B. Howlett and B. Mühlherr, Reflections in abstract Coxeter groups, *Comment. Math. Helv.* **81** (2006), no. 3, 665–697.

[19] X. Hou, Hurwitz equivalence in tuples of generalized quaternion groups and dihedral groups, *Electron. J. Combin.* **15** (2008), no. 1, Paper No. R80.

[20] A. Hubery and H. Krause, A categorification of non-crossing partitions, preprint (2013), `http://arxiv.org/abs/1310.1907`; to appear in *J. Eur. Math. Soc.*

[21] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Stud. Adv. Math. 29, Cambridge University Press, Cambridge, 1990.

[22] S. P. Humphries, Finite Hurwitz braid group actions on sequences of Euclidean reflections, *J. Algebra* **269** (2003), no. 2, 556–588.

[23] A. Hurwitz, Über Riemannsche Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **39** (1891), 1–61.

[24] K. Igusa and R. Schiffler, Exceptional sequences and clusters, *J. Algebra* **323** (2010), 2183–2202.

[25] A. James, K. Magaard and S. Shpectorov, The lift invariant distinguishes components of Hurwitz spaces for A_5, *Proc. Amer. Math. Soc.* **143** (2015), no. 4, 1377–1390.

[26] R. Kane, *Reflection Groups and Invariant Theory*, CMS Books Math./Ouvrages Math. SMC 5, Springer, New York, 2001.

[27] P. Kluitmann, *Geometrische Basen des Milnorgitters einer einfach elliptischen Singularität*, Diploma thesis, Universität Bonn, Bonn, 1983.

[28] S. V. Kulikov and M. Teicher, Braid monodromy factorizations and diffeomorphism types, *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), no. 2, 89–120.

[29] E. Liberman and M. Teicher, The Hurwitz equivalence problem is undecidable, preprint (2004), http://arxiv.org/abs/math/0511153.

[30] H. Matsumoto, Générateurs et relations des groupes de Weyl généralisés, *C. R. Math. Acad. Sci. Paris* **258** (1964), 3419–3422.

[31] J. Michel, Hurwitz action on tuples of Euclidean reflections, *J. Algebra* **295** (2006), no. 1, 289–292.

[32] K. Nuida, On the direct indecomposability of infinite irreducible Coxeter groups and the isomorphism problem of Coxeter groups, *Comm. Algebra* **34** (2006), no. 7, 2559–2595.

[33] D. Qi, A note on parabolic subgroups of a Coxeter group, *Expo. Math.* **25** (2007), no. 1, 77–81.

[34] V. Reiner, V. Ripoll and C. Stump, On non-conjugate Coxeter elements in well-generated reflection groups, preprint (2014), http://arxiv.org/abs/1404.5522.

[35] C. Sia, Hurwitz equivalence in tuples of dihedral groups, dicyclic groups, and semidihedral groups, *Electron. J. Combin.* **16** (2009), no. 1, Paper No. R95.

[36] J. Tits, *Buildings of Spherical Type and Finite Bn-Pairs*, Lecture Notes in Math. 386, Springer, Berlin, 1974.

[37] E. Voigt, Ausgezeichnete Basen von Milnorgittern einfacher Singularitäten, *Bonner Math. Schriften* **160** (1985).

[38] The GAP Group, GAP-Groups, Algorithms, and Programming, Version 4.7.7, 2015, http://www.gap-system.org.

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