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A note on Factoring groups into dense subsets

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Abstract. Let G be a group of cardinality $\kappa > \aleph_0$ endowed with a topology $\mathcal{T}$ such that $|U| = \kappa$ for every non-empty $U \in \mathcal{T}$ and $\mathcal{T}$ has a base of cardinality κ. We prove that G can be factorized $G = AB$ (i.e. each $g \in G$ has a unique representation $g = ab$, $a \in A$, $b \in B$) into dense subsets A, B, $|A| = |B| = \kappa$. We do not know if this statement holds for $\kappa = \aleph_0$ even if G is a topological group.

1 Introduction

For a cardinal κ, a topological space X is called *κ-resolvable* if X can be partitioned into κ dense subsets [1]. In the case $\kappa = 2$, these spaces were defined by Hewitt [4] as *resolvable spaces*. If X is not κ-resolvable, then X is called *κ-irresolvable.*

In topological groups, the intensive study of resolvability was initiated by the following remarkable theorem of Comfort and van Mill [2]: every countable non-discrete Abelian topological group G with finite subgroup $B(G)$ of elements of order 2 is 2-resolvable. In fact [11], every infinite Abelian group G with finite $B(G)$ can be partitioned into ω subsets dense in every non-discrete group topology on G. On the other hand, under Martin's Axiom, the countable Boolean group G, $G = B(G)$ admits a maximal (hence, 2-irresolvable) group topology [5]. Every non-discrete ω-irresolvable topological group G contains an open countable Boolean subgroup provided that G is Abelian [6] or countable [10], but the existence of a non-discrete ω-irresolvable group topology on the countable Boolean group implies that there is a P-point in ω^* (see [6]). Thus, in some models of ZFC (see [8]), every non-discrete Abelian or countable topological group is ω-resolvable. For a systematic exposition of resolvability in topological and left topological groups see [3, Chapter 13].

Recently, a new kind resolvability of groups was introduced in [7]. A group G provided with a topology $\mathcal{T}$ is called *box κ-resolvable* if there is a factorization $G = AB$ such that $|A| = \kappa$ and each subset aB is dense in $\mathcal{T}$. If G is left topological (i.e. each left shift $x \mapsto gx$, $g \in G$ is continuous), then this is equivalent to B being dense in $\mathcal{T}$. We recall that a product AB of subsets of a group G

is a *factorization* if $G = AB$ and the subsets $\{aB : a \in A\}$ are pairwise disjoint (equivalently, each $g \in G$ has a unique representation $g = ab$, $a \in A$, $b \in B$). For factorizations of groups into subsets see [9]. By [7, Theorem 1], if a topological group G contains an injective convergent sequence, then G is box ω-resolvable.

The aim of this note is to find some conditions under which an infinite group G of cardinality κ provided with a topology can be factorized into two dense subsets of cardinality κ. To this goal, we propose a new method of factorization based on filtrations of groups.

2 Theorem and question

We recall that the weight $w(X)$ of a topological space X is the minimal cardinality of bases of the topology of X.

Theorem. *Let G be an infinite group of cardinality κ, $\kappa > \aleph_0$, endowed with a topology $\mathcal{T}$ such that $w(G, \mathcal{T}) \le \kappa$ and $|U| = \kappa$ for each non-empty $U \in \mathcal{T}$. Then there is a factorization $G = AB$ into dense subsets A, B, $|A| = |B| = \kappa$.*

We do not know whether or not this Theorem is true for $\kappa = \aleph_0$ even if G is a topological group.

Question. Let G be a non-discrete countable Hausdorff topological group G of countable weight. Can G be factorized $G = AB$ into two countable dense subsets?

In Section 4, we give a positive answer in the following cases: each finitely generated subgroup of G is nowhere dense, the set $\{x^2 : x \in U\}$ is infinite for each non-empty open subset of G, G is Abelian.

3 Proof

We begin with some general constructions of factorizations of a group G via filtrations of G.

Let G be a group with the identity e. Let κ be a cardinal. A family $\{G_\alpha : \alpha < \kappa\}$ of subgroups of G is called a *filtration* if

(1) $G_0 = \{e\}$, $G = \bigcup_{\alpha<\kappa} G_\alpha$,

(2) $G\alpha \subset G_\beta$ for all $\alpha < \beta$,

(3) $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ for every limit ordinal β.

Every ordinal $\alpha < \kappa$ has the unique representation $\alpha = \gamma(\alpha) + n(\alpha)$, where $\gamma(\alpha)$ is either a limit ordinal or 0 and $n(\alpha) \in \omega$, $\omega = \{0, 1, \dots\}$. We partition κ into two subsets

$$E(\kappa) = \{\alpha < \kappa : n(\alpha) \text{ is even}\}$$

and

$$O(\kappa) = \{\alpha < \kappa : n(\alpha) \text{ is odd}\}.$$

For each $\alpha \in E(\kappa)$, we choose some system L_α of representatives of left cosets of $G_{\alpha+1} \setminus G_\alpha$ by G_α so $G_{\alpha+1} \setminus G_\alpha = L_\alpha G_\alpha$. For each $\alpha \in O(\kappa)$, we choose some system R_α of representatives of right cosets of $G_{\alpha+1} \setminus G_\alpha$ by G_α so we have $G_{\alpha+1} \setminus G_\alpha = G_\alpha R_\alpha$.

We take an arbitrary element $g \in G \setminus \{e\}$ and choose the smallest subgroup G_γ such that $g \in G_\gamma$. By (3), $\gamma = \alpha(g)+1$ so $g \in G_{\alpha(g)+1} \setminus G_{\alpha(g)}$. If $\alpha(g) \in E(\kappa)$, we choose $x_0(g) \in L_\alpha(g)$ and $g_0 \in G_\alpha(g)$ such that $g = x_0(g)g_0$. If $\alpha(g) \in O(\kappa)$, we choose $y_0(g) \in R_\alpha(g)$ and $g_0 \in G_\alpha(g)$ such that $g = g_0 y_0(g)$. If $g_0 = e$, we stop. Otherwise we repeat the argument for g_0 and so on. Since the set of ordinals less than κ is well ordered, after a finite number of steps we get the representation

$$g = x_0(g)x_1(g) \dots x_{\lambda(g)}(g) y_{\rho(g)} \dots y_1(g) y_0(g), \tag{4}$$

where

$$x_i \in L_{\alpha_i(g)}, \quad \alpha_0(g) > \alpha_1(g) > \cdots > \alpha_{\lambda(g)(g)},$$
$$y_i \in R_{\beta_i(g)}, \quad \beta_0(g) > \beta_1(g) > \cdots > \beta_{\rho(g)(g)}.$$

If either $\{\alpha_0(g), \dots, \alpha_{\lambda(g)}(g)\} = \varnothing$ or $\{\beta_0(g), \dots, \beta_{\rho(g)}(g)\} = \varnothing$, then we write $g = y_{\rho(g)} \dots y_1(g) y_0(g)$ or $g = x_0(g)x_1(g) \dots x_{\lambda(g)}(g)$. Thus, $G = AB$ where A is the set of all elements of the form $x_0(g)x_1(g) \dots x_{\lambda(g)}$ and B is the set of all elements of the form $y_{\rho(g)} \dots y_1(g) y_0(g)$. To show that the product AB is a factorization of G, we assume that, besides (4), g has a representation

$$g = z_0 z_1 \dots z_\lambda t_\rho \dots t_1 t_0.$$

If $g \in G_{\alpha+1} \setminus G_\alpha$ and $\alpha \in O(\kappa)$, then $z_0 z_1 \dots z_\lambda t_\rho \dots t_1 \in G_\alpha$ so $t_0 = y_0(g)$. If $\alpha \in E(\kappa)$, then $z_1 \dots z_\lambda t_\rho \dots t_1 t_0 \in G_\alpha$ so $z_0 = x_0(g)$. We replace g by $g t_0^{-1}$ or by $z_0^{-1} g$ respectively and repeat the same arguments.

Now we are ready to prove the Theorem. Let $\{U_\alpha : \alpha < \kappa\}$ be a κ-sequence of non-empty open sets such that each non-empty $U \in \mathcal{T}$ contains some U_α. Since $|U_\alpha| = \kappa$ for every $\alpha < \kappa$, we can construct inductively a filtration $\{G_\alpha : \alpha < \kappa\}$, $|G_\alpha| = \max\{\aleph_0, |\alpha|\}$ such that for each $\alpha \in E(\kappa)$ (resp. $\alpha \in O(\kappa)$) there is a system L_α (resp. R_α) of representatives of left (resp. right) cosets of $G_{\alpha+1} \setminus G_\alpha$ by G_α such that $L_\alpha \cap U_\gamma \neq \varnothing$ (resp. $R_\alpha \cap U_\gamma \neq \varnothing$) for each $\gamma \le \alpha$. Then the subsets A, B of the above factorization of G are dense in $\mathcal{T}$ because $L_\alpha \subset A$, $R_\beta \subset B$ for each $\alpha \in E(\kappa)$, $\beta \in O(\kappa)$.

4 Comments

1. Analyzing the proof, we see that the Theorem holds under the weaker condition: G has a family $\mathcal{F}$ of subsets such that $|\mathcal{F}| = \kappa$, $|F| = \kappa$ for each $F \in \mathcal{F}$ and, for every non-empty $U \in \mathcal{T}$, there is $F \in \mathcal{F}$ such that $F \subseteq U$.

If $\kappa = \aleph_0$ but each finitely generating subgroup of G is nowhere dense, we can choose a family $\{G_n : n \in \omega\}$ such that the corresponding A, B are dense. Thus, we get a positive answer to the Question if each finitely generated subgroup H of G is nowhere dense (equivalently the closure of H is not open).

2. Let G be a group and let A, B be subsets of G. We say that the product AB is a *partial factorization* if the subsets $\{aB : a \in A\}$ are pairwise disjoint (equivalently, $\{Ab : b \in B\}$ are pairwise disjoint).

We assume that AB is a partial factorization of G into finite subsets and that X is an infinite subset of G. Then the following statements are easily verified

(5) there is $x \in X$ such that $x \notin B$ and $A(B \cup \{x\})$ is a partial factorization;

(6) if the set $\{x^2 : x \in X\}$ is infinite, then there is an element $x \in X$ such that $(A \cup \{x, x^{-1}\})B$ is a partial factorization.

3. Let G be a non-discrete Hausdorff topological group, let AB be a partial factorization of G into finite subsets, $A = A^{-1}$, $e \in A \cap B$ and $g \notin AB$. Then

(7) there is a neighbourhood V of e such that, for $U = V \setminus \{e\}$ and for any $x \in U$, the product $(A \cup \{x, x^{-1}\})(B \cup \{x^{-1}g\})$ is a partial factorization (so $g \in (A \cup \{x, x^{-1}\})(B \cup \{x^{-1}g\})$).

It suffices to choose V so that $V = V^{-1}$ and

$$AUg \cap AB = \varnothing, \quad UB \cap (AB \cup AUg) = \varnothing, \quad U^2 g \cap AB = \varnothing, \quad U \cap A = \varnothing.$$

We use $A = A^{-1}$ only in $UB \cap AUg = \varnothing$.

4. Let G be countable non-discrete Hausdorff topological group such that the set $\{x^2 : x \in U\}$ is infinite for every non-empty open subset U of G. We enumerate $G = \{g_n : n \in \omega\}$, $g_0 = e$ and choose a countable base $\{U_n : n \in \omega\}$ for non-empty open sets. We put $A_0 = \{e\}$, $B_0 = \{e\}$ and use (5), (6), (7) to choose inductively two sequences $(A_n)_{n\in\omega}$ and $(B_n)_{n\in\omega}$ of finite subsets of G such that for every $n \in \omega$, $A_n \subset A_{n+1}$, $B_n \subseteq B_{n+1}$, $A_n = A_n^{-1}$, $A_n B_n$ is a partial factorization, $g_n \in A_n B_n$, $A_n \cap U_n \neq \varnothing$, $B_n \cap U_n \neq \varnothing$. We put

$$A = \bigcup_{n\in\omega} A_n, \quad B = \bigcup_{n\in\omega} B_n$$

and note that AB is a factorization of G into dense subsets.

5. Let G be a countable Abelian non-discrete Hausdorff topological group of countable weight. We suppose that G contains a non-discrete finitely generated subgroup H. Given any non-empty open subset U of G, we choose a neighborhood X of e in H and $g \in S$ such that $Xg \subset U$. Since H is finitely generated, the set $\{x^2 : x \in X\}$ is infinite so we can apply comment 4. If each finitely generated subgroup of G is discrete then, to answer the Question, we use comment 1.

6. Let G be a countable group endowed with a topology $\mathcal{T}$ of countable weight such that U is infinite for every $U \in \mathcal{T}$. Applying the inductive construction from comment 5 to $A_n B_n$ and $B_{n+1}^{-1} A_n^{-1}$, we get a partial factorization of G into two dense subsets.

7. Let G be a group satisfying the assumption of the Theorem and let γ be an infinite cardinal, $\gamma < \kappa$. We take a subgroup A of cardinality γ and choose inductively a dense set B of representatives of right cosets of G by A. Then we get a factorization $G = AB$. In particular, if G is left topological, then G is box γ-resolvable.

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