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# On projectively Krylov transitive and projectively weakly transitive Abelian $p$-groups

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**Abstract.** We define the concepts of *projectively Krylov transitive* and *strongly projectively Krylov transitive* Abelian $p$-groups. These concepts are non-trivial expansions of the projectively fully transitive and strongly projectively fully transitive Abelian $p$-groups, respectively, defined by the second author and Goldsmith in [13]. Some results concerning them and various other notions of transitivity are established as well, thus also generalizing achievements of Danchev–Goldsmith from [11]. Certain relationships between these classes are also obtained.

## 1 Introduction and main definitions

Throughout this paper, all groups under consideration are *additive $p$-torsion Abelian groups*, where $p$ is a fixed prime integer. All basic notions and notation are standard and follow those from [13, 16–18, 23, 24]. All other ones, not explicitly defined herein, are explicitly stated below.

For instance, for any group $G$, the symbol $\mathrm{E}(G)$ denotes the ring of all endomorphisms of $G$ with additive group $\mathrm{End}(G)$ and multiplicative group of units $\mathrm{Aut}(G)$ consisting of all automorphisms in $\mathrm{E}(G)$. Likewise, $\mathrm{Proj}(G)$ denotes the subring of $\mathrm{E}(G)$ generated by all projections in $\mathrm{E}(G)$ and $\Pi(G)$ is its additive group. Thus $\mathrm{Proj}(G) \leq \mathrm{E}(G)$ and $\Pi(G) \leq \mathrm{End}(G) \cap \mathrm{Proj}(G)$.

An endomorphism $\varphi : G \to G$ of the group $G$ is said to be a *projection* or, equivalently, an *idempotent endomorphism* if $\varphi^2 = \varphi$. These projection endomorphisms have been of great interest and importance and were studied very intensively by some authors (see, e.g., [2, 7, 12, 13, 20, 25]). They allow us to define numerous new concepts which play an important role in Abelian group theory. Some other valuable sources in that subject, the reader can see respectively in [1, 4, 5, 9, 10, 21].

The motivation of writing this article is to substantially generalize almost all of the results from [13] to the class of (projectively) Krylov transitive groups and (projectively) weakly transitive groups. In doing that, we organize the work as follows: In the first section, we state all new terminology and concepts used in the further work. In the second section, we provide some preliminary assertions and also give a series of non-elementary constructions that show that the notions defined in Section 1 are independent of each other. We proceed in the third section with the main results and their consequences. And finally, we close the work in the fourth section with a series of significant problems that remain unanswered yet.

Our major machinery is the following:

**Definition 1.** *A group $G$ is said to be projectively Krylov transitive if, given elements $x, y \in G$ with $U_G(x) = U_G(y)$, there exists $\phi \in \mathrm{Proj}(G)$ with $\phi(x) = y$.*

Note also that by symmetry there is $\psi \in \mathrm{Proj}(G)$ with $\psi(y) = x$.

**Definition 2.** *A group $G$ is said to be strongly projectively Krylov transitive if, given $x, y \in G$ with $U_G(x) = U_G(y)$, there exists $\phi \in \Pi(G)$ with $\phi(x) = y$.*

Note also that by symmetry there is $\psi \in \Pi(G)$ with $\psi(y) = x$.

**Definition 3.** *A group $G$ is called projectively transitive if, given $x, y \in G$ with $U_G(x) = U_G(y)$, there exists $\phi \in \mathrm{Aut}(G) \cap \mathrm{Proj}(G)$ with $\phi(x) = y$.*

Note also that by symmetry there is $\psi \in \mathrm{Aut}(G) \cap \mathrm{Proj}(G)$ with $\psi(y) = x$. However, is it true that $\psi^{-1} \in \mathrm{Aut}(G) \cap \mathrm{Proj}(G)$?

**Definition 4.** *A group $G$ is called strongly projectively transitive if, given $x, y \in G$ with $U_G(x) = U_G(y)$, there exists $\phi \in \mathrm{Aut}(G) \cap \Pi(G)$ with $\phi(x) = y$.*

Note also that by symmetry there is $\psi \in \mathrm{Aut}(G) \cap \Pi(G)$ with $\psi(y) = x$. However, is it true that $\psi^{-1} \in \mathrm{Aut}(G) \cap \Pi(G)$?

**Definition 5.** *We say that the group $G$ is projectively weakly transitive if, for any elements $x, y \in G$ and $\phi, \psi \in \mathrm{Proj}(G)$ with $\phi(x) = y$, $\psi(y) = x$, there exists $\theta \in \mathrm{Aut}(G) \cap \mathrm{Proj}(G)$ with $\theta(x) = y$.*

Note also that by symmetry there is $\eta \in \mathrm{Aut}(G) \cap \mathrm{Proj}(G)$ with $\eta(y) = x$. However, is it true that $\eta^{-1} \in \mathrm{Aut}(G) \cap \mathrm{Proj}(G)$?

**Definition 6.** *We say that the group $G$ is strongly projectively weakly transitive if, for any elements $x, y \in G$ and $\phi, \psi \in \Pi(G)$ with $\phi(x) = y, \psi(y) = x$, there exists $\theta \in \mathrm{Aut}(G) \cap \Pi(G)$ with $\theta(x) = y$.*

Note also that by symmetry there is $\eta \in \mathrm{Aut}(G) \cap \Pi(G)$ with $\eta(y) = x$. However, is it true that $\eta^{-1} \in \mathrm{Aut}(G) \cap \Pi(G)$?

Evidently, any (strongly) projectively transitive group is both (strongly) projectively weakly transitive and (strongly) projectively Krylov transitive, and conversely. Also, if the above maps $\psi, \eta$ are both involutions, i.e., $\psi^2 = 1 = \eta^2$, then these four questions hold in the affirmative.

It is worth noting that some of the results below are announced and reported in [8].

## 2 Preliminaries and examples

We start here with a reminder of some old results related to our current context.

**Proposition 2.1.** *If $G$ is a $p$-group with $p \neq 2$, then $G$ is projectively Krylov transitive if, and only if, $G$ is projectively fully transitive.*

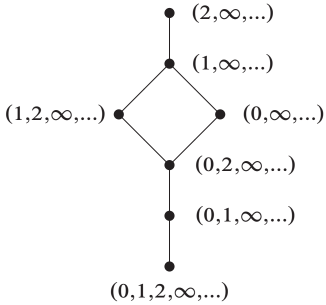
*Proof.* The sufficiency being elementary, we concentrate on the necessity. Reworking the proof of [19, Proposition 2.3], we observe that the same arguments work. □

So, the question of when a $p$-group is proper projectively Krylov transitive reduces to the case $p = 2$. This motivates us to ask whether or not the concepts of projective full transitivity and Krylov full transitivity coincide in general. However, the following example manifestly shows that they are independent notions. Note that Corner constructed in [6] a transitive (and hence Krylov transitive) 2-group $G$ which is not fully transitive such that $2^{\omega}G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8$, but his construction is not directly applicable in our situation and thus a more detailed exhibition is determinably needed here.

**Example 2.2.** There exists a projectively Krylov transitive 2-group $G$ which is not projectively fully transitive and such that $2^{\omega}G$ is a direct sum of three cyclic groups.

*Proof.* Let $H = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$, where $o(a) = 2$, $o(b) = 4$, $o(c) = 8$. Whence $|H| = 64$.

The lattice diagram below shows the relationships between the Ulm sequences of the elements of the group $H$.



Define the projections $\varphi a = a + 2b + 4c$, $\varphi b = a + 2c$, $\varphi c = a + b + 7c$; $\psi a = a$, $\psi b = 0$, $\psi c = c$; $\alpha a = 0$, $\alpha b = a + b$, $\alpha c = 0$. Set $\Phi$ to be the subring of $\mathrm{E}(H)$ generated by $I, \varphi, \psi, \alpha$. Furthermore,

$$c \xrightarrow{3I} 3c \xrightarrow{7I} 5c \xrightarrow{3I} 7c \xrightarrow{\varphi-\psi+I} a + c \xrightarrow{3I} a + 3c \xrightarrow{7I} a + 5c \xrightarrow{3I} a + 7c$$

$$\xrightarrow{5\varphi} b + c \xrightarrow{2I+2\alpha+\varphi} b + 3c \xrightarrow{5\varphi-2I} b + 5c \xrightarrow{3I-2\alpha} b + 7c$$

$$\xrightarrow{\psi\alpha\varphi+\psi-2I} a + b + c \xrightarrow{3I-2\alpha} a + b + 3c \xrightarrow{7I-2\alpha} a + b + 5c$$

$$\xrightarrow{3I-2\alpha} a + b + 7c \xrightarrow{5\alpha-\psi-2\psi\alpha} 2b + c \xrightarrow{3I} 2b + 3c \xrightarrow{7I} 2b + 5c$$

$$\xrightarrow{3I} 2b + 7c \xrightarrow{3(\psi-\varphi-I)} a + 2b + c \xrightarrow{3I} a + 2b + 3c \xrightarrow{7I} a + 2b + 5c$$

$$\xrightarrow{3I} a + 2b + 7c \xrightarrow{\varphi+\alpha} 3b + c \xrightarrow{3\psi} 3b + 3c \xrightarrow{7I+2\alpha} 3b + 5c \xrightarrow{3I+2\alpha} 3b + 7c$$

$$\xrightarrow{7I+7\psi\alpha+2\alpha} a + 3b + c \xrightarrow{3\psi} a + 3b + 3c \xrightarrow{7\psi+2\alpha} a + 3b + 5c$$

$$\xrightarrow{3I+2\alpha} a + 3b + 7c \xrightarrow{7(I-\alpha)} c$$

and thus all elements of the Ulm sequence $(0, 1, 2, \infty, \ldots)$ are translated to elements from $\Phi$.

Moreover,

$$2c \xrightarrow{\varphi} 2b + 6c \xrightarrow{I-\alpha} 6c \xrightarrow{\varphi} 2b + 2c \xrightarrow{I-\alpha} 2c$$

are elements of the Ulm sequence $(1, 2, \infty, \ldots)$;

$$b \xrightarrow{3I} 3b \xrightarrow{3\alpha} a + b \xrightarrow{3I} a + 3b \xrightarrow{I+\varphi} a + b + 2c \xrightarrow{2I+3\alpha} a + b + 4c$$

$$\xrightarrow{\alpha+\varphi\alpha+\alpha\varphi\alpha} a + b + 6c \xrightarrow{\psi+3\alpha} b + 6c \xrightarrow{3I-\alpha\varphi\alpha} b + 2c \xrightarrow{3(I+\psi)} b + 4c$$

$$\xrightarrow{\varphi+3I} a + 3b + 2c \xrightarrow{I+2\psi} a + 3b + 6c \xrightarrow{2I-\alpha} a + 3b + 4c$$

$$\xrightarrow{3(\varphi\alpha+I)+\psi\alpha} 3b + 2c \xrightarrow{3(I+\psi)} 3b + 4c \xrightarrow{3\varphi+\alpha} 3b + 6c \xrightarrow{I-\psi} b$$

are elements of the Ulm sequence $(0, 1, \infty, \ldots)$;

$$2b \xrightarrow{I+\varphi} 2b + 4c \xrightarrow{\alpha} 2b$$

are elements of the Ulm sequence $(1, \infty, \ldots)$;

$$a \xrightarrow{I+\alpha\varphi} a + 2b \xrightarrow{I-\varphi\psi+\psi} a + 4c \xrightarrow{I+\alpha\varphi} a + 2b + 4c \xrightarrow{I+\alpha+\varphi\alpha} a$$

are elements of the Ulm sequence $(0, \infty, \ldots)$;

$$a + 2c \xrightarrow{3I} a + 6c \xrightarrow{\psi} a + 2b + 6c \xrightarrow{3I} a + 2b + 2c \xrightarrow{I+\alpha} a + 2c$$

are elements of the Ulm sequence $(0, 2, \infty, \ldots)$ and $4c \xrightarrow{I} 4c$. One sees that all elements of equal Ulm sequences are sent to elements from $\Phi$.

Taking into account the action on the element $a + 2c$, we obtain

$$\varphi = I, \quad \alpha = \alpha\varphi = \psi\alpha\psi = \varphi\alpha\psi = 0, \quad \psi\varphi\psi = \varphi\psi + \alpha\psi.$$

Consequently, $\Phi(a + 2c) = \Psi(a + 2c)$, where $\Psi$ is the subring of $\Phi$, generated by $I, \psi, \alpha\psi, \varphi\psi$. In view of the above equalities it is routinely observed that $\Psi = \{xI + y\psi + z(\alpha\psi) + t(\varphi\psi) : 0 \leqslant x, y, z, t \leqslant 8\}$. In fact, dropping off the coefficients in the corresponding endomorphisms, we deduce that

$$\begin{aligned}\Phi = \{I &+ \varphi + \psi + \alpha + \varphi\psi + \psi\varphi + \varphi\alpha + \alpha\varphi + \psi\alpha + \alpha\psi + \varphi\psi\varphi \\ &+ \psi\varphi\psi + \varphi\alpha\varphi + \alpha\varphi\alpha + \psi\alpha\psi + \alpha\psi\alpha + \varphi\psi\alpha + \alpha\psi\varphi + \psi\varphi\alpha \\ &+ \psi\alpha\varphi + \alpha\varphi\psi + \varphi\alpha\psi + \cdots\};\end{aligned}$$

note that in the $\cdots$ space, we have products consisting of four elements, etc. Again considering the action on the element $a + 2c$, one can simply infer that

$$\begin{aligned}&\varphi\psi\varphi = \varphi\psi, \quad \psi\varphi\psi = 3\psi, \quad \varphi\alpha\varphi = \alpha\varphi\alpha = \psi\alpha\psi = \alpha\psi\alpha = \varphi\psi\alpha = 0, \\ &\alpha\psi\varphi = \alpha\psi, \quad \psi\varphi\alpha = \psi\alpha\varphi = \alpha\varphi\psi = 0, \quad \varphi\alpha\psi = 2\varphi,\end{aligned}$$

which substantiates our claim that

$$\Psi = \{xI + y\psi + z(\alpha\psi) + t(\varphi\psi) : 0 \leqslant x, y, z, t \leqslant 8\}.$$

If now $(xI + y\psi + z(\alpha\psi) + t(\varphi\psi))(a + 2c) = a$, then we conclude that

$$(x + y + t)a = a, \quad 2(y + z)b = 0, \quad 2(x + y + 3t)c = 0,$$

whence $x + y + t = 2n + 1$, $y + z = 2m$, $x + y + 3t = 4k$ for some integers $n, m, k$ which leads to the contradiction $2(n+t)+1 = 4k$. Hence $a \notin \Psi(a+2c)$. However, $(0, 2, \infty, \ldots) = U(a + 2c) < U(a) = (0, \infty, \ldots)$, which guarantees that if $G$ is a 2-group such that $2^\omega G = H$ and $E(G) \restriction H = \Phi$, then $G$ is projectively Krylov transitive but not projectively fully transitive, as wanted. □

**Remark.** It is worth noting that in the case when $p^\omega G$ is a direct sum of two cyclic groups the situation is totally different. To demonstrate this, we first need the following technicality.

**Lemma 2.3.** *Let $A = \bigoplus_{i \in I} A_i$ be a group with projections $\pi_i \colon A \to A_i$, $R$ is a ring with* 1 *and all $\pi_i \in R$. Then:*

(1) *if $R$ acts Krylov transitively on $A$ and fully transitively on $A_i$ for each $i \in I$, then $R$ acts fully transitively on $A$,*

(2) *if $R$ acts projectively Krylov transitively on $A$ and projectively fully transitively on $A_i$ for each $i \in I$, then $R$ acts projectively fully transitively on $A$.*

*Proof.* (1) It suffices to show the result for the direct summand $\bigoplus_{j \in J} A_j$ of $A$, where $J \subseteq I$ is a finite set, and thus by induction for the direct sum of only two groups. Let $U_G(a_1 + a_2) \leqslant U_G(b_1 + b_2)$. According to [19, Lemma 2.2] it is enough to consider that $b_1 + b_2 \in A[p]$. Since $\mathrm{ht}(a_1 + a_2) = \min\{\mathrm{ht}(a_1), \mathrm{ht}(a_2)\}$, we may with no loss of generality assume that $\mathrm{ht}(a_1 + a_2) = \mathrm{ht}(a_1)$, whence $\mathrm{ht}(a_1) \leqslant \mathrm{ht}(b_1 + b_2) \leqslant \mathrm{ht}(b_1)$. Since $b_1 \in A_1[p]$, we have $U_{A_1}(a_1) \leqslant U_{A_1}(b_1)$. Write $\varphi(a_1) = b_1$ for some $\varphi \in R$. Observing that $U_{A_1}(a_1) \leqslant U_{A_2}(b_2)$, we deduce $U_G(a_1 + b_2) = U_G(a_1)$. So, $\alpha(a_1) = a_1 + b_2$ and $(\alpha - I)(a_1) = b_2$, where $I$ is the identity of $A$ (a unit of $R$). Therefore, $\pi_1(a_1 + a_2) = a_1$, $(\varphi + (\alpha - I))\pi_1 \in R$ and $(\varphi + (\alpha - I))\pi_1(a_1 + a_2) = \varphi(a_1) + (\alpha - I)a_1 = b_1 + b_2$.

The proof of (2) is similar. □

We are now ready to illustrate the following somewhat surprising statement. In [6] a construction was given of a transitive (whence Krylov transitive) 2-group $G$ such that $2^{\omega}G$ is a direct sum of two cyclic groups of orders 2 and 8, respectively, and such that $G$ is not fully transitive. The next assertion contrasts this.

**Corollary 2.4.** *Any projectively Krylov transitive group $G$ for which $p^{\omega}G$ is a direct sum of two cyclic groups is projectively fully transitive.*

*Proof.* The action of $\mathrm{E}(G)$ on $p^{\omega}G$ contains at least one non-trivial projection $\pi$. Then $I - \pi$ is an orthogonal projection to $\pi$, so $p^{\omega}G = \pi(p^{\omega}G) \oplus (I - \pi)(p^{\omega}G)$, where each summand is cyclic. The identity map of any cyclic group acts projectively fully transitively. So, the result follows from Lemma 2.3. □

It is worth noting that by [13, Corollary 4.2], any separable group and any group $G$ with cyclic subgroup $p^{\omega}G$ are both strongly projectively fully transitive.

**Example 2.5.** For every prime number $p$ there exist strongly projectively fully transitive $p$-groups $G$ with the property $p^{\omega}G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.

*Proof.* Let $H = \langle a \rangle \oplus \langle b \rangle$, where $o(a) = o(b) = p$. We consider two cases.

**Case (1): $p \geqslant 3$.** Define the idempotents $\varphi$ and $\psi$ of $H$ by $\varphi(a) = a - b$, $\varphi(b) = 0$ and $\psi(a) = 0$, $\psi(b) = a + b$. Let $G$ be a group with $p^{\omega}G \cong H$ and $\mathrm{E}(G) \restriction p^{\omega}G = \Phi$, where $\Phi$ is a subring of $\mathrm{E}(H)$, generated by $I, \varphi, \psi$ and $I$ is the identity on $H$. What remains to show is that for all nonzero elements $xa + yb$ and $\alpha a + \beta b$ in $H$ there exists $\phi \in \Pi(\Phi)$ with the property that

$$\phi(xa + yb) = \alpha a + \beta b.$$

If $x = 0$ and $y \neq 0$, then

$$[\alpha y^{-1}(\psi - I) + \beta y^{-1} I](yb) = \alpha a + \beta b;$$

if $y = 0$ and $x \neq 0$, then

$$[\alpha x^{-1} I + \beta x^{-1}(I - \varphi)](xa) = \alpha a + \beta b.$$

So, let $x, y \neq 0$. Since $(kI + m\varphi + n\psi)(xa + yb) = \alpha a + \beta b$, we derive

$$(kx + mx + ny)a + (ky - mx + ny)b = \alpha a + \beta b$$

or

$$kx + mx + ny = \alpha, \quad ky - mx + ny = \beta.$$

From the second equation we solve $k = \frac{\beta + mx - ny}{y}$. Substituting $k$ in the first equation, we obtain $-(xy + x^2)m + (y^2 - xy)n = \alpha y - \beta x$. The coefficients of $m$ and $n$ cannot simultaneously be zero, because if $x(y + x) = 0$ and $y(y - x) = 0$, then $2x = 2y = 0$. Since $p \neq 2$, we conclude $x = y = 0$, which contradicts our assumption. Consequently, if, for instance, $xy + x^2 \neq 0$, then

$$m = \frac{\beta x - \alpha y + (y^2 - xy)n}{xy + x^2}.$$

Since $n = 0, \ldots, p - 1$, we can find such a number $m$, which means that

$$\phi = kI + m\varphi + n\psi.$$

**Case (2): $p = 2$.** Write $H = \{0, a, b, a + b\}$ and if $\varphi(a) = b$, $\varphi(b) = b$, $\psi(a) = a$, $\psi(b) = a$, $\theta(a) = a$, $\theta(b) = 0$ and $\Phi$ is a subring of $\mathrm{E}(H)$ generated by $I, \varphi, \psi, \theta$. Thus we have $I(a) = a$, $\varphi(a) = b$, $(I + \varphi)(a) = a + b$; $\psi(b) = a$, $I(b) = b$, $(I + \psi)b = a + b$; $\theta(a + b) = a$, $(I + \theta)(a + b) = b$, $I(a + b) = a + b$. It is now easily seen that $G$ is a 2-group with $2^\omega G \cong H$ and $\mathrm{E}(G) \restriction 2^\omega G = \Phi$. Hence in both cases $G$ is a strongly projectively fully transitive group, as promised. □

The next example sheds some more concrete light on the case of 2-groups.

**Example 2.6.** There exists a 2-group $G$, which is both transitive and fully transitive but which is neither projectively transitive nor projectively fully transitive, such that $2^\omega G$ is a homogeneous group of rank 4.

*Proof.* Let $H = \langle a_1\rangle \oplus \langle a_2\rangle \oplus \langle a_3\rangle \oplus \langle a_4\rangle$, where $o(a_j) = 2$ with $j = 1, 2, 3, 4$. Define an endomorphism $\varphi$ of the group $H$ as follows: $\varphi a_1 = a_2$, $\varphi a_2 = a_3$, $\varphi a_3 = a_4$ and $\varphi a_4 = a_1 + a_2 + a_3 + a_4$. Then

$$\begin{cases} \varphi^2 a_1 = a_3, \\ \varphi^2 a_2 = a_4, \\ \varphi^2 a_3 = a_1 + a_2 + a_3 + a_4, \\ \varphi^2 a_4 = a_1, \end{cases} \qquad \begin{cases} \varphi^3 a_1 = a_4, \\ \varphi^3 a_2 = a_1 + a_2 + a_3 + a_4, \\ \varphi^3 a_3 = a_1, \\ \varphi^3 a_4 = a_2, \end{cases}$$

$$\begin{cases} \varphi^4 a_1 = a_1 + a_2 + a_3 + a_4, \\ \varphi^4 a_2 = a_1, \\ \varphi^4 a_3 = a_2, \\ \varphi^4 a_4 = a_3. \end{cases}$$

Thus $\varphi^4 = I + \varphi + \varphi^2 + \varphi^3$, where $I$ is the identity on $H$. Let $G$ be a 2-group with $2^\omega G \cong H$ and $E(G) \upharpoonright 2^\omega G = \Phi$, where $\Phi$ is a subring of $E(H)$, generated by $I, \varphi$. Therefore, $\Phi = \{y_0 I + y_1\varphi + y_2\varphi^2 + y_3\varphi^3 : 0 \leqslant y_i \leqslant 1,\ i = 0, 1, 2, 3\}$. It is not hard to check that if $y_0 I + y_1\varphi + y_2\varphi^2 + y_3\varphi^3 = 0$, then $y_i = 0$. It follows then that $(x_1 I + x_2\varphi + x_3\varphi^2 + x_4\varphi^3)a_1 = x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4$. To prove that $G$ is fully transitive, it suffices to show that each nonzero element $x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4$ will be mapped to the element $a_1$ by the endomorphism $\phi = y_0 I + y_1\varphi + y_2\varphi^2 + y_3\varphi^3$. In fact,

$$\begin{aligned} &\phi(x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4) \\ &\quad = (y_0x_1 + y_1x_4 + y_2(x_3 + x_4) + y_3(x_2 + x_3))a_1 \\ &\qquad + (y_0x_2 + y_1(x_1 + x_4) + y_2x_3 + y_3(x_2 + x_4))a_2 \\ &\qquad + (y_0x_3 + y_1(x_2 + x_4) + y_2(x_1 + x_3) + y_3x_2)a_3 \\ &\qquad + (y_0x_4 + y_1(x_3 + x_4) + y_2(x_2 + x_3) + y_3(x_1 + x_2))a_4 = a_1, \end{aligned}$$

as desired. Furthermore, we obtain a system of linear equations in the variables $y_i$, where $i = 0, 1, 2, 3$:

$$\begin{cases} x_1y_0 + x_4y_1 + (x_3 + x_4)y_2 + (x_2 + x_3)y_3 = 1, \\ x_2y_0 + (x_1 + x_4)y_1 + x_3y_2 + (x_2 + x_4)y_3 = 0, \\ x_3y_0 + (x_2 + x_4)y_1 + (x_1 + x_3)y_2 + x_2y_3 = 0, \\ x_4y_0 + (x_3 + x_4)y_1 + (x_2 + x_3)y_2 + (x_1 + x_2)y_3 = 0. \end{cases}$$

Any choice of the coefficients $x_1, x_2, x_3, x_4$ of this system corresponds to one of the sixteen elements $x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4$ of the group $H$.

It is also just a routine technical matter to verify that the determinant

$$\begin{vmatrix} x_1 & x_4 & x_3+x_4 & x_2+x_3 \\ x_2 & x_1+x_4 & x_3 & x_2+x_4 \\ x_3 & x_2+x_4 & x_1+x_3 & x_2 \\ x_4 & x_3+x_4 & x_2+x_3 & x_1+x_2 \end{vmatrix}$$

of that system is manifestly nonzero at least for one nonzero $x_j$. So, the given above system can be successfully solved for these $x_j$ and, by what we have just shown above, this means that $G$ is fully transitive, indeed, as wanted.

Moreover, one sees that $\ker\phi = 0$ for every $\phi \neq 0$, i.e., all nonzero endomorphisms of the group $H$ are automorphisms. This also leads us to the fact that $G$ is transitive.

Now we assume that $\phi^2 = \phi$. Consequently,

$$\phi^2 = (y_0^2 + y_2^2)I + (y_2^2 + y_3^2)\varphi + (y_1^2 + y_2^2)\varphi^2 + y_2^2\varphi^3.$$

Hence

$$\begin{cases} y_0^2 + y_2^2 = y_0 \\ y_2^2 + y_3^2 = y_1 \\ y_1^2 + y_2^2 = y_2 \\ y_2^2 = y_3. \end{cases}$$

Taking into account that $y_0^2 = y_0$ and $y_2^2 = y_2$, the only nonzero solutions of that system are the following: $y_0 = 1$, $y_1 = y_2 = y_3 = 0$. That is why the only nonzero idempotent of the ring $\Phi$ is its identity $I$, so that the group $G$ is neither projectively fully transitive nor projectively transitive, as asserted. □

Recall that the *n-th Ulm–Kaplansky* invariant $f_n(G)$ of a group $G$ is the rank of the factor-group $(p^nG)[p]/(p^{n+1}G)[p]$.

**Lemma 2.7.** *The following statements hold.*

(1) *Let $G$ be a separable group such that if $f_n(G) \neq 0$, then $f_n(G) \geqslant 2$ and $1 \in R$ is a subring of* $\mathrm{E}(G)$ *which acts on $G$ Krylov transitively. Then $R$ acts on $G$ fully transitively.*

(2) *Let $G$ be a group such that $p^\omega G$ is separable and if $f_n(p^\omega G) \neq 0$, then $f_n(p^\omega G) \geqslant 2$. If $G$ is Krylov transitive (resp., projectively Krylov transitive), then $G$ is fully transitive (resp., projectively fully transitive).*

*Proof.* (1) Let $y \in G[p]$ and $U(x) < U(y)$ for some $x \in G$ (see Lemma 2.3). If $\mathrm{ht}(x) < \mathrm{ht}(y)$, then $U(x+y) = U(x)$ and thus $f(x) = x + y$ for $f \in R$, whence $(f - I)(x) = y$. Assume $\mathrm{ht}(x+y) > \mathrm{ht}(y) = k$ and hence $\mathrm{ht}(x) = \mathrm{ht}(y)$. Let

$G = G_0 \oplus G_1 \oplus \cdots \oplus G_k \oplus G_{k+1} \oplus G_{k+1}^*$, where if $G_i \neq 0$ then $G_i$ is a direct sum of cyclic groups of order $p^i$ for each $n \geq 1$. Since $\mathrm{ht}(x) = \mathrm{ht}(y) = k$, we have $x, y \in G_{k+1} \oplus G_{k+1}^*$. Let $x = a + b$ and $y = c + d$, where $a, c \in G_{k+1}$ and $b, d \in G_{k+1}^*$. Since $\mathrm{ht}(x + y) > k$, we have $a = -c$. So $a$ and $c$ are contained in the same cyclic direct summand of $G_{k+1}$. Furthermore, let $z$ be some element from the additional direct summand of $(G_{k+1})[p]$. So, $U(z) = U(y)$ and $U(x + z) = U(x)$. Consequently, $\eta(x) = z$ for some $\eta \in R$. If now $\delta(z) = y$, then $\delta\eta\colon x \to y$, as required.

Part (2) follows immediately from (1). □

The next examples also demonstrate that the new concepts are different from the corresponding old ones, and so the group structure is rather complicated.

**Example 2.8.** There exists a Krylov transitive group (namely, a fully transitive 2-group) which is not projectively Krylov transitive.

*Proof.* In [13, Proposition 3.5] an example was constructed of a fully transitive $p$-group $G$ which is not projectively fully transitive such that (1) $p^\omega G$ is an elementary group of infinite rank; (2) $p^\omega G$ is an elementary $p$-group of rank 2 with $p = 5n + 2$ or $p = 2$; (3) $p = 2$ and $2^\omega G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$. We shall construct an example of a fully transitive 2-group that is not projectively Krylov transitive for which $2^\omega G$ is an elementary group of rank 3.

Let $H = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$, where $2a = 2b = 2c = 0$, and define the endomorphism $\varphi$ by $\varphi(a) = c$, $\varphi(c) = b$ and $\varphi(b) = a + c$. Then it is easily checked that $\varphi^3 = I + \varphi$, where $I$ is the identity on $H$. Let $\Phi$ be the subring of $E(H)$ generated by $I$ and $\varphi$. Therefore, $\Phi = \{xI + y\varphi + z\varphi^2 : x, y, z = 0, 1\}$. Assuming that $xI + y\varphi + z\varphi^2$ is an idempotent, we get the system of comparisons mod 2, namely: $x^2 = x$, $z^2 = y$ and $z^4 + z^2 = z$, which has only one nonzero solution $x = 1$, $y = z = 0$, i.e., the unique nonzero idempotent of $\Phi$ is $I$. Now, using Corner's realization theorem, we exhibit a group $G$ such that $2^\omega G = H$ and $E(G) \restriction H = \Phi$. It is clear that $E(G)$ does not act projectively Krylov transitively on $2^\omega G$ and so $G$ is not projectively Krylov transitive by Lemma 3.1 below. However, $G$ is fully transitive. It is enough to show that $\Phi$ acts fully transitively on $H$. Observe that $(xI + y\varphi + z\varphi^2)a = xa + yb + zc$. If $z \neq 0$, then $z = 1$ and $((x+y)I - \varphi)(xa + yb + c) = (x(x+y) - y)a + (y(x+y) - 1)b \neq 0$. In fact, the system of comparisons $x(x + y) - y = 0$ and $y(x + y) = 1$ is insoluble by mod 2. Thus, it is shown that any element $xa + yb + zc \neq 0$ is an image of $a$. Hereafter, $(I + \varphi + \varphi^2)b = a$ and $\varphi(a + b) = a$ that completes the construction of the example. □

**Example 2.9.** There exists a transitive group which is not projectively transitive.

*Proof.* As in [13, Proposition 3.5 (ii)], let $H = \langle a \rangle \oplus \langle b \rangle$, where $o(a) = o(b) = p$; $\varphi(a) = b$, $\varphi(b) = a + b$ which implies $\varphi^2 = I + \varphi$; note that $\varphi$ is a monomorphism of $H$; $\Phi$ is the subring of $E(H)$ generated by $I, \varphi$:

$$\Phi = \{xI + y\varphi : 0 \leqslant x, y \leqslant p\};$$

and $G$ is a $p$-group such that $p^\omega G = H$ with $E(G) \restriction H = \Phi$. Furthermore, we consider two possibilities for the prime number $p$:

(1) It is of the form of $p = 5n + 2$ for some natural $n$. In [13] it was shown that $G$ is fully transitive. We shall also prove that it is transitive. To this end, it is sufficient to show that any of its nonzero endomorphisms is a monomorphism. Assuming $(xI + y\varphi)(\alpha a + \beta b) = 0$, where $x, y \neq 0$, we have $\alpha x + \beta y = 0$, $\beta x + (\alpha + \beta)y = 0$, whence $\alpha = -(\beta y)/x$ and $\beta(x^2 - y^2 + xy) = 0$. If $\beta = 0$, then $\alpha = 0$. But if $\beta \neq 0$, then we have $t^2 + t - 1 = 0$, where $t = y/x$. However, it was demonstrated in [13] that the equation $t^2 + t - 1 = 0$ is unresolved modulo $p = 5n + 2$.

(2) Let $p = 2$. It was also directly proved in [6] that $G$ is a transitive group.

However, in both cases, $G$ is not manifestly projectively transitive, because by Lemma 3.2 below $\mathrm{E}(G)$ does not act projectively transitively on $H = p^\omega G$. □

Clearly any projectively transitive group is projectively weakly transitive, but the converse need not be true (see Example 2.11 below). In the same way, as in [19, Lemma 3.4], any projectively fully transitive and projectively weakly transitive group is projectively transitive.

**Theorem 2.10** ([19, Theorem 3.10]). *Let $G$ be a finite group and $1 \in R$ a subring of $\mathrm{E}(G)$. Then $R$ acts weakly transitively on $G$.*

**Example 2.11.** There exist projectively weakly transitive groups which are neither projectively Krylov transitive nor fully transitive.

*Proof.* Choose any finite group $H$ and any subring $1 \in R$ of $\mathrm{E}(H)$ which acts neither projectively Krylov transitively nor fully transitively on $H$; for instance, $R$ can be taken as a subring, generated by the identity map $I$ of $H$, that is, the identity of $\mathrm{E}(H)$, where $H$ is a decomposable elementary group. If now $G$ is a group such that $p^\omega G = H$ and $E(G) \restriction H = R$, then by Theorem 2.10 the group $G$ is projectively weakly transitive, as claimed. □

At this stage, it is impossible to find a weakly transitive group which is not projectively weakly transitive. However, we suspect that such a group exists, and even more that there is a transitive group which is not projectively weakly transitive. The difficulty arises from the fact of [13, Proposition 2.3] that the square $G \oplus G$ has only endomorphisms generated by projections, that is, $\mathrm{E}(G \oplus G) = \mathrm{Proj}(G \oplus G)$.

Thereby, utilizing the main result of [15], if $G$ is fully transitive, then $G \oplus G$ is projectively transitive, and vice versa. That $G \oplus G$ is projectively fully transitive, provided $G$ is fully transitive, was established in [13, Theorem 3.7]. Moreover, as in [13], we are currently unable to decide whether or not there is a projectively (Krylov transitive, transitive, weakly transitive) group that is not strongly projectively (Krylov transitive, transitive, weakly transitive). And finally, we also ask whether there do exist a projectively fully transitive group which is not (projectively) weakly transitive as well as a projectively Krylov transitive group that is not fully transitive. So, we remain with these four questions still left-open.

## 3 Basic results and corollaries

We begin here with an attack on the new results. First, four technical but very useful statements are needed. It is worth noting that the first three have proofs which are very similar to those in [13, Lemma 3.12] and [13, Lemma 4.1], so we omit the details and leave them to the interested reader (see also [3, Lemmas 3.12 and 3.26] and [14]).

**Lemma 3.1.** *A group $G$ is (strongly) projectively Krylov transitive if, and only if,* $\mathrm{E}(G)$ *acts (strongly) projectively Krylov transitive on $p^{\omega}G$..*

**Lemma 3.2.** *A group $G$ is (strongly) projectively transitive if, and only if,* $\mathrm{E}(G)$ *acts (strongly) projectively transitive on $p^{\omega}G$..*

**Lemma 3.3.** *A group $G$ is (strongly) projectively weakly transitive if, and only if,* $\mathrm{E}(G)$ *acts (strongly) projectively weakly transitive on $p^{\omega}G$..*

Since divisible groups are both transitive and fully transitive, according to [13, Theorem 3.1] combined with Lemma 3.2, one can derive the following assertion..

**Lemma 3.4.** *Divisible groups are strongly projectively transitive (and so strongly projectively weakly transitive)..*

Next, we concentrate on how projective Krylov transitivity is situated on Ulm subgroups and Ulm factors..

**Proposition 3.5.** *If $G$ is (strongly) projectively Krylov transitive, then $p^{\beta}G$ is (strongly) projectively Krylov transitive for any ordinal $\beta$..*

*Proof.* Letting $H = p^{\beta}G$, we observe that if $x, y \in H$ with $U_H(x) = U_H(y)$, then we have $U_G(x) = U_G(y)$. So, there is a map $\phi \in \mathrm{Proj}(G)$ with $\phi(x) = y$. However, since $H$ is fully invariant in $G$, it is readily seen that $\phi \restriction H \in \mathrm{Proj}(H)$, as required.

The case of a strongly projectively Krylov transitive group can be handled similarly. □

Under a standard limitation on the quotient group, the following statement somewhat reverses the above implication.

**Theorem 3.6.** *Suppose that $\alpha$ is an ordinal strictly less than $\omega^2$, and that $G/p^\alpha G$ is totally projective. Then $G$ is (strongly) projectively Krylov transitive if, and only if, $p^\alpha G$ is (strongly) projectively Krylov transitive.*

*Proof.* We will treat only the case of projectively Krylov transitive groups, because its “strong” version holds identically with Lemma 3.1 at hand.

So, the necessity follows directly from Proposition 3.5.

As for the sufficiency, we consider three main cases:

**Case 1: $\boldsymbol{\alpha = n}$ is natural.** Specifically, we will show that if $p^n G$ is (strongly) projectively Krylov transitive for some finite $n$, then $G$ is (strongly) projectively Krylov transitive.

In fact, by induction, we may restrict our attention to $n = 1$. Set $H = pG$. Using Lemma 3.1 we need to prove that E$(G)$ acts projectively Krylov transitively on $p^\omega G$. So, let $x, y \in p^\omega G$ with $U_G(x) \leqslant U_G(y)$. Clearly, $p^\omega G = p^\omega H$ and $U_G(x) \leqslant U_G(y)$. By assumption, $\varphi(x) = y$ for some $\varphi \in$ Proj(H). It follows from [12, Theorem 1.11] that every idempotent in E$(H)$ lifts to an idempotent in E$(G)$, and so each element of Proj(H) lifts to an element of Proj(G), as required.

**Case 2: $\boldsymbol{\alpha = \omega}$.** Let $x, y \in p^\omega G$ and $U_G(x) \leqslant U_G(y)$. Then we have that $U_{p^\omega G}(x) \leqslant U_{p^\omega G}(y)$. By assumption, there is a $\varphi \in$ Proj(p$^{!}$G) with $\varphi(x) = y$. It follows either from [22, Theorem 11] or from [7] that every idempotent in E$(p^\omega G)$ lifts to an idempotent in E$(G)$, so the map lifts to a map $\psi \in$ Proj(G) with $\psi(x) = y$, as required.

**Case 3: $\boldsymbol{\alpha = \omega \cdot m + n}$ where $\boldsymbol{m, n < \omega}$ are positive integers.** We will use again an induction on the ordinal $\alpha$. To that end, if $\alpha \leqslant \omega$, then we can use Cases 1 and 2. Now let $\alpha > \omega$ and $\alpha = \beta + 1$ for some ordinal $\beta$. Put $X = p^\beta G$ and note that by assumption $pX = p^\alpha G$ is projectively Krylov transitive. It follows at once from Case 1 that $X$ is projectively Krylov transitive. Moreover,

$$G/p^\beta G \cong (G/p^\alpha G)/(p^\beta G/p^\alpha G) \cong (G/p^\alpha G)/p^\beta(G/p^\alpha G),$$

and hence $G/p^\beta G$ is totally projective by a well-known result of Nunke (see, e.g., [16, Exercise 82.3]). So, by the induction hypothesis, $G$ is projectively Krylov transitive, as desired.

Let finally $\alpha = \beta + \omega$ for some $\beta$. Set $X = p^\beta G$, so that $p^\omega X = p^\alpha G$ is projectively Krylov transitive. Notice once again that $X/p^\omega X \cong p^\beta G/p^\alpha G$ is totally projective. By Case 2 we have that $X = p^{\beta}G$ is projectively Krylov transitive. Since $G/p^{\beta}G$ is totally projective and $\beta < \alpha$, then by the induction hypothesis we infer that $G$ is projectively Krylov transitive, as wanted. □

**Proposition 3.7.** *If $G$ is (strongly) projectively transitive, then $p^{\beta}G$ is (strongly) projectively transitive for any ordinal $\beta$.*

*Proof.* Given $x, y \in H = p^{\beta}G$ with $U_H(x) = U_H(y)$, it routinely follows that $U_G(x) = U_G(y)$. Thus, there is a map $\phi \in \mathrm{Aut}(G) \cap \mathrm{Proj}(G)$ with $\phi(x) = y$. Since the subgroup $H$ is known to be characteristic in $G$, it is obviously seen that $\phi \restriction H \in \mathrm{Aut}(H) \cap \mathrm{Proj}(H)$, as required.

The "strongly" case follows in the same manner. □

**Theorem 3.8.** *Suppose that $\alpha$ is an ordinal strictly less than $\omega^2$, and that $G/p^{\alpha}G$ is totally projective. Then $G$ is (strongly) projectively transitive if, and only if, $p^{\alpha}G$ is (strongly) projectively transitive.*

*Proof.* We will deal only with the case of projectively transitive groups, because its "strong" variant can be processed analogically with Lemma 3.2 at hand.

So, the necessity follows directly from Proposition 3.7.

As for the sufficiency, it follows in the same way as that in Theorem 3.6 taking into account Lemma 3.2 and [22]. □

**Proposition 3.9.** *If $G$ is (strongly) projectively weakly transitive, then $p^{\beta}G$ is (strongly) projectively weakly transitive for any ordinal $\beta$.*

*Proof.* This follows in a very similar way to that of Propositions 3.5 and 3.7, so we omit the details. □

**Theorem 3.10.** *Suppose that $\alpha$ is an ordinal strictly less than $\omega^2$, and that $G/p^{\alpha}G$ is totally projective. Then $G$ is (strongly) projectively weakly transitive if, and only if, $p^{\alpha}G$ is (strongly) projectively weakly transitive.*

*Proof.* We will consider only the case of projectively weakly transitive groups, because its "strong" part can be proved similarly with Lemma 3.3 at hand.

So, the necessity follows directly from Proposition 3.9.

As for the sufficiency, it follows in the same manner as that in Theorem 3.6 bearing in mind Lemma 3.3 and [22]. □

We now act on some generalizations of results from [11], showing when projective Krylov transitivity coincides with projective full transitivity and projective transitivity.

**Proposition 3.11.** *Let $G$ be a projectively Krylov transitive group such that $p^\omega G$ is a direct sum of cyclic groups of order $p^n$, where $n \in \mathbb{N}$. Then the following two conditions hold:*

(i) *$G$ is projectively fully transitive.*

(ii) *If $G = G_1 \oplus G_2$ with $p^\omega G_1 \neq 0$ and $p^\omega G_2 \neq 0$, then $G$ is projectively transitive.*

*Proof.* (i) Supposing $x, y \in p^\omega G$ with $U_G(x) \leqslant U_G(y)$, one can write $x = p^r x'$ and $y = p^s y'$, where $x'$ and $y'$ are elements of order $p^n$ in $p^\omega G$ and $r \leqslant s$. Choose $\varphi \in \mathrm{Aut(G)}$ with $\varphi(x') = y'$. Then $p^{s-r}\varphi(x) = y$.

(ii) Utilizing [6, Proposition 2.2], it suffices to show that each endomorphism, represented by the matrix

$$\begin{pmatrix} 1 + q\sigma_2\sigma_1 & \sigma_2 \\ q\sigma_1 & 1 \end{pmatrix},$$

where $q$ is an integer and $\sigma_1 \colon G_1 \to G_2$, $\sigma_2 \colon G_2 \to G_1$ are arbitrary homomorphisms which belong to Proj(G), again lies in Proj(G). In fact,

$$\begin{pmatrix} 0 & 0 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \sigma_1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Pi(G).$$

Analogously,

$$\begin{pmatrix} 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} \in \Pi(G).$$

Consequently,

$$\begin{pmatrix} 1 & \sigma_2 \\ \sigma_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \sigma_1 & 0 \end{pmatrix} \in \Pi(G),$$

while

$$\begin{pmatrix} \sigma_2\sigma_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \sigma_1 & 0 \end{pmatrix} \in \mathrm{Proj(G)},$$

as needed. □

**Remark.** It is perhaps worth noting that Example 2.2 unambiguously shows that this property may fail if $p^\omega G$ is not the direct sum of cyclic groups of the same order $p^n$ for some fixed natural number $n$.

As usual, for any group $A$ define $\mathrm{supp}(A) = \{\sigma < \mathrm{length}(A) : f_A(\sigma) \neq 0\}$.

**Proposition 3.12.** *If $G = G_1 \oplus G_2$ is projectively Krylov transitive, $G_2$ is projectively transitive and* $\mathrm{supp}(p^\omega G_1) \subseteq \mathrm{supp}(p^\omega G_2)$, *then G is projectively transitive.*

*Proof.* The arguments are based on [15, Lemma 2 and Proposition 2], which claimed that the full transitivity of the group forces its transitivity. In these two statements there exist compositions of automorphisms of the sort $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$. Since $G_2$ is projectively transitive, we have $\beta \in \mathrm{Proj}(G_2)$. Furthermore, appealing to Proposition 3.11, the existing automorphisms belong to Proj(G), as required. □

We next will need the following well-known result, which appeared as [11, Theorem 2.4].

**Theorem 3.13.** *Suppose that G is a Krylov transitive group and that G has at most two Ulm invariants equal to* 1*, and if it has exactly two ones, let they correspond to successive ordinals. Then G is fully transitive.*

So, we are ready to state and prove the following:

**Theorem 3.14.** *Suppose $G = G_1 \oplus G_2$ and* $\mathrm{supp}(p^\omega G_1) = \mathrm{supp}(p^\omega G_2)$. *Then the following are equivalent:*

(i) *$G$ is projectively Krylov transitive.*

(ii) *$G$ is projectively fully transitive.*

(iii) *$G$ is projectively transitive.*

*Proof.* (i) ⇒ (ii). According to [13, Lemma 3.12], it is sufficient to show that $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$. Indeed, no Ulm invariant of $p^\omega G$ is equal to 1 and it then follows from Theorem 3.13 and the remarks above that $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$. Thus $G$ is also projectively fully transitive, as required.

(ii) ⇒ (iii). The proof is analogous to [15, Theorem 1] accomplished with some simple arguments.

(iii) ⇒ (i). This implication is always true. □

As an immediate consequence, we yield:

**Corollary 3.15.** *For any group $G$ the following conditions are equivalent:*

(a) *$G \oplus G$ is (strongly) projectively Krylov transitive.*

(b) *$G \oplus G$ is (strongly) projectively fully transitive.*

(c) *$G \oplus G$ is (strongly) projectively transitive.*

Notice also that, owing to [13, Proposition 4.12], there is a non-strongly projectively fully transitive group $G$ such that $G \oplus G$ is strongly projectively fully transitive. A direct summand of a projectively fully transitive group is not necessarily a projectively fully transitive group by [13, Corollary 3.9] (see Proposition 3.16, too).

Before proving our next two results, the following observation could be useful: Suppose that $G \neq 0$ is a group such that all its elements have comparable Ulm sequences. Then it can be decomposed as $G = G_1 \oplus G_2$, where $G_1$ is a direct sum of cyclic groups of order $p^n$ for some finite $n$ and $G_2$ (if $G_2 \neq 0$) is a direct sum of cyclic groups of order $p^{n+1}$. In fact, the sufficiency is trivial. As for the necessity, assume that $A \oplus B$ is such a direct summand of the group $G$ that $A = \langle a \rangle$ is a cyclic group of order $p^n$, whereas $B = \langle b \rangle$ is a cyclic group of order $p^m$ with $m > n + 1$. Thus $a$ and $pb$ have incomparable Ulm sequences because $U(a) = (0, 1, \dots, n-1, \infty, \dots)$ and $U(b) = (1, 2, \dots, m-1, \infty, \dots)$.

The following assertion slightly strengthens [13, Theorem 3.7 and Proposition 4.10] (see also [13, Remark 4.11]).

**Proposition 3.16.** *Suppose that $G$ is a group such that all elements of $p^\omega G$ have comparable Ulm sequences. Then $G$ is fully transitive if, and only if, $G \oplus G$ is strongly projectively fully transitive.*

*Proof.* The sufficiency is immediate since summands of fully transitive groups are again fully transitive. As for the necessity, let $H = G \oplus G$ and $x = (a, b) \in p^\omega G$, $y = (c, d) \in p^\omega G$ with $U(x) \leqslant U(y)$, where $py = 0$. Moreover, as in [19, Lemma 2.2], it is sufficient to demonstrate that there exists an endomorphism from $\Pi(H)$, mapping $x$ to $y$. For instance, assume that $a \neq 0$ and $U(a) \leqslant U(b)$. Then $U(a) \leqslant U(c), U(d)$. So, if $\alpha(a) = c$, $\beta(a) = d - b$ and $\alpha(b) = \beta(b) = 0$, then the matrix

$$\Delta = \begin{pmatrix} \alpha & \alpha \\ \beta & 1-\alpha \end{pmatrix}$$

represents an endomorphism of $H$ which maps $x$ to $y$ and $\Delta \in \Pi(H)$ since

$$\Delta = \begin{pmatrix} \alpha & \alpha \\ 1-\alpha & 1-\alpha \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \beta+\alpha-1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where one easily checks that each of these matrices is an idempotent. □

As a valuable consequence, we derive the following improvement of [11, Theorem 2.13].

**Theorem 3.17.** *Suppose that $G$ is a group such that all elements of $p^\omega G$ have comparable Ulm sequences. Then $G$ is Krylov transitive if, and only if, $G \oplus G$ is strongly projectively Krylov transitive.*

*Proof.* Note that with the Remark at the end of [13] in hand, if $G$ is a Krylov transitive group such that all elements of $p^\omega G$ have comparable Ulm sequences, then $G$ is fully transitive. So, we just need to combine Proposition 3.16 with Corollary 3.15. □

If $p^\omega G$ is bounded by $p$, then all its elements have comparable Ulm sequences and so by Theorem 3.17 we deduce:

**Proposition 3.18.** *Let $G$ be a group with $p^{\omega+1}G = \{0\}$. Then $G$ is Krylov transitive if, and only if, $G \oplus G$ is strongly projectively Krylov transitive.*

Arguing as above, the next worthwhile consequence, which is not explicitly stated in [11], can be also deduced:

**Corollary 3.19.** *Let $G$ be a group with all elements of $p^\omega G$ having comparable Ulm sequences (in particular, if $p^{\omega+1}G = \{0\}$). Then $G$ is Krylov transitive if, and only if, $G$ is fully transitive.*

The following statement extends [13, Proposition 3.14] in the new context (see also [11, Proposition 3.5]). Recall that a group $G$ is said to be *projectively socle-regular* if, for any projection-invariant subgroup $P$ of $G$, there exists an ordinal $\tau$ depending on $P$ such that $P[p] = (p^\tau G)[p]$ – see [12] for the original version.

**Proposition 3.20.** *If $G$ is projectively Krylov transitive, then $G$ is projectively socle-regular.*

*Proof.* Let $P$ be an arbitrary projection-invariant subgroup of $G$ and let $\alpha = \min\{\mathrm{ht}_G(v) : v \in P[p]\}$, whence $P[p] \le (p^\alpha G)[p]$. Next choose $x \in P[p]$ with $\mathrm{ht}_G(x) = \alpha$ such that $U_G(x) = (\alpha, \infty, \dots)$. Letting $y \in (p^\alpha G)[p]$ be an arbitrary element, we have that $U_G(y) = (\beta, \infty, \dots)$ where $\beta \ge \alpha$. If $\beta = \alpha$, then $U_G(x) = U_G(y)$ and thus, with Krylov transitivity at hand, there is an endomorphism $\phi \in \mathrm{Proj}(G)$ such that $\phi(x) = y$. Hence $y \in \phi(P[p]) \le P[p]$ because $P[p]$ is projection-invariant in $G$. If now $\beta > \alpha$, it follows that $\mathrm{ht}_G(x + y) = \alpha$ and so $U_G(x+y) = (\alpha, \infty, \dots) = U_G(x)$. Again, using Krylov transitivity, there exists an endomorphism $\psi \in \mathrm{Proj}(G)$ with the property $\psi(x) = x + y$. But then $\psi - 1_G \in \mathrm{Proj}(G)$ and $(\psi - 1_G)(x) = \psi(x) - 1_G(x) = x + y - x = y$, forcing that $y \in P[p]$ because $y = (\psi - 1_G)(x) \in (\psi - 1_G)(P[p]) \le P[p]$. Thus in either situation we conclude that $(p^\alpha G)[p] \le P[p]$ which by what we have shown above is tantamount to the desired equality $P[p] = (p^\alpha G)[p]$. □

## 4 Open questions

In closing, we shall state some unresolved problems that still elude us:

It follows from [13, Proposition 2.3] that if $G$ is (Krylov, fully, weakly) transitive, then the square $G \oplus G$ is projectively (Krylov, fully, weakly) transitive. So, the following seems to be adequate:

**Problem 1.** *Find some suitable conditions on the (Krylov, fully, weakly) transitive group $G$ under which $G \oplus G$ will be strongly projectively (Krylov, fully, weakly) transitive.*

**Problem 2.** *Find some non-trivial conditions under which the following are equivalent:*

(1) *$G$ is (strongly) projectively Krylov transitive.*

(2) *$G$ is (strongly) projectively fully transitive.*

(3) *$G$ is (strongly) projectively transitive.*

**Problem 3.** *Find some natural conditions on the group $G$ such that $p^\omega G$ being projectively Krylov transitive (projectively fully transitive or projectively transitive, respectively) implies the same for $G$.*

**Problem 4.** *Does it follow that the square $G \oplus G$ is projectively weakly transitive, where $G$ is the projectively Krylov transitive* 2*-group constructed in Example* 2.2*?*

**Problem 5.** *Is it true that Theorems* 3.6*,* 3.8 *and* 3.10 *remain valid without the restriction on the ordinal $\alpha$?*

**Problem 6.** *Suppose that $A = G \oplus S$ is a group, where $S$ is a separable subgroup. Is $A$ projectively (Krylov, fully, weakly) transitive if, and only if, $G$ is?*

We finish off the work with the following.

**Remark.** In [12, Problem 2] the word “finite” should be written and read as “finite homogeneous”. For socle-regular groups with only finite (but non homogeneous) first Ulm subgroup, the question is settled in the negative via [12, Proposition 1.13].

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