J. Group Theory **19** (2016), 983–992
DOI 10.1515/jgth-2016-0013

The influence of maximal quotient groups on the normalizer conjecture of integral group rings

Zhengxing Li, Yuanlin Li and Hongwei Gao

Communicated by Pavel A. Zalesskii

Abstract. Let N be a minimal normal subgroup of a finite group G. The aim of the present paper is to investigate the influence of the quotient group G/N on the normalizer conjecture of the integral group ring $\mathbb{Z}G$. Some conditions on G/N are obtained under which the normalizer conjecture holds for $\mathbb{Z}G$.

1 Introduction

All groups considered are assumed to be finite. Let G be a group and $\mathbb{Z}G$ its integral group ring. The normalizer conjecture states that

$$\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G) = G \cdot \mathrm{Z}(\mathrm{U}(\mathbb{Z}G)),$$

where $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G)$ denotes the normalizer of G in the unit group $\mathrm{U}(\mathbb{Z}G)$ (see [23, Problem 43]). If the conjecture is valid for the integral group ring $\mathbb{Z}G$, then we sometimes say that G has the normalizer property. Recently, the normalizer conjecture has been extensively studied by many authors, see [3, 5, 9–20].

The normalizer conjecture intimately relates with special automorphisms of groups. Denote by $\mathrm{Aut}_{\mathbb{Z}}(G)$ the group of all automorphisms of G each of which is induced by some $u \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G)$ via conjugation. Denote by $\mathrm{Aut}_{\mathrm{Col}}(G)$ the group of all Coleman automorphisms of G (an automorphism σ of G is called a Coleman automorphism if its restriction to each Sylow subgroup of G equals the restriction of some inner automorphism of G, see [6]). It is easy to check that $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G) = G \cdot \mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$ if and only if $\mathrm{Aut}_{\mathbb{Z}}(G) = \mathrm{Inn}(G)$. In addition,

Supported by National Natural Science Foundation of China (71171120, 71571108, 11401329), Projects of International (Regional) Cooperation and Exchanges of NSFC (71411130215), Specialized Research Fund for the Doctoral Program of Higher Education of China (20133706110002), Natural Science Foundation of Shandong Province (ZR2015GZ007), the Doctoral Fund of Shandong Province (BS2012SF003), the Project of Shandong Province Higher Educational Science and Technology Program (J14LI10) and a Discovery Grant from the Natural Science and Engineering Research Council of Canada.

The corresponding author is Zhengxing Li.

Coleman's Lemma (see [6, Introduction]) implies that

$$\mathrm{Aut}_{\mathbb{Z}}(G) \leq \mathrm{Aut}_{\mathrm{Col}}(G).$$

Write $\mathrm{Out}_{\mathbb{Z}}(G) := \mathrm{Aut}_{\mathbb{Z}}(G)/\mathrm{Inn}(G)$ and $\mathrm{Out}_{\mathrm{Col}}(G) := \mathrm{Aut}_{\mathrm{Col}}(G)/\mathrm{Inn}(G)$. By Krempa's result $\mathrm{Out}_{\mathbb{Z}}(G)$ is an elementary abelian 2-group. So, if $\mathrm{Out}_{\mathrm{Col}}(G) = 1$ or, more generally, $\mathrm{Out}_{\mathrm{Col}}(G)$ is an odd order group, then the normalizer conjecture holds for $\mathbb{Z}G$. In this direction, many results have appeared in the literature, see [4, 6, 9, 10, 14, 16] for instance.

We should mention that in [1] Hertweck constructed a metabelian group of order $2^{25} \cdot 97^2$ for which the normalizer property fails to hold. Obviously, the normalizer property holds for any abelian group. So Hertweck's counterexample demonstrates that if a group G is an extension of a group A by a group B, then G does not necessarily have the normalizer property even if both A and B do. It is interesting to determine under what conditions G has the normalizer property provided that both A and B have.

It is well known (see [6]) that the normalizer property holds for any simple group. Let N be a minimal normal subgroup of a group G. Then N is a direct product of copies of a simple group. Since the normalizer property is closed under taking direct products (see [10, Proposition 3]), it follows that N has the normalizer property. The aim of this paper is to present some conditions on G/N under which the normalizer conjecture holds for $\mathbb{Z}G$. Our main results are as follows.

Theorem A. *Let N be a minimal normal subgroup of G such that $\mathbb{Z}(G/N)$ has only trivial central units. Then the normalizer conjecture holds for $\mathbb{Z}G$.*

Theorem B. *Let N be a minimal normal subgroup of G such that G/N is a nilpotent group with a Dedekind Sylow 2-subgroup. Then the normalizer conjecture holds for $\mathbb{Z}G$.*

We note that in [10] Kimmerle investigated the influence of composition factors of a group on the normalizer conjecture. Therein he developed some useful techniques which will be used in our proof of Theorem B when we tackle the case in which N is non-abelian. We note also that in some sense our results could be regarded as extensions of [11, Proposition 2.20].

Now we fix some notation. Let $N \trianglelefteq G$ and $\sigma \in \mathrm{Aut}(G)$. Suppose that N is fixed by σ (note that this is always the case if $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(G)$ or, more generally, $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, see [3, Remark 4.3]). Denote by $\sigma|_N$ the restrictions of σ to N and $\sigma|_{G/N}$ the automorphism of G/N induced by σ in the natural way. Denote by $\mathrm{conj}(x)$ the inner automorphism of G induced by $x \in G$ via conjugation. Let p be a prime. Denote by $\mathrm{O}_p(G)$ the largest normal p-subgroup of G. Unless stated otherwise, other notation and terminology follow those in [22].

2 Some known results

In this section, we recall some known results relating with the normalizer conjecture which will be used in the sequel. The following result is due to Coleman. For its proof the reader may refer to that of [23, Theorem 9.1].

Theorem 2.1. *Let G be a group and let $v \in \mathbb{Z}G$ be an element with augmentation* 1*. Let φ be an automorphism of G such that $v = g^{-1}vg^{\varphi}$ for all $g \in G$. Then for any Sylow subgroup P of G there is an element g in the support of v such that φ coincides with conjugation by g on P.*

Theorem 2.2 (Krempa). *Let G be a group. Then* $\mathrm{Out}_{\mathbb{Z}}(G)$ *is an elementary abelian* 2*-group.*

By using Krempa's result, Jackowski and Marciniak proved the following result.

Theorem 2.3 ([8, Theorem 3.6]). *If G has a normal Sylow* 2*-subgroup, then* $\mathrm{Out}_{\mathbb{Z}}(G) = 1$.

Recall that an automorphism σ of G is said to be p-central if $\sigma|_P = \mathrm{id}|_P$ for some Sylow p-subgroup P of G, where p is a prime. As far as p-central automorphisms are concerned, Hertweck and Kimmerle proved the following result.

Theorem 2.4 ([6, Theorem 14]). *Let G be a simple group. Then there is a prime $p \in \pi(G)$ such that every p-central automorphism of G is inner. In particular,* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Juriaans, de Miranda and Robério proved the following result, which generalizes an early result obtained by Marciniak and Roggenkamp in [17].

Theorem 2.5 ([9, Theorem 3.1]). *Let G be a group with abelian Sylow* 2*-subgroups. Suppose that there exists a nilpotent normal subgroup N of G such that G/N has a normal Sylow* 2*-subgroup. Then every class-preserving Coleman automorphism of G of* 2*-power order is inner. In particular,* $\mathrm{Out}_{\mathbb{Z}}(G) = 1$.

3 Proofs of Theorems A and B

In this section, we present proofs of Theorems A and B. To do this, the following technical result ([13, Proposition 3.1]) and its proof are needed. For the reader's convenience, we include an outline of its original proof below.

Proposition 3.1. *Let G be a group with a normal subgroup N such that the central units of $\mathbb{Z}(G/N)$ are trivial. Let $\phi \in \mathrm{Aut}_{\mathbb{Z}}(G)$. Then there is an element $b \in G$ such that for any Sylow subgroup P of G there exists an element $a \in N$ with $\phi\,\mathrm{conj}(b^{-1})|_P = \mathrm{conj}(a)|_P$. Furthermore, $\phi\,\mathrm{conj}(b^{-1})|_{G/N} = \mathrm{id}|_{G/N}$.*

Proof. Let $H = G/N$ and let $s : H \to G$ be a section such that $s(e_H) = e_G$, where e_M is the identity element of a group M. Let u be a unit of $\mathrm{N}_{\mathrm{U}(\mathbb{Z}G)}(G)$ inducing ϕ via conjugation, i.e., $g^\phi = g^u$ for all $g \in G$. We may assume that the augmentation $\epsilon(u) = 1$. Then by [13, Lemma 2.1] the image $\bar{u}$ of u in $\mathbb{Z}H$ is an element t of H. Let $b = s(t)$, $w = ub^{-1}$ and let $\varphi = \phi\,\mathrm{conj}(b^{-1})$. Then $\bar{w} = 1$ and $w = g^{-1}wg^\varphi$ for any $g \in G$. We may write $w = \sum_{h\in H} w_h s(h)$, where $w_h \in \mathbb{Z}N$ for each $h \in H$. Then we have $1 = \bar{w} = \sum_{h\in H} \epsilon(w_h)h$. It follows that $\epsilon(w_e) = 1$ and $\epsilon(w_h) = 0$ for all $h \neq e$, where $e = e_H$.

Let $g \in G$ and let $\bar{g}$ be its image in H. Then for any $h \in H$ there exists $c_h \in N$ such that $g^{-1}s(h)g^\varphi = c_h s(\bar{g}^{-1}h\bar{g})$. In particular, $c_e = g^{-1}g^\varphi$. Then we have

$$\sum_{h\in H} w_h s(h) = w = g^{-1}wg^\varphi = \sum_{h\in H} g^{-1}w_h g c_h s(\bar{g}^{-1}h\bar{g}).$$

Note that $g^{-1}w_h g c_h \in \mathbb{Z}N$ for any $h \in H$. It follows that $w_{\bar{g}^{-1}h\bar{g}} = g^{-1}w_h g c_h$ for all $h \in H$. Taking $h = e = e_H$, we get $w_e = g^{-1}w_e g c_e = g^{-1}w_e g^\varphi$. Let P be a Sylow subgroup of G. Note that the support of w_e is contained in N. So by Theorem 2.1 there exists some $a \in N$ such that φ coincides with conjugation by a on P. This means $\phi\,\mathrm{conj}(b^{-1})|_P = \mathrm{conj}(a)|_P$. Since $\varphi = \mathrm{conj}(w)$ and $\bar{w} = 1$, the second assertion follows immediately. This completes the proof of Proposition 3.1. □

As a direct result of Proposition 3.1, we have the following ([13, Theorem 3.2]).

Corollary 3.2. *Let G be a group with a normal subgroup A of odd order such that the central units of $\mathbb{Z}(G/A)$ are trivial. Then $\mathrm{Out}_{\mathbb{Z}}(G) = 1$.*

In addition, the following well-known result is also needed in the sequel. For its proof the reader may refer to [2].

Lemma 3.3. *Let $\varphi \in \mathrm{Aut}(G)$ be of p-power order, where p is a prime. Suppose that there is a normal subgroup N of G such that $\varphi|_N = \mathrm{id}|_N$, $\varphi|_{G/N} = \mathrm{id}|_{G/N}$. Then $\varphi|_{G/\mathrm{O}_p(\mathrm{Z}(N))} = \mathrm{id}|_{G/\mathrm{O}_p(\mathrm{Z}(N))}$. Furthermore, if φ fixes element-wise a Sylow p-subgroup of G, then φ is an inner automorphism of G.*

Proof of Theorem A. Let G and N be as in Theorem A. Let $\phi \in \mathrm{Aut}_{\mathbb{Z}}(G)$. We have to show that $\phi \in \mathrm{Inn}(G)$. The proof is divided into two cases according to whether N is abelian or not.

Case 1: N is abelian. If N is of odd order, then the assertion follows from Corollary 3.2. Note that N is an elementary abelian p-group. So it remains to consider the case when N is of 2-power order. Take a Sylow 2-subgroup P of G. Then $N \leq P$. By Proposition 3.1, there exists $b \in G$ such that $\varphi := \phi\,\mathrm{conj}(b^{-1})$ acts as conjugation by some element $x \in N$ on P and induces identity on G/N. Namely, we have $\varphi|_P = \mathrm{conj}(x)|_P$ and $\varphi|_{G/N} = \mathrm{id}|_{G/N}$. Write $\sigma := \varphi\,\mathrm{conj}(x^{-1})$. Then we have $\sigma|_P = \mathrm{id}|_P$ and $\sigma|_{G/N} = \mathrm{id}|_{G/N}$. Note that N is contained in P. So we also have $\sigma|_N = \mathrm{id}|_N$. Since by Theorem 2.2 σ^2 is inner, without loss of generality, we may assume that σ is of 2-power order. Applying Lemma 3.3, we have $\sigma \in \mathrm{Inn}(G)$, yielding that $\phi \in \mathrm{Inn}(G)$.

Case 2: N is non-abelian. Since $\phi \in \mathrm{Aut}_{\mathbb{Z}}(G)$, it follows from Proposition 3.1 that there exists an element $b \in G$ such that $\varphi := \phi\,\mathrm{conj}(b^{-1})$ acts as conjugation by some element in N on every Sylow subgroup of G. It follows that φ, when restricts to N, is a Coleman automorphism of N. Since N is a non-abelian minimal normal subgroup of G, it follows that N is a direct product of isomorphic non-abelian simple groups. It is known that $\mathrm{Out}_{\mathrm{Col}}(\cdot)$ is closed under taking direct products (see [3, Remark 4.3]). So by Theorem 2.4, $\mathrm{Out}_{\mathrm{Col}}(N) = 1$ and thus $\varphi|_N = \mathrm{conj}(x)|_N$ for some $x \in N$. In addition, by Proposition 3.1, we have $\varphi|_{G/N} = \mathrm{id}|_{G/N}$. Write $\sigma := \varphi\,\mathrm{conj}(x^{-1})$. Then we have $\sigma|_N = \mathrm{id}|_N$ and $\sigma|_{G/N} = \mathrm{id}|_{G/N}$. As before, we may assume that σ is of 2-power order. Note that $\mathrm{Z}(N) = 1$. So by Lemma 3.3, we have $\sigma \in \mathrm{Inn}(G)$, implying $\phi \in \mathrm{Inn}(G)$. This completes the proof of Theorem A. □

We note that Ritter and Sehgal presented a characterization of finite groups whose integral group rings have only trivial central units; in particular, the integral ring of the symmetric group S_n of degree n possesses this property, for this see [21].

Corollary 3.4. *Let G be an extension of a simple group by a group whose integral group ring has only trivial central units. Then the normalizer conjecture holds for $\mathbb{Z}G$. In particular, this is the case when G is an extension of a simple group by the symmetric group S_n of degree n.*

Before proving Theorem B, we recall a result ([7, Theorem 11]) due to Higman which states that the integral group ring $\mathbb{Z}G$ has trivial units if and only if G is either an abelian group of exponent $2, 3, 4, 6$ or $G \cong Q_8 \times E$, where Q_8 denotes the quaternion group of order 8 and E is an elementary abelian 2-group.

Proof of Theorem B. Let G, N be as in Theorem B. Write $H := G/N$. Then by hypothesis, H is a nilpotent group with a Dedekind Sylow 2-subgroup, say P.

Let $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(G)$. We have to show $\sigma \in \mathrm{Inn}(G)$. The proof is divided into two cases according to whether P is abelian or Hamiltonian 2-group.

Case 1: P is abelian. The proof splits into two subcases according to whether N is abelian or not.

Subcase 1.1: N is abelian. Then N is an elementary abelian p-group for some prime p. If $p = 2$, then G has a normal Sylow 2-subgroup and thus the assertion follows from Theorem 2.3. If $p \neq 2$, then G has an abelian Sylow 2-subgroup. Note that G is a nilpotent-by-nilpotent group. So the assertion follows from Theorem 2.5.

Subcase 1.2: N is non-abelian. As $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(G)$, it follows that $\sigma|_N \in \mathrm{Aut}(N)$. Let $N = S \times S \times \cdots \times S$, where S is a non-abelian simple group. By Theorem 2.4, there exists a prime q such that every q-central automorphism of S is inner. Let Q be a Sylow q-subgroup of N. Then there exists some element $g \in G$ such that $\sigma|_Q = \mathrm{conj}(g)|_Q$. Write $\varphi := \sigma \,\mathrm{conj}(g^{-1})$. Then $\varphi|_N \in \mathrm{Aut}(N)$. It follows that $\varphi|_N$ permutes on the set of all minimal normal subgroups of N. On the other hand, note that $\varphi|_Q = \mathrm{id}|_Q$; in particular, $\varphi|_{S \cap Q} = \mathrm{id}|_{S \cap Q}$ for each S. We conclude that $\varphi|_N$ must fix every S, i.e., $\varphi|_S \in \mathrm{Aut}(S)$. Consequently, $\varphi|_S$ is a q-central automorphism of S and thus $\varphi|_S \in \mathrm{Inn}(S)$. As N is the direct product of copies of S, it follows that $\varphi|_N \in \mathrm{Inn}(N)$. Let $x \in N$ such that $\varphi|_N = \mathrm{conj}(x)|_N$. Write $\phi := \varphi \,\mathrm{conj}(x^{-1})$. Then $\phi|_N = \mathrm{id}|_N$. Since H is nilpotent and $\phi|_H \in \mathrm{Aut}_{\mathbb{Z}}(H)$, it follows from Theorem 2.1 that $\phi|_H = \mathrm{conj}(h)|_H$ for some $h \in H$. Note that $\phi \in \mathrm{Aut}_{\mathbb{Z}}(G)$. So, without loss of generality, we may assume that ϕ is of 2-power order and that h is of 2-power order. Since H is a nilpotent, it follows that $h \in P$. Keep in mind that in our case P is abelian. So actually $\phi|_H = \mathrm{conj}(h)|_H = \mathrm{id}|_H$. Now by Lemma 3.3 we have $\phi|_{G/\mathrm{O}_2(\mathrm{Z}(N))} = \mathrm{id}|_{G/\mathrm{O}_2(\mathrm{Z}(N))}$. Note that in our case $\mathrm{Z}(N) = 1$. So the previous equality yields that $\phi = \mathrm{id}$, implying $\sigma \in \mathrm{Inn}(G)$, as desired.

Case 2: P is Hamiltonian. Let $P = Q_8 \times E$, where E denotes an elementary abelian 2-group. The proof of this case splits into two subcases according to whether N is abelian or not.

Subcase 2.1: N is abelian. Then N is an elementary abelian p-group for some prime p. In the case when $p = 2$, the group G has a normal Sylow 2-subgroup and thus the assertion follows from Theorem 2.3. It remains to consider the case when $p \neq 2$. Let L be the normal subgroup of G such that $L/N = \mathrm{O}(H)$, where $\mathrm{O}(H)$ denotes the maximal normal subgroup of H of odd order. Then L is of odd order and $G/L \cong G/N/L/N \cong P$. This shows that G may be regarded as an extension of an odd order group L by $Q_8 \times E$. Since $\mathbb{Z}(Q_8 \times E)$ has only trivial units, it follows from Corollary 3.2 that the normalizer conjecture holds for $\mathbb{Z}G$.

Subcase 2.2: N is non-abelian. Denote by M the normal subgroup of G such that $M/N = \mathrm{O}(H)$. We claim that $\mathrm{Out}_{\mathrm{Col}}(M)$ is a $2'$-group. Let $\rho \in \mathrm{Aut}_{\mathrm{Col}}(M)$ be of 2-power order. We have to show that $\rho \in \mathrm{Inn}(M)$. As in the proof of Subcase 1.2 above, without loss of generality, we may assume that $\rho|_N = \mathrm{id}|_N$. Since $\rho \in \mathrm{Aut}_{\mathrm{Col}}(M)$, it follows that $\rho|_{M/N} \in \mathrm{Aut}_{\mathrm{Col}}(M/N)$. Note that M/N is of odd order. So, on the one hand, by [6, Proposition 1], $\rho|_{M/N}$ is of odd order. On the other hand, by hypothesis, ρ is of 2-power order, so $\rho|_{M/N}$ is of 2-power order. Consequently, we must have $\rho|_{M/N} = \mathrm{id}|_{M/N}$. Now, applying Lemma 3.3, we obtain that $\rho|_{M/\mathrm{O}_2(\mathrm{Z}(N))} = \mathrm{id}|_{M/\mathrm{O}_2(\mathrm{Z}(N))}$. Since N is a non-abelian minimal normal subgroup of G, it follows that $\mathrm{Z}(N) = 1$. Hence the previous equality yields $\rho = \mathrm{id}$. As ρ is arbitrary, $\mathrm{Out}_{\mathrm{Col}}(M)$ is a $2'$-group, as desired.

Note that $G/M \cong P = Q_8 \times E$. So $\mathbb{Z}(G/M)$ has only trivial units. Thus by Proposition 3.1, there exists some $b \in G$ such that for any Sylow subgroup Q of G there is $s \in M$ with $\sigma\,\mathrm{conj}(b)|_Q = \mathrm{conj}(s)|_Q$. In particular, this implies that $\sigma\,\mathrm{conj}(b)|_M \in \mathrm{Aut}_{\mathrm{Col}}(M)$. As before, we may assume that $\sigma\,\mathrm{conj}(b)$ is of 2-power order. Since $\mathrm{Out}_{\mathrm{Col}}(M)$ is a $2'$-group, there exists some element $x \in M$ such that $\sigma\,\mathrm{conj}(b)|_M = \mathrm{conj}(x)|_M$. Applying Proposition 3.1 again, we have $\sigma\,\mathrm{conj}(b)|_{G/M} = \mathrm{id}|_{G/M}$. Write $\gamma = \sigma\,\mathrm{conj}(bx^{-1})$. Then we have $\gamma|_M = \mathrm{id}|_M$ and $\gamma|_{G/M} = \mathrm{id}|_{G/M}$. Without loss of generality, we may assume that γ is of 2-power order. Then by Lemma 3.3, we have $\gamma|_{G/\mathrm{O}_2(\mathrm{Z}(M))} = \mathrm{id}|_{G/\mathrm{O}_2(\mathrm{Z}(M))}$. It is clear that $\mathrm{O}_2(\mathrm{Z}(M)) = 1$ since $\mathrm{O}_2(\mathrm{Z}(M))$ is contained in N and N is a non-abelian minimal normal subgroup of G. Thus we actually have $\gamma = \mathrm{id}$. Namely, $\sigma = \mathrm{conj}(xb^{-1}) \in \mathrm{Inn}(G)$. The proof of Theorem B is finished. □

As a direct consequence of Theorem B, we have the following result.

Corollary 3.5. *Let G be an extension of a simple group by a Dedekind group. Then the normalizer conjecture holds for $\mathbb{Z}G$.*

We note that if the quotient group G/N in Theorem B is abelian, then a stronger result can be stated as follows.

Corollary 3.6. *Let N be a minimal normal subgroup of a group G such that G/N is an abelian group. Then* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. *In particular, this is the case when G is an extension of a simple group by an abelian group.*

Proof. Let $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$. We have to show that $\sigma \in \mathrm{Inn}(G)$. We first consider the case when N is abelian. In this case, N is an elementary abelian p-group for some prime p. Let P be the Sylow p-subgroup of G. Then by the definition of σ there exists some $x \in G$ such that $\sigma|_P = \mathrm{conj}(x)|_P$. Without loss of generality, we may assume that $\sigma|_P = \mathrm{id}|_P$. In particular, we have $\sigma|_N = \mathrm{id}|_N$. Since G/N is abelian, it follows that $\sigma|_{G/N} = \mathrm{id}|_{G/N}$. Thus $o(\sigma)$ divides the order N and hence σ is of p-power order. Applying Lemma 3.3, we have $\sigma \in \mathrm{Inn}(G)$, as desired. It remains to consider the case when N is non-abelian. The proof for this case is similar to that of Subcase 1.2 in Theorem B. So we leave it to the reader. □

We would like to point out that Corollary 3.5 may be restated in terms of the derived subgroup of G.

Corollary 3.7. *Suppose that the derived subgroup G' is a minimal normal subgroup of G. Then* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. *In particular, this is the case when G' is a simple group.*

We note that the condition that G/N is nilpotent in Theorem B is used to guarantee that the Sylow 2-subgroup of G/N is a direct factor of the whole group. So a more general result than Theorem B can be stated as follows.

Theorem 3.8. *Let N be a minimal normal subgroup of G such that the Sylow 2-subgroup of G/N is both Dedekind and a direct factor of G/N. Then the normalizer conjecture holds for $\mathbb{Z}G$.*

We close this paper by stating the following remark. Before that, we recall a result ([16, Theorem B]) which says that if G is a group whose non-trivial normal subgroups have the same order, then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Remark 3.9. In [10] Kimmerle proved that if G has no composition factor of order 2, then $\mathrm{Out}_{\mathrm{Col}}(G)$ is a $2'$-group (see [10, Proposition 3] and [6, Theorem 14]). In this spirit, we shall record the following result: if G has a chief series of length 2, then $\mathrm{Out}_{\mathrm{Col}}(G) = 1$; in particular, this is the case when G has a composition series of length 2. In fact, if all non-trivial normal subgroups of G have the same order, then the assertion follows from [16, Theorem B]. Otherwise, G has two minimal normal subgroups N and M say, such that $|N| < |M|$. Since G has a chief series of length 2, it follows that $\{N, G/N\} = \{M, G/M\}$ (up to isomorphism). So $|N| = |G/M|$ and $N \cap M = 1$. We obtain that $G = M \times N$. Since $\mathrm{Out}_{\mathrm{Col}}(M) = \mathrm{Out}_{\mathrm{Col}}(N) = 1$ and $\mathrm{Out}_{\mathrm{Col}}(\cdot)$ is closed under taking direct products (see [3, Remark 4.3]), it follows that $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Acknowledgments. The authors would like to thank the referee for carefully reading the paper and for his (or her) valuable comments.

Bibliography

[1] M. Hertweck, A counterexample to the isomorphism problem for integral group rings, *Ann. of Math. (2)* **154** (2001), 115–138.

[2] M. Hertweck, Class-preserving automorphisms of finite groups, *J. Algebra* **241** (2001), 1–26.

[3] M. Hertweck, Local analysis of the normalizer problem, *J. Pure Appl. Algebra* **163** (2001), 259–276.

[4] M. Hertweck, Class-preserving Coleman automorphisms of finite groups, *Monatsh. Math.* **136** (2002), 1–7.

[5] M. Hertweck and E. Jespers, Class-preserving automorphisms and the normalizer property for Blackburn groups, *J. Group Theory* **12** (2009), 157–169.

[6] M. Hertweck and W. Kimmerle, Coleman automorphisms of finite groups, *Math. Z.* **242** (2002), 203–215.

[7] G. Higman, The units of group rings, *Proc. Lond. Math. Soc. (2)* **46** (1940), 231–248.

[8] S. Jackowski and Z. S. Marciniak, Group automorphisms inducing the identity map on cohomology, *J. Pure Appl. Algebra* **44** (1987), 241–250.

[9] S. O. Juriaans, J. M. de Miranda and J. R. Robério, Automorphisms of finite groups, *Comm. Algebra* **32** (2004), 1705–1714.

[10] W. Kimmerle, On the normalizer problem, in: *Algebra*, Trends Math., Birkhäuser, Basel (1999), 89–98.

[11] Y. Li, The normalizer property of a metabelian group in its integral group ring, *J. Algebra* **256** (2002), 343–351.

[12] Y. Li, S. K. Sehgal and M. M. Parmenter, On the normalizer property for integral group rings, *Comm. Algebra* **27** (1999), 4217–4223.

[13] Z. Li and J. Hai, The normalizer property for integral group rings of finite solvable T-groups, *J. Group Theory* **15** (2012), 237–243.

[14] Z. Li and J. Hai, Coleman automorphisms of standard wreath products of nilpotent groups by groups with prescribed Sylow 2-subgroups, *J. Algebra Appl.* **13** (2014), Article ID 1350156.

[15] Z. Li and J. Hai, The normalizer property for integral group rings of a class of complete monomial groups, *Comm. Algebra* **42** (2014), 2502–2509.

[16] Z. Li, J. Hai and S. Yang, Coleman automorphisms of finite groups with a unique nontrivial normal subgroup, *J. Math. Res. Appl.* **34** (2014), 301–306.

[17] Z. S. Marciniak and K. W. Roggenkamp, The normalizer of a finite group in its integral group ring and Cech cohomology, in: *Algebra-Representation Theory* (Constanta 2000), NATO ASI Ser. II 28, Kluwer Academic, Dordrecht (2001), 159–188.

[18] M. Mazur, The normalizer of a group in the unit group of its group ring, *J. Algebra* **212** (1999), 175–189.

[19] T. Petit Lobão and C. Polcino Milies, The normalizer property for integral group rings of Frobenius groups, *J. Algebra* **256** (2002), 1–6.

[20] T. Petit Lobão and S. K. Sehgal, The normalizer property for integral group rings of complete monomial groups, *Comm. Algebra* **31** (2003), 2971–2983.

[21] J. Ritter and S. K. Sehgal, integral group rings with trivial central units, *Proc. Amer. Math. Soc.* **108** (1990), 327–329.

[22] J. S. Rose, *A Course on Group Theory*, Cambridge University Press, Cambridge, 1978.

[23] S. K. Sehgal, *Units in Integral Group Rings*, Longman Scientific and Technical Press, Harlow, 1993.

Received December 10, 2015; revised February 21, 2016.

Author information

Zhengxing Li, School of Mathematics and Statistics,
Qingdao University, Qingdao 266071, P. R. China;
and Institute of Applied Mathematics of Shandong,
Qingdao 266071, P. R. China.
E-mail: lzxlws@163.com

Yuanlin Li, Department of Mathematics and Statistics, Brock University,
St. Catharines, Ontario L2S 3A1, Canada.
E-mail: yli@brocku.ca

Hongwei Gao, School of Mathematics and Statistics,
Qingdao University, Qingdao 266071, P. R. China;
and Institute of Applied Mathematics of Shandong,
Qingdao 266071, P. R. China.
E-mail: cmgta2007@163.com