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Two-spherical topological Kac–Moody groups are Kazhdan

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Abstract. In this note, we prove that a two-spherical Kac–Moody group over a local field endowed with the Kac–Peterson topology enjoys Kazhdan's property (T).

1 Introduction

Locally compact topological groups with Kazhdan's property (T) are known in abundance. Fewer examples are known for non-locally compact topological Kazhdan groups; among those are the loop groups $\mathrm{LSL}_n(\mathbb{C})$ for $n \geq 3$ ([15]), the unitary group of an infinite-dimensional Hilbert space endowed with the strong topology ([4]), and some other groups of mappings ([6]).

In this note, we demonstrate that the classical local-to-global argument used for establishing Kazhdan's property (T) for semisimple Lie groups over local fields (cf. [5, Section 1.6], [13, Section III.5]) in fact is also applicable to two-spherical Kac–Moody groups over local fields endowed with the Kac–Peterson topology (see [11, Remark (iii)], [12, Section 4.G], [10, Definition 7.8, Remark 7.12]). The key observation is that the Kac–Peterson topology induces the Lie group topology on Levi factors of spherical parabolic subgroups (cf. [10, Corollary 7.16 (iv)]).

We assume the reader is familiar with the concept of Kac–Moody groups and RGD systems; see [14,16], also [2, Chapter 8]. All Kac–Moody groups considered in this note are required to be *irreducible* (i.e., the Dynkin diagram is connected), *two-spherical* (i.e., the Dynkin diagram does not contain edges with the label ∞), and *centred* (i.e., the Kac–Moody group is generated by its root subgroups). Under these assumptions it is well known that Kac–Moody groups over finite fields are Kazhdan, provided the cardinality of the field is sufficiently large [8]. Here we are interested in the case of Kac–Moody groups over local fields. These groups are uncountable, hence cannot be Kazhdan as discrete groups. On the other hand we will show that they enjoy property (T) with respect to a certain non-discrete group topology induced by the non-discrete topology of the ground field.

To state the main result of the note, let G be an almost split Kac–Moody group (cf. [14, Definition 11.3.1]) satisfying property (DCS) (cf. [14, Hypothesis 12.1.1]) over a local field $\mathbb{F}$ with respect to a Galois group of continuous field automorphisms. Note that the class of almost split Kac–Moody groups satisfying property (DCS) contains the classes of split and almost split Kac–Moody groups (cf. [14, Proposition 13.1, Definition 13.2.1]). In [10], we constructed a topology τ on $G(\mathbb{F})$, called the *Kac–Petersen topology*, with the following properties:

(i) τ is Hausdorff ([10, Proposition 7.10]),

(ii) τ is a group topology ([10, Proposition 7.10]),

(iii) τ restricts to the Lie group topology on any Levi factor of a spherical parabolic subgroup ([10, Corollary 7.16 (iv)]),

(iv) τ is the finest topology satisfying these properties ([10, Proposition 7.21]).

Our observation is that such a topology turns $G(\mathbb{F})$ into a Kazhdan group (in the sense that there exist a compact subset Q and $\epsilon > 0$ such that, whenever a continuous unitary representation has a (Q, ϵ)-invariant vector, then it has a non-trivial fixed vector, cf. [5, Definition 1.1.3]):

Theorem. *Let $\mathbb{F}$ be a local field and let G be an irreducible centred two-spherical almost split Kac–Moody group satisfying property* (DCS) *(e.g., split or quasi-split) over $\mathbb{F}$ with respect to a Galois group of continuous field automorphisms of rank $n \geq 2$ and let τ be a topology on $G(\mathbb{F})$ satisfying* (i), (ii) *and* (iii) *above. Then $(G(\mathbb{F}), \tau)$ has Kazhdan's property* (T).

The theorem implies that the group $(G(\mathbb{F}), \tau)$ also has property (FH) by [5, Theorem 2.12.4], i.e., every affine isometric action of $(G(\mathbb{F}), \tau)$ on a real Hilbert space has a fixed point. Note that in the case of non-locally compact groups property (FH) generally does not imply property (T).

Note that, moreover, in the case of non-locally compact groups property (T) generally does not imply compact generation. However, in the present case for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ the topological group $(G(\mathbb{F}), \tau)$ is not only compactly generated but also compactly presented in the sense of [1, Section 1.1], i.e., there exists a compact generating subset of $G(\mathbb{F})$ such that, as an abstract group, it is the quotient of the free group generated by this subset modulo relations of bounded length. Indeed, as each (real or complex) semisimple Lie group is compactly presented (cf. e.g. [7, Proposition 5.1]), this follows from the Curtis–Tits presentation of two-spherical Kac–Moody groups ([3], [10, Theorem 7.22]). In case $\mathbb{F}$ is a non-archimedian local field of characteristic 0, one may apply [1, Theorem 3.1] in order to obtain compact presentability for *split* Kac–Moody groups $(G(\mathbb{F}), \tau)$.

2 Kac–Moody groups are Kazhdan

Two-spherical Kac–Moody groups over local fields by definition contain many subgroups which, as abstract groups, are isomorphic to Lie groups of (relative) rank two. The following well-known property will allow us to use our local-to-global argument.

Lemma 2.1. *Let G be a quasi-simple Lie group of rank two, let T be a maximal split torus of G, let α_1, α_2 be simple roots and let G_{α_1}, G_{α_2} be the corresponding fundamental rank one subgroups of G with maximal split tori $T_{\alpha_1}, T_{\alpha_2} \leq T$. Then there exists an element $x \in T_{\alpha_2}$ such that for each $y \in U_{\alpha_1}$ one has*

$$\lim_{n\to\infty} x^n y x^{-n} = 1$$

and for each $y \in U_{-\alpha_1}$ one has

$$\lim_{n\to\infty} x^{-n} y x^n = 1.$$

Proof. This follows from the action of the torus on the root subgroups; cf. e.g. [13, Proposition I.2.4.1]. □

The preceding lemma will allow us to apply Mautner's lemma:

Lemma 2.2 ([13, Lemma II.3.2]). *Let H be a topological group and let $x, y \in H$ be elements such that*

$$\lim_{n\to\infty} x^n y x^{-n} = 1.$$

If ρ is a unitary representation of the group H on a Hilbert space W such that there exists an element $w \in W$ with $\rho(x)w = w$, then $\rho(y)w = w$.

Proof of the theorem. We proceed by induction on n. For $n = 2$ the assumption of two-sphericity implies that the group G in fact is a Lie group which, by [9, Corollary 2.3] (also (iii) above), is endowed with its Lie group topology. Therefore G has property (T) by [13, Theorem III.5.3].

Let now $n \geq 3$ and assume that the claim has already been proved for each irreducible centred two-spherical Kac–Moody group of lower rank. Let $\alpha_1, \dots, \alpha_n$ be the simple roots of G and let H be the fundamental subgroup of rank $n-1$ corresponding to the set $\alpha_2, \dots, \alpha_n$ of simple roots, i.e., $H := \langle U_{\pm\alpha_i} \mid 2 \leq i \leq n \rangle$. Since G is irreducible, up to a change of enumeration, we can assume that also H is irreducible and that, moreover, the fundamental rank two subgroup G_{α_1,α_2} of G is irreducible, i.e., G_{α_1,α_2} is isomorphic to a quasi-simple Lie group as an abstract group. The subspace topology induced on H certainly satisfies properties (i)–(iii) above; furthermore, by (iii) above the subspace topology induced on G_{α_1,α_2} equals the Lie group topology.

Let ρ be a unitary representation of G on some Hilbert space W almost having invariant vectors. Since H is Kazhdan by induction hypothesis, there exists a non-zero $\rho(H)$-invariant vector $w \in W$. Hence any element x of the torus T_{α_2} of the fundamental rank one subgroup $G_{\alpha_2} \leq H$ satisfies $\rho(x)w = w$. By Lemma 2.1 there exists one such $x \in T_{\alpha_2}$ such that for each $y \in U_{\alpha_1}$ one has

$$\lim_{n\to\infty} x^n y x^{-n} = 1$$

and for each $y \in U_{-\alpha_1}$ one has

$$\lim_{n\to\infty} x^{-n} y x^n = 1.$$

Mautner's Lemma 2.2 implies $\rho(U_{\alpha_1})w = w = \rho(U_{-\alpha_1})w$. Therefore, as G is centred and, thus, generated by H and $G_{\alpha_1} = \langle U_{\alpha_1}, U_{-\alpha_1}\rangle$, we have $\rho(G)w = w$. Hence, indeed, G has property (T). □

Remark 2.3. Since property (T) passes to finite direct products by [5, Proposition 1.7.8], the theorem extends to the case where G is not necessarily irreducible, but all irreducible factors are two-spherical and of rank at least 2. The latter condition is necessary by [5, Proposition 1.7.8] and [13, Theorem III.5.6].

The condition that G be two-spherical is necessary by the following argument. The group $SL_2(\mathbb{R}[t, t^{-1}])$ is a non-two-spherical Kac–Moody group. It does not have property (T): indeed, it admits $SL_2(\mathbb{R})$ as a congruence quotient (modulo the ideal generated by $t - 1$). Moreover, the evaluation map is continuous. Since $SL_2(\mathbb{R})$ does not have property (T) by [13, Theorem III.5.6], the claim follows by [5, Theorem 1.3.4].

The condition that G be centered is necessary as well: it excludes examples like $GL_n(\mathbb{R})$, which are not Kazhdan; cf. [5, Corollary 1.3.6]

Remark 2.4. At the time of writing of this note not a single continuous unitary representation with a (Q, ϵ)-invariant vector has been known for non-affine two-spherical Kac–Moody groups. It is therefore a possibility that begs further investigation whether these Kac–Moody groups are Kazhdan simply because they do not admit such representations at all.

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