J. Group Theory **15** (2012), 311–321
DOI 10.1515/JGT.2011.110 © de Gruyter 2012

2-Blocks with minimal nonabelian defect groups II

Charles W. Eaton, Burkhard Külshammer and Benjamin Sambale

Communicated by Chris Parker

Abstract. We determine the structure of 2-blocks with minimal nonabelian defect groups, by making use of the classification of finite simple groups.

In [29], the third author of this paper investigated 2-blocks B of finite groups whose defect groups D are minimal nonabelian; this means that D is nonabelian but all proper subgroups of D are abelian. In most cases, it was possible to determine the numerical invariants $k(B)$, $l(B)$ and $k_i(B)$, for $i \geq 0$. Here, as usual, $k(B)$ denotes the number of irreducible ordinary characters in B, $l(B)$ denotes the number of irreducible Brauer characters in B, and $k_i(B)$ denotes the number of irreducible ordinary characters of height i in B, for $i \geq 0$.

However, for one family of 2-blocks only partial results were obtained in [29]. Here we deal with this remaining family of 2-blocks, by making use of the classification of finite simple groups. Our main result is as follows:

Theorem 1. *Let B be a nonnilpotent 2-block of a finite group G with defect group*

$$D = \langle x, y : x^{2^r} = y^{2^r} = [x, y]^2 = [x, x, y] = [y, x, y] = 1\rangle, \tag{1}$$

where $[x, y] := xyx^{-1}y^{-1}$, $[x, x, y] := [x, [x, y]]$ and $|D| = 2^{2r+1} \geq 32$. Then B is Morita equivalent to $\mathcal{O}[D \rtimes E]$, where E is a subgroup of $\mathrm{Aut}(D)$ *of order* 3. *In particular, we have*

$$l(B) = 3, \qquad k(B) = \frac{5 \cdot 2^{2r-2} + 16}{3},$$
$$k_0(B) = \frac{2^{2r} + 8}{3}, \qquad k_1(B) = \frac{2^{2r-2} + 8}{3}.$$

Here $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ denotes a splitting 2-modular system for G. Let D be a 2-group as in (1). If B is a nilpotent 2-block of a finite group G with defect group D, then, by the main result of [28], B is Morita equivalent to $\mathcal{O}D$. So we have the following consequence of Theorem 1.

Corollary 2. *Let D be a 2-group as in* (1). *Then Donovan's conjecture* [24] *holds for 2-blocks of finite groups with defect group D.*

Combining Theorem 1 with results in [29], we obtain the following.

Corollary 3. *Let B be a 2-block of a finite group with minimal nonabelian defect groups. Then B satisfies Dade's ordinary conjecture* [13]*, Alperin's weight conjecture* [2]*, the Alperin–McKay conjecture* [1]*, Brauer's $k(B)$-conjecture* [6]*, Olsson's $k_0(B)$-conjecture* [27]*, Eaton's conjecture* [14]*, Brauer's height-zero conjecture* [6]*, and the Eaton–Moreto conjecture* [15].

We gather together some useful facts about blocks with defect groups as in (1), all of which may be found in or easily deduced from results in [29].

Lemma 4. *Let B be a block of a finite group G with defect group D as in* (1)*. Let (D, b_D) be a maximal B-subpair. Then*

(i) $\mathrm{N}_G(D, b_D)$ *controls fusion of subpairs contained in* (D, b_D).

(ii) *Either B is nilpotent or* $|\mathrm{N}_G(D, b_D) : D\,\mathrm{C}_G(D)| = 3$*, and in the latter case $z := [x, y]$ is the only nontrivial fixed point of* $\mathrm{Z}(D)$ *under the action of* $\mathrm{N}_G(D, b_D)$.

(iii) *If B is not nilpotent, then* $\mathrm{O}_2(\mathrm{Z}(G)) \leq \langle z \rangle$.

(iv) *If $Q \leq \mathrm{Z}(D)$ and $Q \not\leq D'$, then there is a B-subpair (Q, b_Q) with b_Q nilpotent.*

(v) *If $D \in \mathrm{Syl}_2(G)$, then G is solvable.*

In our proof of Theorem 1, the following result will be very useful.

Lemma 5. *Let G, B, D be as in Theorem* 1*. Moreover, let b be a 2-block of a normal subgroup H of G which is covered by B. If a defect group d of b satisfies $|d| < |D|$, then b is nilpotent.*

Proof. It is well known that d is conjugate to $D \cap H$ (cf. [22, Theorem E]). Replacing D by a conjugate, if necessary, we may assume that $d = D \cap H$. If $d < D$ then also $d\Phi(D) < D$. By Lemma 4, B has inertial index $t(B) = 3$. Since $|D : \Phi(D)| = 4$, this implies that $\mathrm{N}_G(D)$ permutes the three maximal subgroups of D transitively. Since $d\Phi(D)$ is normal in $\mathrm{N}_G(D)$, we must have $|D : d\Phi(D)| \geq 4$. But then $d \subseteq \Phi(D)$, and

$$[\mathrm{N}_H(D), D] \subseteq D \cap H = d \subseteq \Phi(D).$$

Thus, $\mathrm{N}_H(D)$ acts trivially on $D/\Phi(D)$. Hence, $\mathrm{N}_H(D)/\mathrm{C}_H(D)$ is a 2-group. Let β be the unique 2-block of DH covering b. Then D is a defect group of β, by Theorem E in [22]. Let β_D be a 2-block of $D\,\mathrm{C}_{DH}(D)$ such that $(\beta_D)^{DH} = \beta$.

Then $\mathrm{N}_H(D, \beta_D)/\mathrm{C}_H(D)$ and $\mathrm{N}_{DH}(D, \beta_D)/\mathrm{C}_{DH}(D)$ are also 2-groups, i.e. β has inertial index $t(\beta) = 1$. Since β is a controlled block, by Lemma 4 this implies that β is a nilpotent block. But now Proposition 6.5 in [25] shows that b is also nilpotent. □

Corollary 6. *Let G, B, D be as in Theorem* 1. *If $H \lhd G$ has index a power of* 2, *then $D \leq H$.*

Proof. There is a block b of H covered by B with defect group $D \cap H$. If $D \not\leq H$, then by Lemma 5, b is nilpotent. But then by [25, Proposition 6.5], B is nilpotent, a contradiction. □

We will apply Lemma 5 in connection with the results in [25]. We are almost in a position to start our proof of Theorem 1. First we prove a general result which is presumably well known, but whose proof we sketch for the convenience of the reader.

Lemma 7. *Let $G = G_1 * G_2$ be a central product of finite groups G_1 and G_2 and let B be a block of G. Let B_i be the (unique) block of $G_i \lhd G$ covered by B. Then B is nilpotent if and only if both B_1 and B_2 are.*

Proof. We may write $G = E/Z$, where $E = G_1 \times G_2$ and $Z \leq \mathrm{Z}(E)$. Let B_E be the unique block of E dominating B, so $\mathrm{O}_{p'}(Z)$ is in the kernel of B_E and B_E has defect group D_E such that $D_E Z/Z$ is a defect group for B. By [4, Proposition 2.6] B_E is nilpotent if and only if B is. Note that B_E is a product of blocks of G_1 and G_2 which are nilpotent if and only if B_1 and B_2 are. Hence it suffices to consider the case $G = G_1 \times G_2$. However, the result follows easily in this case since the normalizer and centralizer of a subgroup Q of $G_1 \times G_2$ are $\mathrm{N}_{G_1}(\pi_1(Q)) \times \mathrm{N}_{G_2}(\pi_2(Q))$ and $\mathrm{C}_{G_1}(\pi_1(Q)) \times \mathrm{C}_{G_2}(\pi_2(Q))$, where $\pi_i(Q)$ is the image of the projection onto G_i (we leave the details to the reader). □

Proof of Theorem 1. We assume that Theorem 1 fails, and choose a counterexample G, B, D such that $|G : \mathrm{Z}(G)|$ is as small as possible. Moreover, among all such counterexamples, we choose one where $|G|$ is minimal. Then, by the first Fong reduction, the block B is quasiprimitive, i.e. for every normal subgroup N of G, there is a unique block of N covered by B; in particular, this block of N is G-stable. Moreover, by the second Fong reduction $\mathrm{O}_{2'}(G)$ is cyclic and central.

We claim that $Q := \mathrm{O}_2(G) \subseteq D'$. Since $Q \trianglelefteq G$ we certainly have $Q \subseteq D$. If $Q = D$ then B has a normal defect group, and B is Morita equivalent to $\mathcal{O}[D \rtimes E]$, by the main result of [23]. Thus, we may assume that $1 < Q < D$; in particular, Q is abelian. Let B_Q be a block of $Q\,\mathrm{C}_G(Q) = \mathrm{C}_G(Q)$ such that $(B_Q)^G = B$.

Since $\mathrm{C}_G(Q) \trianglelefteq G$, the block B_Q has defect group $\mathrm{C}_D(Q)$, and either $\mathrm{C}_D(Q) = D$ or $|D : \mathrm{C}_D(Q)| = 2$. Since B has inertial index $t(B) = 3$, $\mathrm{N}_G(D)$ permutes the maximal subgroups of D transitively. Since $\mathrm{C}_D(Q) \trianglelefteq \mathrm{N}_G(D)$, we must have $\mathrm{C}_D(Q) = D$, i.e. $Q \subseteq \mathrm{Z}(D)$.

Thus, B_Q is a 2-block of $\mathrm{C}_G(Q)$ with defect group D. If $Q \nsubseteq D'$ then B_Q is nilpotent, by Lemma 4. Then, by the main result of [25], B is Morita equivalent to a block of $\mathrm{N}_G(D)$ with defect group D, and we are done by the main result of [23].

This shows that we have indeed $\mathrm{O}_2(G) \subseteq D'$; in particular, $|\mathrm{O}_2(G)| \leq 2$ and thus $\mathrm{O}_2(G) \subseteq \mathrm{Z}(G)$. Hence, also $\mathrm{F}(G) = \mathrm{Z}(G)$.

Let b be a block of $\mathrm{E}(G)$ covered by B. If b is nilpotent, then, by the main result of [25], B is Morita equivalent to a 2-block $\widetilde{B}$ of a finite group $\widetilde{G}$ having a nilpotent normal subgroup $\widetilde{N}$ such that $\widetilde{G}/\widetilde{N} \cong G/\mathrm{E}(G)$, and the defect groups of $\widetilde{B}$ are isomorphic to D. Thus by minimality, we must have $\mathrm{E}(G) = 1$. Then $\mathrm{F}^*(G) = \mathrm{F}(G) = \mathrm{Z}(G)$, and

$$G = \mathrm{C}_G(\mathrm{Z}(G)) = \mathrm{C}_G(\mathrm{F}^*(G)) = \mathrm{Z}(\mathrm{F}^*(G)) = \mathrm{Z}(G),$$

a contradiction.

Thus, b is not nilpotent. By Lemma 5, b has defect group D. Let $L_1, \ldots, L_n$ be the components of G and, for $i = 1, \ldots, n$, let b_i be a block of L_i covered by b. If $b_1, \ldots, b_n$ were nilpotent, then b would also be nilpotent by Lemma 7, a contradiction. Thus, we may assume that b_1 is a nonnilpotent 2-block (of the quasisimple group L_1). By Lemma 5, D is a defect group of b_1. But now the following proposition gives a contradiction. □

Proposition 8. *Let D be a 2-group as in* (1)*, and let G be a quasisimple group. Then G does not have a 2-block B with defect group D.*

Note that the proposition holds for classical groups by [3], where blocks whose defect groups have derived subgroup of prime order are classified. However, since our situation is less general we are able to give new and more direct arguments here.

Proof. We assume the contrary. Then we may also assume that B is faithful. Note that by [5], B cannot be nilpotent since D is nonabelian. By Lemma 4, D is not a Sylow 2-subgroup of G, in particular, $64 = 2^6$ divides $|G|$.

Suppose first that $\overline{G} := G/\mathrm{Z}(G) \cong A_n$ for some $n \geq 5$. If $|\mathrm{Z}(G)| > 2$, then $n \in \{6, 7\}$ and $|\mathrm{Z}(G)|$ divides 6, by [19]. But then $|G|$ is not divisible by 64, a contradiction. Thus, we must have $|\mathrm{Z}(G)| \leq 2$. Then $\mathrm{Z}(G) \subseteq D$, and B dominates a unique 2-block $\overline{B}$ of $\overline{G}$ with defect group $\overline{D} := D/\mathrm{Z}(G) \neq 1$. Let

$\mathcal{B}$ be a 2-block of S_n covering $\overline{B}$. Then $\mathcal{B}$ has a defect group $\mathcal{D}$ such that $\overline{D} \subseteq \mathcal{D}$ and $|\mathcal{D} : \overline{D}| = 2$, by results in [21]. Let w denote the weight of $\mathcal{B}$. Then, by a result in [21], $\mathcal{D}$ is conjugate to a Sylow 2-subgroup of S_{2w}. We may assume that $\mathcal{D}$ is a Sylow 2-subgroup of S_{2w}. Then $\overline{D} = \mathcal{D} \cap A_n = \mathcal{D} \cap S_{2w} \cap A_n = \mathcal{D} \cap A_{2w}$ is a Sylow 2-subgroup of A_{2w}, and D is a Sylow 2-subgroup of A_{2w} or $2.A_{2w}$. Thus, A_{2w} is solvable by Lemma 4, so that $w \leq 2$ and $|\overline{D}| \leq 4$, $|D| \leq 8$. Since $|D| \geq 32$, this is a contradiction.

Suppose next that $\overline{G}$ is a sporadic simple group. Then, using Table 1 in [5], we get a contradiction immediately unless $G = \mathrm{Ly}$ and $|D| = 2^7$. In this remaining case, we get a contradiction since, by [26], D is a Sylow 2-subgroup of $2.A_8$, and A_8 is nonsolvable.

Now suppose that G is a group of Lie type in characteristic 2. Then, by a result of Humphreys [20], the 2-blocks of G have either defect zero or full defect. Thus, again Lemma 4 leads to a contradiction.

It remains to deal with the groups of Lie type in odd characteristic. We use three strategies to deal with the various subcases.

Suppose first that $\overline{G} \cong \mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$, where $1 < n \in \mathbb{N}$ and q is odd. Except in the cases $\mathrm{PSL}_2(9)$ and $\mathrm{PSU}_4(3)$, there is $E \cong \mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$ such that G is a homomorphic image of E with kernel W say. We may rule out the cases $G/\mathrm{Z}(G) \cong \mathrm{PSL}_2(9)$ or $\mathrm{PSU}_4(3)$ using [18]. Let $H \cong \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$ with $E \lhd H$. There is a block B_E of E with defect group D_E such that $D_E W/W \cong D$. Let B_H be a block of H covering B_E with defect group D_H such that $D_H \cap E = D_E$. Now B_H is labeled by a semisimple element $s \in H$ of odd order such that $D_H \in \mathrm{Syl}_2(\mathrm{C}_H(s))$ (see, for example, [7, Proposition 3.6]). It follows that $D \in \mathrm{Syl}_2(\mathrm{C}_E(s)/W)$ and so $\mathrm{C}_E(s)/W$ is solvable by Lemma 4. Now W and H/E are solvable, so $\mathrm{C}_H(s)$ is also solvable. By [17, Proposition 1A] $\mathrm{C}_H(s)$ is a direct product of groups of the form $\mathrm{GL}_{n_i}(q^{m_i})$ and $\mathrm{GU}_{n_i}(q^{m_i})$. Write

$$\mathrm{C}_H(s) \cong \Big(\prod_{i=1}^{t_1} \mathrm{GL}_{n_i}(q^{m_i})\Big) \times \Big(\prod_{i=t_1+1}^{t_2} \mathrm{GU}_{n_i}(q^{m_i})\Big),$$

where $t_1, t_2 \in \mathbb{N}$, $n_1, \ldots, n_{t_2} \in \mathbb{N}$, and $m_1, \ldots, m_{t_2} \in \mathbb{N}$, with $n_i \geq 3$ for $i > t_1$. Solvability implies that $t_2 = t_1$ and that for $i = 1, \ldots, t_1$ we have either $n_i = 1$ or $n_i = 2$, where in the latter case $m_i = 1$ and $q = 3$. Since D, D_E, and D_H are nonabelian, we cannot have $n_i = 1$ for all $i = 1, \ldots, t_1$. Thus, we must have $q = 3$ and, without loss of generality, $n_1 = 2$, $m_1 = 1$. Then D_H is a direct product of factors which are either cyclic or isomorphic to SD_{16}. Moreover, we have $|D_H : D_E| \leq 2$ and $|W| \leq 2$. Since $|D : \Phi(D)| = 4$, we also have $|D_E : \Phi(D_E)| \leq 8$ and $|D_H : \Phi(D_H)| \leq 16$.

Suppose first that $|D_H : \Phi(D_H)| = 16$. Then we have $|D_E : \Phi(D_E)| = 8$, $|D_H : D_E| = 2$, and $|W| = 2$. Since $W \not\subseteq \Phi(D_E)$, $D_E \cong D \times W$. If $D_H \cong SD_{16} \times SD_{16}$, then $|D_H| = 2^8$ and $|D| = 2^6$ which is impossible.

Thus, we must have $D_H \cong SD_{16} \times C_k \times C_l$, where k and l are powers of 2. Observe that $\Phi(D_E) \subseteq \Phi(D_H)$ and $|D_H : \Phi(D_H)| = 16 = |D_H : \Phi(D_E)|$. So we must have $\Phi(D_E) = \Phi(D_H)$. Since

$$\Phi(D_E) \cong \Phi(D) \cong C_{2^{r-1}} \times C_{2^{r-1}} \times C_2$$

and $\Phi(D_H) \cong C_4 \times C_{k/2} \times C_{l/2}$, this implies that $4 = 2^{r-1}$, i.e. $r = 3$ and $\Phi(D) \cong \Phi(D_E) \cong C_4 \times C_4 \times C_2$. So we may assume that $k = 8$, $l = 4$. Thus, $D_E \cong D \times C_2$ and $D_H \cong SD_{16} \times C_8 \times C_4$. Hence, $D'_E = D' \times 1$, $|D'_E| = 2$ and

$$D'_E \subseteq D'_H \cap \mathrm{Z}(D_H) \cong \mathrm{Z}(SD_{16}) \times 1 \times 1,$$

so that $D'_E = \mathrm{Z}(SD_{16}) \times 1 \times 1$. Moreover, $D_E/D'_E \cong C_8 \times C_8 \times C_2$ is a subgroup of $D_H/D'_E \cong D_8 \times C_8 \times C_4$. Hence $\mho_2(C_8 \times C_8 \times C_2) \cong C_2 \times C_2$ is isomorphic to a subgroup of $\mho_2(D_8 \times C_8 \times C_4) \cong C_2$ which is impossible.

Next we consider the case $|D_H : \Phi(D_H)| = 8$. In this case $D_H \cong SD_{16} \times C_k$, where k is a power of 2. Then we have $\Phi(D_E) \subseteq \Phi(D_H) \cong C_4 \times C_{k/2}$ and $\Phi(D) \cong \Phi(D_E W/W) = \Phi(D_E)W/W$. However, this contradicts the fact that $\Phi(D) \cong C_{2^{r-1}} \times C_{2^{r-1}} \times C_2$.

The case $|D_H : \Phi(D_H)| \leq 4$ is certainly impossible.

A similar argument applies to the other classical groups, at least when they are defined over fields with $q > 3$ elements, and we give this now. Suppose that G is a classical quasisimple group of type $B_n(q)$, $C_n(q)$, $D_n(q)$ or ${}^2D_n(q)$, where $q > 3$ is a power of an odd prime. Note that in these cases there is no exceptional cover.

Let E be the Schur cover of $G/\mathrm{Z}(G)$, so that G is a homomorphic image of E with kernel W say. Note that $\mathrm{Z}(E)$, and so W, is a 2-group. There is a block B_E of E with defect group D_E such that $D \cong D_E/W$. Details of the following may be found in [10] and [8]. We may realize E as $\mathbf{E}^F$, where $\mathbf{E}$ is a simple, simply-connected group of Lie type defined over the algebraic closure of a finite field, $F : \mathbf{E} \to \mathbf{E}$ is a Frobenius map and $\mathbf{E}^F$ is the group of fixed points under F. Write $\mathbf{E}^*$ for the group dual to $\mathbf{E}$, with corresponding Frobenius map F^*. Note that if $\mathbf{H}$ is an F-stable connected reductive subgroup of $\mathbf{E}$, then $\mathbf{H}$ has dual $\mathbf{H}^*$ satisfying $|\mathbf{H}^F| = |(\mathbf{H}^*)^{F^*}|$.

By [16, Proposition 1.5] there is a semisimple element $s \in \mathbf{E}^*$ of odd order such that D_E is a Sylow 2-subgroup of $\mathbf{L}^F$, where $\mathbf{L} \leq \mathbf{E}$ is dual to $\mathrm{C}^0_{\mathbf{E}^*}(s)$, the connected component of $\mathrm{C}_{\mathbf{E}^*}(s)$ containing the identity element. Now $W \leq \mathrm{Z}(E) \leq D_E \leq \mathbf{L}^F$. Hence $D_E/W \in \mathrm{Syl}_2(\mathbf{L}^F/W)$. By Lemma 4, $\mathbf{L}^F/W$, and

so $\mathbf{L}^F$, is solvable. Now by [9], $\mathrm{C}_{\mathbf{E}^*}(s)$ factorizes as $\mathbf{MT}$, where $\mathbf{T}$ is a torus and $\mathbf{M}$ is semisimple, $\mathrm{C}_{(\mathbf{E}^*)^{F^*}}(s) = \mathrm{C}_{\mathbf{E}^*}(s)^{F^*} = \mathbf{M}^{F^*}\mathbf{T}^{F^*}$ and the components of $\mathbf{M}^{F^*}$ are classical groups defined over fields of order a power of q. Hence $\mathrm{C}_{(\mathbf{E}^*)^{F^*}}(s)$ is either abelian or nonsolvable. It follows that $\mathbf{L}^F$ is either abelian or nonsolvable, in either case a contradiction.

Let G be a quasisimple finite group of Lie type with $|G|$ minimized such that there is a block B of G with defect group D as in (1). We have shown that G cannot be defined over a field of characteristic two, of type $A_n(q)$ or ${}^2A_n(q)$ or of classical type for $q > 3$.

We group the remaining cases into two.

Case 1. Suppose that G is a quasisimple finite group of Lie type with center of odd order, and further that $q = 3$ if G is classical. We analyze $\mathrm{C}_G(z)$, where we recall that $D' = \langle z \rangle$. There is a nonnilpotent block b_z of $\mathrm{C}_G(z)$ with defect group D. As z is semisimple, $\mathrm{C}_G(z)$ may be described in detail. By [19, Theorem 4.2.2] $\mathrm{C}_G(z)$ has a normal subgroup C^0 such that $\mathrm{C}_G(z)/C^0$ is an elementary abelian 2-group and $C^0 = LT$, where $L = L_1 * \cdots * L_m \lhd C^0$ is a central product of quasisimple groups of Lie type and T is an abelian group acting on each L_i by inner-diagonal automorphisms.

If G is a classical group or any exceptional group of Lie type except $E_6(q)$, ${}^2E_6(q)$ or $E_7(q)$, then by [19, Tables 4.5.1, 4.5.2], T is a 2-group. In particular $\mathrm{C}_G(z)/L$ is a 2-group, so by Corollary 6, $D \leq L$. Let b_L be a block of L covered by b_z with defect group D. If b_L is nilpotent, then by [25, Proposition 6.5], b_z is also nilpotent since $\mathrm{C}_G(z)/L$ is a 2-group, a contradiction. Hence b_L is not nilpotent. By Lemma 5, for each i we have that b_L either covers a nilpotent block of L_i, or $D \leq L_i$. It follows that either $D \leq L_i$ for some i or b_L covers a nilpotent block of each L_i. In the latter case by Lemma 7, b_L would be nilpotent, a contradiction. Hence $D \leq L_i$ for some i and there is a nonnilpotent block of L_i with defect group D. But $|L_i| < |G|$ and L_i is quasisimple, contradicting minimality.

If G is of type $E_6(q)$ or ${}^2E_6(q)$, then in the notation of [19, Table 4.5.1] G has (up to isomorphism of centralizers) two conjugacy classes of involutions, with representatives t_1 and t_2. Suppose first of all that z is of type t_1. In this case $\mathrm{C}_G(z)$ has a normal subgroup X of index a power of 2 such that X is a central product of $L = L_1$ and a cyclic group A. Arguing as above, b_z either covers a nilpotent block of X, and so is itself nilpotent (a contradiction) or $D \leq X$. So b_z covers a nonnilpotent block b_X of X with defect group D. Applying the argument again, either b_X covers nilpotent blocks of L and A, in which case b_X would be nilpotent by Lemma 7 (a contradiction), or $D \leq L$. We have $|L| < |G|$ and L is quasisimple, so by minimality we obtain a contradiction. Consider now the

case that z has type t_2. Then $\mathrm{C}_G(z)$ has a normal subgroup of index 2 which is a central product of quasisimple groups, and we can argue as above to again get a contradiction.

If G is of type $E_7(q)$, then in the notation of [19, Table 4.5.1] G has (up to isomorphism of centralizers) five conjugacy classes of involutions, with representatives t_1, t_4, t_4', t_7 and t_7'. In the first three of these cases T is a 2-group and we may argue exactly as above. In case t_7 and t_7', we have $|\mathrm{C}_G(z) : C^0| = 2$ and by a now familiar argument $D \leq C^0$ and b_z covers a nonnilpotent block of C^0 with defect group D. There is $X \lhd C^0$ of index 3 such that $X = LA$, where $L = L_1$ and A is cyclic of order $q \pm 1$. Now by Lemma 4, $\mathrm{O}_2(\mathrm{Z}(A)) = \langle z \rangle$, so $|A|_2 = 2$ and $D \leq L$. By minimality this situation cannot arise since L is quasisimple, and we are done in this case.

Case 2. Suppose that G is a quasisimple group of Lie type with center of even order, and further that $q = 3$ if G is classical. Note that G cannot be of type $A_n(q)$ or ${}^2A_n(q)$. Here we must use a different strategy since we may have $\mathrm{C}_G(z) = G$. Let $u \in \mathrm{Z}(D)$ be an involution with $u \neq z$. By Lemma 4 there is a nilpotent block b_u of $\mathrm{C}_G(u)$ with $b_u^G = B$. As before we refer to [19, Table 4.5.2] for the structure of $\mathrm{C}_G(u)$, and $\mathrm{C}_G(u) \cong LT$, where L is a central product of either one or two quasisimple groups and T is an abelian group acting on L by inner-diagonal automorphisms. We take a moment to discuss types $D_n(3)$ for $n \geq 4$ even and ${}^2D_n(3)$. In these two cases the universal version of the group has center of order 4, and the information given in [19, Table 4.5.2] applies only to the full universal version. In order to extract the required information when $|\mathrm{Z}(G)| = 2$ it is necessary to use [19, Table 4.5.1], taking advantage of the fact that if Y is a finite group, $X \leq \mathrm{Z}(Y)$ with $|X| = 2$ and $y \in Y$ is an involution, then $|\mathrm{C}_{Y/X}(yX) : \mathrm{C}_Y(y)/X|$ divides 2. Note also that [19, Table 4.5.2] gives the fixed point group of an *automorphism* of order 2 acting on G, and that not every such automorphism is realized by an involution in G (this information is contained in the column headed $|\hat{t}|$). We will make no further reference to this fact.

Now $\mathrm{Z}(\mathrm{C}_G(u))$ and T are both 2-groups, and in each case there is a direct product E of quasisimple groups of Lie type and abelian 2-groups, with $W \leq \mathrm{Z}(E)$ such that $L \cong E/W$ and W is a 2-group, and there is a direct product H of finite groups of Lie type such that $E \leq H$ has index a power of 2 and H/W has a subgroup isomorphic to $\mathrm{C}_G(u)$ of index a power of 2. Since W and H/E are 2-groups, by [25, Proposition 6.5] there are nilpotent blocks B_E of E and B_H of H with defect groups D_E and D_H such that $D_E \leq D_H$ and D_E/W has a subgroup isomorphic to D. By Lemma 7, B_E is a product of nilpotent blocks of finite groups of Lie type, and so by [5], D_E is abelian. But then D is abelian, a contradiction. □

Proposition 9. *Let B be as in Theorem* 1. *Then D is the vertex of the simple B-modules.*

Proof. First we consider the situation in the group $D \rtimes E$. Here the three irreducible Brauer characters are linear and can be extended to irreducible ordinary characters. By Theorem 1 there is a Morita equivalence between $\mathcal{O}[D \rtimes E]$ and B. Under this equivalence the three ordinary linear characters map to irreducible characters of height 0 in B. These characters are again extensions of three distinct Brauer characters, since the decomposition matrix is also preserved under Morita equivalence. Now the claim follows from [12, Theorem 19.26]. □

Acknowledgments. The first author is supported by a Royal Society University Research Fellowship, and was partially supported in this research by the "Deutsche Forschungsgemeinschaft". The third author is also supported by the "Deutsche Forschungsgemeinschaft".

Bibliography

[1] J. L. Alperin, The main problem of block theory, in: *Proceedings of the Conference on Finite Groups* (Univ. Utah, Park City, Utah, 1975), Academic Press, New York (1976), 341–356.

[2] J. L. Alperin, Weights for finite groups, in: *The Arcata Conference on Representations of Finite Groups* (Arcata, 1986), American Mathematical Society, Providence (1987), 369–379.

[3] J. An and C. W. Eaton, Blocks with extraspecial defect groups of finite quasisimple groups, *J. Algebra* **328** (2011), 301–321.

[4] J. An and C. W. Eaton, Nilpotent blocks of quasisimple groups for odd primes, *J. reine angew. Math.* **656** (2011), 131–177.

[5] J. An and C. W. Eaton, Nilpotent blocks of quasisimple groups for the prime two, *Algebr. Repr. Theory*, to appear.

[6] R. Brauer, Representations of finite groups, in: *Lectures on Modern Mathematics. I*, Wiley, New York (1963), 133–175.

[7] M. Broué, Les l-blocs des groups $\mathrm{GL}(n,q)$ et $\mathrm{U}(n,q^2)$ et leurs structures locales, *Astérisque* **133–134** (1986), 159–188.

[8] M. Cabanes and M. Enguehard, *Representation Theory of Finite Reductive Groups*, Cambridge University Press, Cambridge, 2004.

[9] R. W. Carter, Centralizers of semisimple elements in finite classical groups, *Proc. LMS (3)* **42** (1981), 1–41.

[10] R. W. Carter, *Finite Groups of Lie Type. Conjugacy Classes and Complex Characters*, John Wiley & Sons, New York, 1985.

[11] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[12] C. W. Curtis and I. Reiner, *Methods of Representation Theory. I*, John Wiley & Sons, New York, 1990.

[13] E. C. Dade, Counting characters in blocks. II, *J. reine angew. Math.* **448** (1994), 97–190.

[14] C. W. Eaton, Generalisations of conjectures of Brauer and Olsson, *Arch. Math. (Basel)* **81** (2003), 621–626.

[15] C. W. Eaton and A. Moreto, Extending Brauer's height zero conjecture to blocks with nonabelian defect groups, preprint.

[16] M. Enguehard, Vers une décomposition de Jordan des blocs des groupes réductifs finis, *J. Algebra* **319** (2008), 1035–1115.

[17] P. Fong and B. Srinivasan, The blocks of finite general linear and unitary groups, *Invent. Math.* **69** (1982), 109–153.

[18] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12, 2008, `www.gap-system.org`.

[19] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups*, Number 3, Part I, Chapter A, Mathematical Surveys and Monographs, American Mathematical Society, Providence, 1998.

[20] J. E. Humphreys, Defect groups for finite groups of Lie type, *Math. Z.* **119** (1971), 149–152.

[21] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Co., Reading, 1981.

[22] B. Külshammer, On p-blocks of p-solvable groups, *Comm. Algebra* **9** (1981), 1763–1785.

[23] B. Külshammer, Crossed products and blocks with normal defect groups, *Comm. Algebra* **13** (1985), 147–168.

[24] B. Külshammer, Donovan's conjecture, crossed products and algebraic group actions, *Israel J. Math.* **92** (1995), 295–306.

[25] B. Külshammer and L. Puig, Extensions of nilpotent blocks, *Invent. Math.* **102** (1990), 17–71.

[26] P. Landrock, The non-principal 2-blocks of sporadic simple groups, *Comm. Algebra* **6** (1978), 1865–1891.

[27] J. B. Olsson, On the number of characters in blocks of finite general linear, unitary and symmetric groups, *Math. Z.* **186** (1984), 41–47.

[28] L. Puig, Nilpotent blocks and their source algebras, *Invent. Math.* **93** (1988), 77–116.

[29] B. Sambale, 2-Blocks with minimal nonabelian defect groups, *J. Algebra* **337** (2011), 261–284.

Received August 17, 2011.

Author information

Charles W. Eaton, School of Mathematics, University of Manchester, Manchester, UK.
E-mail: `charles.eaton@manchester.ac.uk`

Burkhard Külshammer, Mathematical Institute, Friedrich Schiller University, Jena, Germany.
E-mail: `kuelshammer@uni-jena.de`

Benjamin Sambale, Mathematical Institute, Friedrich Schiller University, Jena, Germany.
E-mail: `benjamin.sambale@uni-jena.de`