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Some groups of exponent 72

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Abstract. Local finiteness is proved for groups of exponent dividing 72 with no elements of order 6.

1 Introduction

Our goal is to find conditions which guarantee local finiteness of a periodic group containing elements of orders 3 and 4 and no elements of order 6. The structure of finite groups with the same condition was described in [1].

Suppose that G is a periodic group. The spectrum $\omega(G)$ is the set of orders of elements of G. If $\omega(G)$ is finite then we define $\mu(G)$ as the set of maximal elements of $\omega(G)$ with respect to division.

In the first place, we are interested in $\{2,3\}$-groups with no elements of order 6. The next theorem is the central result of the paper.

Theorem 1.1. *Suppose that $\mu(G) = \{8, 9\}$. Then G is locally finite.*

This result is a step forward after works by B. H. Neumann [10], I. N. Sanov [12], V. D. Mazurov [9], A. K. Zhurtov and V. D. Mazurov [16] and E. Jabara and D. V. Lytkina [5] where local finiteness is shown of groups with $\mu(G)$ equal to $\{2,3\}$, $\{4,3\}$, $\{8,3\}$, $\{2,9\}$ and $\{4,9\}$ respectively.

The structure of locally finite $\{2,3\}$-groups with no elements of order 6 can be described easily.

Theorem 1.2. *Suppose that G is a locally finite $\{2,3\}$-group without elements of order 6. Then one of the following statements holds:*

(a) *$G = O_3(G)T$ where $O_3(G)$ is Abelian and T is a locally cyclic or locally quaternion group acting freely on $O_3(G)$.*

(b) *$G = O_2(G)R$ where $O_2(G)$ is nilpotent of class at most 2 and R is a locally cyclic 3-group acting freely on $O_2(G)$.*

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(c) *$G = O_2(G)D$ where D contains a subgroup R of index* 2 *and $O_2(G)R$ satisfies* (b).

(d) *G is a* 2*-group or a* 3*-group.*

In (a)–(c)*, G is soluble of length at most* 4.

Here $O_p(G)$, where p is a prime, is defined as the largest normal p-subgroup of G. A group B acting on a non-trivial group A is said to be acting freely on A if $a^b \neq a$ for every $a \in A \setminus \{1\}$ and every $b \in B \setminus \{1\}$. A 2-group Q is called locally quaternion if every finite subset of Q is contained in a finite subgroup of Q isomorphic to a generalized quaternion group. Note that an infinite locally quaternion group is isomorphic to a group

$$\langle a_i, b,\ i = 0, 1, 2, \dots \mid a_0^2 = 1,\ a_{i+1}^2 = a_i\ (i \geqslant 0),\ b^2 = a_0,\ a_i^b = a_i^{-1}\ (i > 0)\rangle.$$

In the general case we manage to prove local finiteness under some additional conditions.

Theorem 1.3. *Let G be a $\{2, 3\}$-group without elements of order* 6.

(1) *If G contains a non-trivial normal p-subgroup ($p = 2$ or 3), then either G is locally finite, or G is a p-group, or G is an extension of a nilpotent* 2*-group of class* 2 *by a* 3*-group which contains a unique subgroup of order* 3.

(2) *If G contains an involution and no elements of order* 27*, then either G is locally finite, or G is an extension of a simple non-abelian group by a* 2*-group.*

Note also that there exist groups G with $\mu(G) = \{2, 3^n\}$ that are not locally finite, where $n \geqslant 7$ (see [3, 14]), and $\mu(G) = \{2^m, 3\}$, where $m \geqslant 54$ (see [9]).

We say that a periodic group G is 3-isolated if and only if $\omega(G) = \{3\} \cup w'$, where none of the numbers in w' is divisible by 3.

Theorem 1.4. *Let G be a* 3*-isolated group. Then one of the following statements holds:*

(1) *G is an extension of a non-abelian simple group by a $3'$-group.*

(2) *G is an extension of a non-trivial* 3*-group of exponent* 3 *by a $3'$-group.*

(3) *G is an extension of a nilpotent group of class* 2 *by a subgroup of S_3.*

(4) *$G = NA$ where A is isomorphic to $A_5 \simeq \mathrm{SL}_2(4)$ and N is the direct product of subgroups of order* 16 *every one of which is invariant in G and isomorphic to the natural $\mathrm{SL}_2(4)$-module of dimension* 2 *over a field of order* 4.

2 Notation and known facts

We say that a group G has period n if the identity $x^n = 1$ holds whenever $x \in G$. The minimal n with this property is called the exponent of G.

For a periodic group G denote by $\Gamma_n(G)$ the set of its elements of order n. Elements of $\Gamma_2(G)$ will be called involutions. Let $\Delta(G) = \{x^2 \mid x \in \Gamma_4(G)\}$.

Let a be an automorphism of a group G. If there exist integers m and n such that $g^{a^2}(g^a)^m g^n = 1$ for all $g \in G$ then a is said to be a quadratic automorphism of G.

If the order of a is s and $gg^a \dots a^{a^{s-1}} = 1$ for all $g \in G$, then a is said to be a splitting automorphism of G.

Lemma 2.1 (V. P. Shunkov [13]). *If a periodic group G contains an involution t with finite centralizer $C_G(t)$, then G is locally finite.*

Lemma 2.2 ([4, Theorem 16.8.7]). *A group of order $p^a q^b$ for primes p, q and $a, b \in \mathbb{N}$ is soluble.*

The next result is well known.

Lemma 2.3. *Suppose that $G = \langle t, x \mid t^2 = x^3 = (tx)^m = 1\rangle$. If $m = 2, 3$ or 4, then G is isomorphic to the symmetric group S_3 of degree 3, the alternating group A_4 of degree 4 or the symmetric group S_4 of degree 4 respectively.*

Lemma 2.4 ([15, Lemma 6], [11]). *Suppose that a group X possesses a splitting automorphism of order 3. Then X is nilpotent of class at most 3 and the order of every non-trivial element of $[[X, X], X]$ equals 3. In particular, if a group of order 3 acts freely on a finite group X, then X is nilpotent of class 2.*

Lemma 2.5 ([5, 9, 10, 12, 16]). *If $\mu(G) = \{2^m, 3^n\}$, where*

$$(m, n) \in \{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2)\},$$

then G is locally finite.

Lemma 2.6. *Let G be a periodic group generated by a pair of quadratic automorphisms of an elementary Abelian p-group. Then G is isomorphic to an extension of a finite p-group by a subgroup of the direct product $L_1 \times \cdots \times L_s$ where $L_i \simeq \mathrm{GL}(2, p^{m_i})$, $i = 1, \dots, s$. Here p is a prime and m_i is a natural number for every i.*

Proof. See [6, proof of Theorem 2]. □

Lemma 2.7 ([7]). *Suppose that a group G contains a subgroup X of order 3 such that $C_G(X) = \langle X\rangle$. If for every $g \in G$ the subgroup $\langle X, X^g\rangle$ is finite, then one of the following holds:*

(1) *$G = NN_G(X)$ for a periodic nilpotent subgroup N of class 2 and NX is a Frobenius group with core N and complement X.*

(2) *$G = NA$ where A is isomorphic to $A_5 \simeq \mathrm{SL}_2(4)$ and N is a normal elementary Abelian 2-subgroup; here N is a direct product of subgroups of order 16 that are normal in G and isomorphic to the natural $\mathrm{SL}_2(4)$-module of dimension 2 over a field of order 4.*

(3) *G is isomorphic to $L_2(7)$.*

In particular, G is locally finite.

3 Proof of Theorem 1.1

Let G be a group with $\mu(G) = \{8, 9\}$. Our goal is to prove that G is locally finite. Suppose the contrary, i.e. G is not locally finite.

Lemma 3.1. *Suppose that $v \in \Gamma_4$, $t = v^2$, $x \in \Gamma_3$, $x^t = x^{-1}$. Then $[x, x^v] = 1$.*

Proof. Consider a group

$$H(l,m) = \langle a, b \mid a^2 = b^4 = (ab^2)^3 = (ab)^l = [a,b]^m\rangle$$

for $l, m \in \{8, 9\}$. The mapping $a \to tx = v^2x$, $b \to v$ obviously can be extended to a homomorphism of one of the groups $H(l,m)$ onto the group $\langle x, v\rangle$. On the other hand, computations in GAP ([2]) show that $|H(8,8)| = 4$, $H(9,9) = 1$, $H(9,8) \simeq L_2(17)$ and $H(8,9) \simeq 3^2 : 4$. Since $\langle x, v\rangle$ contains elements of order 2 and 3 and no elements of order 17, it follows that $\langle x, v\rangle \simeq H(8,9)$ and the lemma is proved. □

Lemma 3.2. *Suppose that $t, u \in \Gamma_2$, $t \neq u$, $[t,u] = 1$. If $x \in \Gamma_3$ and $x^t = x^{-1}$, then $\langle x, t, u\rangle \simeq S_4$.*

Proof. The group $K = \langle x, t, u\rangle$ is a homomorphic image of one of the groups

$$\begin{aligned} K(l,m) = \langle a, b, c \mid 1 = a^3 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ac)^l \\ = (abc)^m = (baa^c)^{72}\rangle. \end{aligned}$$

Computations in GAP ([2]) show that we have $|K(8,8)| = 4$, $K(9,9) = 1$ and $K(8,9) \simeq K(9,8) \simeq S_4$. Therefore K is a homomorphic image of the group S_4. On the other hand, $|K|$ is divisible by 3 and 4, hence $K \simeq S_4$. □

Lemma 3.3. *If $t \in \Delta$ and $u \in \Gamma_2$, then $(tu)^8 = 1$.*

Proof. Suppose that $(tu)^9 = 1$. We can assume that $x = tu \in \Gamma_3$. If not, then we change u to $t^{utu} \in \langle t, u \rangle$. By assumption there exists an element $v \in G$ with the property that $v^2 = t$. By Lemma 3.1, $[x, x^v] = 1$. By Lemma 2.1, $C_G(t)$ contains an involution $w \neq t$.

By Lemma 3.2, $\langle t, w, z \rangle \simeq S_4$ for every non-trivial $z \in \langle x, x^v \rangle$, therefore for any involution y of $\langle t, w, z \rangle$ either $(wy)^3 = 1$ or $(wy)^4 = 1$ holds. For every $l_1, l_2, \ldots, l_8 \in \{3, 4\}$ define the group

$$\begin{aligned} L(l_1, \ldots, l_8) = \langle a, b, c, d \mid 1 = a^3 = b^3 = c^2 = d^2 = [a, b] = (ac)^2 \\ = (bc)^2 = [c, d] = (da)^{l_1} = (db)^{l_2} = (dab)^{l_3} \\ = (dab^{-1})^{l_4} = (cda)^{l_5} = (cdb)^{l_6} \\ = (cdab)^{l_7} = (cdab^{-1})^{l_8} \rangle. \end{aligned}$$

Computations in GAP show that the order of any of these groups is at most 24. On the other hand, the mapping $a \to x$, $b \to x^v$, $c \to t$, $d \to w$ can be extended to a homomorphism of one of these groups onto $\langle x, x^v, t, w \rangle$ whose order is divisible by $36 = 9 \cdot 4$. That contradiction proves the lemma. □

Lemma 3.4. *If $t \in \Delta$ and $x \in \Gamma_3$, then $\langle t, x \rangle \simeq A_4$.*

Proof. Set $u = t^x$, $v = u^x$. If $(tx)^8 = 1$, then by Lemma 3.3 the following relations are true:

$$1 = t^2 = x^3 = (tx)^8 = (tu)^8 = (t^u v)^8 = [t, uv]^8 = ((tu)^4 v)^8 = ((ut)^4 v)^8.$$

Besides, for some $l, r, s \in \{8, 9\}$ the following properties hold:

$$\left((xt)^2 \cdot (tx^u)^2\right)^l = ((xt)^2 (x^2 (tx)^2 t)^2)^r = (ux^t)^s = 1.$$

GAP computations show that the order of a group with such properties is at most 2.

Thus $(tx)^8 \neq 1$ and hence

$$(tx)^9 = (tx^{-1})^9 = 1.$$

Since $(tx^{-1})^3 = tt^x t^{x^2} = tuv$, we get $(tuv)^3 = 1$. In the group $K = \langle t, tuv \rangle$ by Lemma 3.3, the equation

$$(t \cdot tuv)^8 = (uv)^8 = 1$$

holds. As is shown in the previous paragraph this is possible only when $tuv = 1$, i.e. $(tx^{-1})^3 = 1$. That implies $\langle t, x \rangle \simeq A_4$ and the lemma is proved. □

Lemma 3.5. *Suppose that $t, u \in \Delta$, $x \in \Gamma_3$. Then $(tu)^4 = (tux)^3 = 1$.*

Proof. By Lemmas 3.3 and 3.4, the group $\langle t, u, x\rangle$ satisfies the properties

$$1 = t^2 = u^2 = x^3 = (tx)^3 = (ux)^3 = (t^u x)^3 = (u^t x)^3 = (tu)^8 = (tux)^s,$$

where $s \in \{8, 9\}$. GAP computations show that in a group with such properties the relations $(tu)^4 = (tux)^3 = 1$ hold. The lemma is proved. □

Lemma 3.6. *If $a, b, c \in \Delta$, then $[[a, b], c] = 1$.*

Proof. Suppose that $x \in \Gamma_3$. Then by Lemmas 3.4 and 3.5, in $K = \langle x, a, b, c\rangle$ the following elements are trivial:

$$a^2, b^2, c^2, x^3, (ab)^4, (ac)^4, (bc)^4, (a^b c)^4, (b^c a)^4, (c^a b)^4,$$

$$(ax)^3, (bx)^3, (cx)^3, (abx)^3, (acx)^3, (bcx)^3, (a^b cx)^3, (b^c ax)^3, (c^a bx)^3,$$

$$((ab)^2c)^4, ((ac)^2b)^4, ((bc)^2a)^4, ((ab)^2cx)^3, ((ac)^2bx)^3, ((bc)^2ax)^3.$$

Besides, $(abcx)^8 = 1$ or $(abcx)^9 = 1$. GAP computations show that in a group with such properties $[[a, b], c] = 1$. The lemma is proved. □

Lemma 3.7. *The group $P = \langle\Delta\rangle$ is nilpotent of class 2 and period 4.*

Proof. Suppose that $a, b \in \Delta$. By Lemma 3.6, $[a, b]$ commutes with every element of Δ and thus belongs to the center of P. Therefore $P/Z(P)$ is Abelian and P is nilpotent of class 2. Since P is generated by involutions, it has period 4. The lemma is proved. □

It is clear that the set $\omega(G/P)$ contains elements of orders 3 and 2 and is distinct from $\omega(G)$. By Lemma 1.5, G/P is locally finite. In this case G is locally finite as well. Theorem 1.1 is proved.

4 Proof of Theorem 1.2

Let G be a locally finite $\{2, 3\}$-group without elements of order 6. Suppose also that G contains elements of orders 2 and 3. The goal of this section is to prove that one of the cases (a)–(c) of Theorem 1.2 holds.

Suppose first that G is finite. By Lemma 2.2, G is soluble. If $P = O_2(G) \neq 1$, then a Sylow 3-subgroup of G acts freely on P, therefore it is cyclic and P is nilpotent of class 2 by Lemma 2.4. It is clear that (b) or (c) holds.

If $O_2(G) = 1$, then $P = O_3(G)$ is a Sylow subgroup of G acted freely by a Sylow 2-subgroup T. In particular, T contains a unique involution which inverts P. Thus P is Abelian and T is a cyclic or a quaternion group, i.e. (a) holds.

Suppose now that G is infinite. As has already been proved, each of its finite subgroups containing elements of orders 2 and 3 is soluble of degree 4. Therefore G is also soluble of degree 4 and in particular $O_p(G) \neq 1$ for some $p \in \{2, 3\}$.

Let $O_2(G) \neq 1$. Since a Sylow 3-subgroup of G acts freely on $O_2(G)$, the group $O_3(G/O_2(G))$ is locally cyclic and

$$|(G/O_2(G))/O_3(G/O_2(G))| \leqslant 2.$$

In this case one of the cases (b), (c) of the theorem holds.

If $O_3(G) \neq 1$, then a Sylow 2-subgroup of G acts freely on $O_3(G)$ and thus contains a unique involution. Hence $O_3(G)$ is Abelian,

$$O_2(G/O_3(G)) = G/O_3(G)$$

and the case (a) of the theorem holds. Theorem 1.2 is proved.

5 Proof of Theorem 1.3

(1) If $p = 3$ and G is not a 3-group, then $G/O_3(G)$ is a (locally finite) 2-group with a unique involution and $O_3(G)$ is Abelian. In particular, G is locally finite.

Suppose that $p = 2$ and G is not a 2-group. Then $O_2(G)$ is nilpotent of class 2. Set $\overline{G} = G/\Phi(O_2(G))$. Then $V = O_2(G)/\Phi(O_2(G))$ coincides with $C_{\overline{G}}(V)$. If x is an element of order 3 in $\overline{G}$, then x induces a quadratic automorphism in V and hence $\langle x, x^g \rangle$ is a finite group for every $g \in \overline{G}$. By Theorem 1.2, $H = \langle x, x^g \rangle V$ is an extension of a 2-group U by $\langle x \rangle$ and $[v, u, u] = 1$ for every $u \in U$, $v \in V$. This means that $[v, u^2] = 1$ and hence $u^2 \in V$. Thus H/V is of order 3 or isomorphic to A_4. By [8, Theorem 1], $\langle x^{\overline{G}} \rangle V/V$ is an extension of a 2-group by $\langle x \rangle$ and hence $\langle x \rangle V \trianglelefteq \overline{G}$. If $\overline{G} \setminus V$ contains an involution, then $\overline{G}/V$ is an extension of a locally cyclic 3-group by a group of order 2 and hence G is locally finite.

(2) Let N be a non-trivial normal subgroup of G. If $G \setminus N$ contains an element of order 3, then by Lemma 2.4, N is nilpotent and according to (1), G is locally finite.

Suppose that G is not locally finite and S is the intersection of all normal subgroups N of G such that G/N is a 2-group. If S is a non-trivial 3-group and G/S is also not trivial, then any involution of G inverts S and G/S contains a unique involution. In particular, G is locally finite.

Suppose that S is non-trivial and not a 3-group. Obviously, S is generated by elements of order 3. If N is a proper non-trivial normal subgroup of S, then $S \setminus N$ contains an element of order 3 and the same arguments as above show that N is nilpotent of class 2. So G contains a non-trivial nilpotent normal subgroup $\langle N^G \rangle$ and by (1) is locally finite. Therefore S is a simple non-abelian group and G/S is a 2-group. The theorem is proved.

6 Proof of Theorem 1.4

Let G be a 3-isolated group. First, suppose that $N = O_3(G) \neq 1$. If all 3-elements of G are contained in N, then G/N is a $3'$-group, i.e. (2) holds. Suppose that $G \backslash N$ contains an element x of order 3 and prove that this is impossible. Without loss of generality, we can assume that N is an elementary Abelian 3-group and G/N acts faithfully on N by conjugation in G.

Since $N\langle x \rangle$ is a group of exponent 3, x induces a quadratic automorphism in N. By the definition of N there exist elements y and z of order 3 such that $\langle z, y \rangle$ is not a 3-group. By Lemma 2.6, the center of $\langle z, y \rangle$ contains an involution which is impossible.

Let N be the largest normal $3'$-subgroup of G. Suppose that $N \neq 1$. By Lemma 2.4, N is nilpotent. We will prove that in this case G is locally finite. We can assume that N is Abelian. If x is an element of order 3 in G, then x induces in N a quadratic automorphism and hence $\langle x, x^g \rangle$ is finite for every $g \in G$. Since $C_G(x)$ is a 3-group of exponent 3 acting freely on N, it follows that $C_G(x) = \langle x \rangle$. By Lemma 2.7, G is locally finite and one of the cases (3)–(4) holds.

Now suppose that G does not contain any non-trivial 3- or $3'$-subgroups. If N is a non-trivial normal subgroup of G such that there exists an element x of order 3 which does not belong to N, then N is nilpotent by Lemma 2.4 and we are back to one of the considered cases. So suppose that every non-trivial normal subgroup of G contains all elements of order 3 of G. Let M be the intersection of all normal subgroups N of G such that G/N is a $3'$-subgroup. Then M is simple, as above, and the case (1) holds. The theorem is proved.

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