

1 Appendix 1. Identification of randomly interventional analogue of mediation parameter $r\Phi(a, a^*)$

Proof A

$$\begin{aligned}
r\Phi(a, a', a'', a''') &\equiv E[Y(a, G_1(a'), G_2(a'', G_1(a''')))] \\
&= \sum_{m_1} E[Y(a, m_1, G_2(a'', G_1(a''')))|G_1(a') = m_1] \Pr(G_1(a') = m_1) \text{ (add } G_1(a')) \\
&= \sum_{m_1} E[Y(a, m_1, G_2(a'', G_1(a''')))] \Pr(G_1(a') = m_1) \\
&\text{(} G_1(a') \text{ independent to all variables, including } G_1(a''')) \\
&= \sum_{m_1, m_2} E[Y(a, m_1, m_2)|G_2(a'', G_1(a''')) = m_2] \Pr(G_2(a'', G_1(a''')) = m_2) \Pr(G_1(a') = m_1) \\
&\text{(add } G_2(a'', G_1(a''')) \\
&= \sum_{m_1, m_2} E[Y(a, m_1, m_2)] \Pr(G_2(a'', G_1(a''')) = m_2) \Pr(G_1(a') = m_1) \\
&\text{(add } G_2(a'', G_1(a''')) \text{ independent to all variables)} \\
&= \sum_{m_1, m_2} E[Y(a, m_1, m_2)] \Pr(M_2(a'', G_1(a''')) = m_2) \Pr(M_1(a') = m_1) \\
&\text{(G and M have the same distribution)} \\
&= \sum_{m_1, m_2, m'_1} E[Y(a, m_1, m_2)] \Pr(M_2(a'', m'_1) = m_2|G_1(a''') = m'_1) \Pr(G_1(a''') = m'_1) \Pr(M_1(a') = m_1) \\
&\text{(add } G_1(a''')) \\
&= \sum_{m_1, m_2, m'_1} E[Y(a, m_1, m_2)] \Pr(M_2(a'', m'_1) = m_2) \Pr(G_1(a''') = m'_1) \Pr(M_1(a') = m_1) \\
&\text{(} G_1(a') \text{ independent to all variables)} \\
&= \sum_{m_1, m_2, m'_1} E[Y(a, m_1, m_2)] \Pr(M_2(a'', m'_1) = m_2) \Pr(M_1(a''') = m'_1) \Pr(M_1(a') = m_1) \\
&\text{(G and M have the same distribution)}
\end{aligned}$$

Proof B

All components in the last equation of proof a can be identified as the expression of g-formula. In order to do that, we need to make four no unmeasured confounding assumptions: (1) no unmeasured exposure-outcome confounding ($Y(a, m_1, m_2) \perp A|V$), (2) no unmeasured mediator-outcome confounding ($Y(a, m_1, m_2) \perp M_1|V, A, L_1$ and $Y(a, m_1, m_2) \perp M_2|V, A, L_1, M_1, L_2$), (3) no unmeasured exposure-mediator confounding ($(M_1(a), M_2(a, m_1)) \perp A|V$), and (4) no unmeasured mediator-mediator confounding ($M_2(a, m_1) \perp M_1|V, A, L_1$). Under the four assumptions, all components can be identified in forms of g-formula as follows.

$$\begin{aligned}
E[Y(a, m_1, m_2)|v] &= \sum_{l_1, l_2} E[Y|v, a, l_1, m_1, l_2, m_2] \Pr(l_1|v, a) \Pr(l_2|v, a, l_1, m_1) \\
\Pr(M_2(a, m_1) = m_2|v) &= \sum_{l_1} \Pr(m_2|v, a, m_1, l_1) \Pr(l_1|v, a)
\end{aligned}$$

$$\Pr(M_1(a) = m_1|v) = \Pr(m_1|v, a)$$

Based on the above three expressions,

$$\begin{aligned} & \sum_{m_1, m_2, m'_1} E[Y(a, m_1, m_2)] \Pr(M_1(a') = m_1) \Pr(M_2(a'', m'_1) = m_2) \Pr(M_1(a''') = m'_1) \\ &= \sum_{v, m_1, m_2, m'_1} E[Y(a, m_1, m_2)|v] \Pr(M_1(a') = m_1|v) \Pr(M_2(a'', m'_1) = m_2|v) \Pr(M_1(a''') = m'_1|v) \Pr(v) \\ &= \sum_{m_1, m_2, m'_1} \sum_{l_1, l_2} E[Y(v, a, l_1, m_1, l_2, m_2)] \Pr(l_1|v, a) \Pr(l_2|v, a, l_1, m_1) \Pr(m_1|v, a') \sum_{l'_1} \Pr(m_2|v, a'', m'_1, l'_1) \Pr(l'_1|v, a''', m'_1) \end{aligned}$$